HARRIS

Hydromagnetic Channel Flows

by
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PREFACE

Recent scientific and technological advances have brought a great awakening of interest in the exploitation of the phenomena encountered in magnetohydrodynamics, the study of flows of electrically conducting fluids in the presence of electromagnetic fields. Ten years ago many of these phenomena interested only a small group of astrophysicists and geophysicists; today nearly every university and many industrial and government laboratories support teams of engineers and scientists working on some aspects of these problems. Present and proposed applications include the pumping of conducting liquids, central-station power generation, plasma confinement for the fusion reaction, and propulsion and flight control for rocket and hypersonic aerodynamic vehicles. In addition, there is the continuing hope that these studies will increase our understanding of gaseous physics and the structure of cosmic bodies.

This book presents analyses for three flows of viscous, incompressible, electrically-conducting fluids in high-aspect-ratio rectangular channels subjected to transverse magnetic fields. The situations considered, turbulent flow in the presence of a d-c magnetic field and both laminar and turbulent conditions in induction-driven flows, are motivated by the first two applications listed above, pumping and power generation. These are among the simplest types of magnetohydrodynamic flows and, because of their close relations to hydrodynamic channel flows, are termed here hydromagnetic channel flows.

The relative simplicity of these flows occurs because the interaction between the magnetic field and the mechanical motion of the fluid is primarily one way. The magnetic field often has a substantial effect on the mechanical motion, but the mechanical motion seldom effects much change in the magnetic field; these flows usually operate with low values of the magnetic Reynolds number. Viewed mechanically, these flows resemble conventional hydrodynamic flows in which are superposed additional pressure gradients and shear stresses caused by electromagnetic forces. Viewed electrically, they resemble conventional rotating machines in which the rotating armature is replaced by the moving fluid and the shaft torque by the mechanical pressure and friction loading.

Because of the close relationships between hydromagnetic channel flows and other well-understood engineering devices, equipments using these flows provide today the technically most advanced applications of magnetohydrodynamics. Several varieties, including both d-c and induction types, of electromagnetic pumps for the liquid metals used as coolants and solvents in nuclear reactors have been technical realities for about five years. Recently, the schemes for magnetohydrodynamic power generation have progressed beyond their long-held status as inventors' dreams for the production, without the intervention of mechanical devices with highly stressed moving parts, of useful electric power from thermal energy to the point where they are subject to hard technical and economic evaluation.

The application of hydromagnetic flows to date have been in lower-power devices where efficiency of energy conversion has not been a major factor. For this reason, the details of the electromechanical interactions within these flows have been more a matter for scientific curiosity than practical concern. Pump designers have done very well by applying a background of experience in electric machine design and the analogy noted here between hydromagnetic pumps and electric motors, without concerning themselves much with the mechanical aspects of the flow. As the power level of hydromagnetic equipment increases, however, the efficiency of these machines and the effects of flow structure on performance will become more important. The major purposes of the analyses presented here are the development of an intuitive picture of the internal structure of a turbulent hydromagnetic flow and of calculation techniques for the estimation of the power division in these flows.

The differences between this work and that of previous investigators concerned with turbulent hydromagnetic flows are caused by a difference in the starting points of the analyses. Earlier workers started with the solutions for laminar hydromagnetic flows provided by J. Hartmann about twenty years ago and tried to estimate conditions causing the onset of turbulence and the ensuing changes in flow structure. Thy physical experiment most relevant to this approach consists of setting up a pressure-driven laminar hydromagnetic flow, then increasing the pressure gradient until the flow becomes turbulent. In this book we start with the solutions for a turbulent hydrodynamic flow, which also have been known for over twenty years, and attempt to estimate the changes in flow structure occurring as an increasing magnetic field is applied to the channel. The experiment relevant to this approach consists in establishing a turbulent pressure-driven flow with an electrically conducting fluid, then applying a continually larger transverse magnetic field until the flow conditions become laminar. The important differences between these approaches, however, are associated not with the corresponding thought experiment, but rather with the analytical techniques natural to each. Investigators taking the first approach relied mainly on the mathematical tools of electromagnetic field theory and classical hydrodynamics. The approach taken here leads naturally to use of the "semi-empirical" techniques of modern fluid mechanics which depend on the combined use of the basic mathematical laws, a dimensional analysis, and the results of experiments. The relative success of this work merely indicates that turbulent hydromagnetic flows are more closely related to turbulent hydrodynamic flows than to laminar hydromagnetic flows.

The results described here are derived from a thesis submitted to the Department of Electrical Engineering in partial fulfillment of the requirements for the degree of Doctor of Science. During the long course of study and research culminating in this thesis, the author has incurred many more debts to teachers and colleagues than can be acknowledged here. Several persons, however, have been particularly helpful in recent times. Professors D. C. White and H. H. Woodson introduced the author to the study of magnetohydrodynamics and freed him from other duties when the course of this research was established. Professor P. F. Chenea suggested the application of the "semi-empirical" techniques of hydrodynamics to turbulent hydromagnetic channel flows when other methods of analysis had proven fruitless. Professor W. D. Jackson, who served as thesis supervisor, contributed valuable advice, encouragement, and criticisms during many discussions. Mr. J. W. Poduska performed most of the calculations and contributed many suggestions that have been incorporated here.

