Mathematics and Its Applications

Ekkehard Krätzel

Lattice Points

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Series Editor's Preface

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exampla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet Union, Eastern Europe, and Japan, it was decided to devote special attention to the work emanating from these particular regions. Thus it was decided to start three regional series under the umbrella of the main MIA programme.

As the author writes in his preface, this book is devoted to a special part of number theory, namely the old and venerable question of finding out how many lattice points there are in (large) closed domains in Euclidean space. That sounds quite specialized and not at all a topic for the MIA series. However, though this book does not deal with them, questions of estimating how many lattice points there are in given domains are likely to crop up in quite varying parts of mathematics, and there are important relations with even more parts e.g. automorphic forms and Kleinian groups, though there the relations seem mostly to point in the direction of applications of these fields to lattice point problems instead of the otherway. Traditionally, lattice points estimates — the geometry, of numbers — are important in exponential sums. They obviously have relevance to packing and covering problems, and hence to the many applications areas (including structure of matter) of these fields. Further, it is not difficult to see that they are important in computational geometry, and combinatorial optimization problems are important potential application areas of 'lattice point theory'. As said already, this book is not about these application fields. Instead it provides a thorough treatise and the conceptional background of the field, indispensable for those who used (to adapt) these and similar results in various situations. Incidentally, the field is also famous for its many unsolved problems and conjectures; and it is a field in which much is happening with some hundreds of papers a year.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, February 1988

Michiel Hazewinkel

Preface

This book is devoted to a special problem of number theory, that is the estimation of the number of lattice points in large closed domains of ordinary Euclidean spaces. This problem is closely related to the problem of estimation of exponential sums. Van der Corput developed an important method of dealing with such sums at the beginning of our century. His method has initiated a new period of intensive development during the last few years. Beside this method some interesting applications of this theory to the circle and sphere problems, to Dirichlet's divisor problem and some generalizations of these problems are considered in this book. The distribution of powerful numbers and finite Abelian groups are also investigated. It is, however, impossible to describe åll the interesting problems. Furthermore, the estimations proved are not the very best in all cases. Further improvements can be obtained, however, by refinements of the method; whereupon many technical difficulties arise. The object of this book is to acquaint the reader with the fundamental results and methods so that he is able to follow up the original papers.

I wish to express my gratitude to G. Horn, H. Menzer and L. Schnabel for reading the manuscript and for correcting a large number of mistakes. I should also like to this opportunity to thank M. Fritsch for valuable assistance in checking the English.

Jena, 1987 **Ekkehard Krätzel**

Notation

The following notations and conventions will be used throughout the book.

If f and g are two functions and $g \geqq 0$, we write

$$f = O(g) \quad \text{or} \quad f \ll g$$

if there is an absolute constant c such that $|f| \leqq cg$, and the inequality will be valid over the entire range of definition of the functions. An equation of the form

$$f = h + O(g)$$

means that $f - h = O(g)$. The expression $g \gg f$ means the same as $f \ll g$, but it will be used only when both f and g are non-negative. If we have both $f \ll g$ and $f \gg g$, then we write

$$f \asymp g .$$

In the next notations functions of a single variable are considered. If

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 ,$$

we write

$$f(x) \sim g(x) \qquad \text{as } x \to \infty .$$

If

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 ,$$

we write

$$f(x) = o(g(x)) \qquad \text{as } x \to \infty ,$$

where $g(x)$ may be supposed to be positive. Again, an equation of the form

$$f(x) = h(x) + o(g(x)) \qquad \text{as } x \to \infty$$

means that $f(x) - h(x) = o(g(x))$ as $x \to \infty$. Whenever it can be done without causing confusion, we omit the additional notation "as $x \to \infty$".

We use the notation

$$f(x) = \Omega(g(x))$$

if the relation $f(x) = o(g(x))$ does not hold. If this is the case, then a positive constant K exists such that

$$|f(x)| > Kg(x),$$

$g(x) > 0$, for values of x surpassing all limit. We write

$$f(x) = \Omega_+(g(x))$$

if

$$f(x) > Kg(x)$$

is satisfied by values of x surpassing all limit. Similarly, we write

$$f(x) = \Omega_-(g(x))$$

in the case of

$$f(x) < -Kg(x).$$

If we have both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$, we write

$$f(x) = \Omega_\pm(g(x)).$$

Also, an equation of the form

$$f(x) = h(x) + \Omega(g(x))$$

means that $f(x) - h(x) = \Omega(g(x))$. Analogous notations are used for the other Ω-symbols.

If x is a real number, $[x]$ denotes the greatest integer not exceeding x, and $||x||$ denotes the distance from x to the nearest integer. This means that

$$||x|| = \min(x - [x], 1 - (x - [x])).$$

The abbreviation

$$\psi(x) = x - [x] - \frac{1}{2}$$

is used throughout the book. Because of the periodicity we obtain at once the so-called trivial estimation

$$\psi(x) = O(1).$$

Sometimes we use the notation $e(x)$ for the function

$$e(x) = e^{2\pi i x}.$$

Let A be a bounded set. The number of elements of A will be denoted by $\# A$ or

$$\# A = \sum_{x \in A} 1 .$$

It may happen that the summation conditions of a sum are rather complicated. Then we write these conditions separately and use the notation $SC(\Sigma)$. For example, instead of

$$F(x) = \sum_{a \leqq n \leqq x} f(n)$$

we write

$$F(x) = \Sigma f(n) , \qquad SC(\Sigma): \quad a \leqq n \leqq x .$$

Similarly, for

$$F(x) = \int_a^x f(t) \, \mathrm{d}t$$

we write the integration conditions separately such that we get the representation

$$F(x) = \int f(t) \, \mathrm{d}t , \qquad IC(\int): \quad a \leqq t \leqq x .$$

It is also possible that we have both summation and integration. Then we use for the expression

$$F(x) = \sum_{a \leqq g(n,t) \leqq x} \int f(n, t) \, \mathrm{d}t$$

the notation

$$F(x) = \Sigma \int f(n, t) \, \mathrm{d}t , \qquad SC(\Sigma \int): \quad a \leqq g(n, t) \leqq x .$$

A sum

$$\sum_{a < n \leqq b}' f(n)$$

means that the possible term $f(b)$, b being an integer, gets the factor $1/2$.

Similarly, in the sum

$$\sum_{a \leqq n \leqq b}'' f(n)$$

both terms, $f(a)$ and $f(b)$, get factors $1/2$.

Chapter 1

Introduction

Let a Euclidean space with respect to a Cartesian coordinate system be given. Then we consider the set of points whose coordinates are integers. These points form a lattice and therefore they are called lattice points. Lattice point theory is then concerned with the estimation of the number of lattice points in large closed domains. Historically, the first problems which were investigated in this connection were the circle problem introduced by C. F. Gauss and the problem of estimation of the number of lattice points under a hyperbola, firstly considered by P. G. L. Dirichlet. At the present time there is a widespread interest not only for two-dimensional but also for many-dimensional problems. In this introduction we shall show that the number of lattice points in a closed domain can be approximated by its volume in general. We shall develop some important summation formulas, and at last it will be seen that the problem of estimation of lattice points is connected with the problem of estimation of some exponential sums.

1.1. LATTICE POINTS IN PLANE DOMAINS

We begin with the classical circle problem. Given the Cartesian coordinate system (ξ, η). Consider the circle $\xi^2 + \eta^2 = x$ $(x \geqq 1)$. In order to estimate the number

$$G = \# \{(n, m): n, m \in \mathbf{Z}, n^2 + m^2 \leqq x\}$$ ($\mathbf{Z}$ denotes the set of integers)

of lattice points within it, let us use a simple geometrical argument.

We associate each lattice point $P_i = (\xi_i, \eta_i)$ of the circle with the square

$$Q_i = \left\{(\xi, \eta): |\xi - \xi_i| < \frac{1}{2}, |\eta - \eta_i| < \frac{1}{2}\right\}$$

having the area 1. Therefore, the set of lattice points of the circle is associated with the union of the squares Q_i (see Fig. 1). These squares are included in the circle $\xi^2 + \eta^2 = \left(\sqrt{x} + \frac{1}{2}\sqrt{2}\right)^2$. Hence, the number of lattice points is smaller than the area of this circle, and we obtain

$$G \leqq \pi\left(\sqrt{x} + \frac{1}{2}\sqrt{2}\right)^2 < \pi x + 2\pi\sqrt{x}.$$