December, 1959 | Lawson P. Harris

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CONTENTS

Chapter 1

THE STUDY OF HYDROMAGNETIC TURBULENCE

Although hydromagnetic channel flows used in engineering devices usually are turbulent rather than laminar, we know very little about the structure of turbulent flows in which hydromagnetic effects are significant. The major purpose of this study is to develop an analysis, based on the semiempirical techniques of fluid mechanics, of the internal structure of the turbulent hydromagnetic flow of an incompressible fluid in a high-aspect-ratio rectangular channel subjected to a d-c transverse magnetic field. A secondary purpose is to indicate that, under certain conditions, the solutions obtained for the d-c turbulent flow should be applicable, with minor modifications, to induction-driven flows. The principal results of this study consist of the figures in Chapter 7, giving normalized mean velocity profiles, current distributions, and velocity correlations for the d-c turbulent flow over a range of operating conditions. These provide a basis for both a qualitative and a quantitative picture of the internal structure of the flow.

Even for laminar hydromagnetic flows, only a few mathematical solutions have been obtained. All of these are for conditions closely related to one of the approximately half-dozen general situations in which useful solutions have been obtained for laminar hydrodynamic flows. Those that are available, however, are sufficient to give at least a qualitative picture of most laminar hydromagnetic flows of technical interest, and the electromechanical interactions in such flows have been well understood for some time. With this background plus the widely held conviction that turbulent flows inevitably are accompanied by higher power losses than laminar flows, we might be tempted to conclude that, as a practical matter, we should try to avoid turbulence and design all devices for laminar flow. A quick survey of hydraulic equipment, however, suggests that such a conclusion would be erroneous. Except for the laminar films always present on solid boundaries, laminar flow in hydraulic equipment is a rarity; viscometers and hydrodynamic bearings probably are the only technically important applications. It is true that, for the flow geometries, fluid properties, and pressure gradients or flow rates present in most hydraulic equipment, laminar flow would yield lower power losses than turbulent flow, if it could exist. But this fact is largely irrelevant to design activity because the nature, laminar or turbulent, of a hydrodynamic flow is determined by a single parameter, the Reynolds number, and cannot be selected independently of the other flow conditions. The design of power-level equipment for laminar-flow conditions apparently always yields bulkier, more complex configurations and/or higher total power losses than reasonable turbulent-flow designs. There is yet no reason to believe that this experience of hydraulic engineers will be reversed in the design of hydromagnetic equipment.

The study of hydromagnetic turbulence, however, is interesting for reasons quite apart from those associated with the design of hydromagnetic machines. A clearer understanding of turbulent hydromagnetic flows will result in greater insight into strictly hydrodynamic turbulence and into the mechanism of transition between laminar and turbulent flow regimes. The addition of the magnetic field provides a parameter that can be varied at will to change the loss mechanism and the distribution of losses across the flow. The plasma dynamicists reportedly have encountered turbulence-like activity in plasmas with very high energy densities in devices such as the "Stellerator." This device represents one application where the onset of turbulence is a disaster causing rapid destruction of the plasma. The continuum formulation outlined in Chapter 3 and used in this study is too crude to yield detailed predictions of the behavior of plasmas in strong magnetic fields. It does appear, however, to present possibilities for a spontaneous turbulent interaction between velocity and magnetic-field fluctuations in an essentially stationary fluid, when the magnetic field and electrical conductivity become sufficiently high.

Attempts to gain some understanding of hydromagnetic turbulence have taken several forms. Some work has been done, notably by Batchelor[2] and Chandrasekhar,[4] toward extending the statistical theory of turbulence to hydromagnetic flows. Unfortunately, this theory is still less developed than the statistical theory of hydrodynamic turbulence, which cannot yet treat effectively any technical flow. Lock[10] has attempted by use of perturbation theory to compute the critical Reynolds number at transition between laminar and turbulent flow for hydromagnetic flow between two parallel planes in the presence of a d-c transverse field. Stewart[14] has performed similar calculations for the flow with a longitudinal magnetic field. To date, these results have not been verified experimentally; Lock's critical Reynolds numbers exceed by a factor near 200 those obtained experimentally by Murgatroyd.[12] These large discrepancies have been attributed to the facts that Lock's results refer to the instability of laminar flow in the presence of only infinitesmal disturbances, while Murgatroyd's apply to the damping out of turbulence in the presence of fairly large disturbances. The critical Reynolds numbers for these two phenomena can differ by large factors in hydrodynamic flows and, probably, in hydromagnetic flows also.

For our present study, the most useful past work consists of two sets of experiments, one by Hartmann and Lazarus[7] performed in 1937 and one by Murgatroyd[12] performed in 1953, in which measurements were made of pressure gradient, flow rate, and magnetic flux density on flows of mercury in rectangular channels subjected to d-c transverse magnetic fields. The results of these two sets of experiments, which are described in detail in Chapter 5, exhibited both similarities and differences that have been informative in the present study. There have been, of course, previous efforts to explain or correlate the trends shown in these experiments from some theoretical bases. The two analyses accompanying the presentations of the data were restricted to the correlation of the pressure drop versus flow data, and were based on rather unsatisfactory applications of dimensional analysis. Hartmann and Lazarus attempted to separate their results into a "turbulence-damping effect," unique to turbulent flows, and a "viscosity effect," similar to that found in laminar flows. The procedures by which they effected this separation, however, involved several arbitrary decisions and were based on a rationale implying that the turbulence-damping effect is independent of the electrical conductivity of the fluid. As Murgatroyd pointed out, there is no justification, either from theory or experiment, for this assumption. Murgatroyd proposed instead a correlation scheme based on the assumption that the friction factor f (defined in Chapter 2) for the turbulent hydromagnetic flow is independent of the fluid viscosity. This assumption also is untenable. Although turbulent flows exhibit a tendency for the local structure in the central portion of the flow to be independent of fluid viscosity, the friction factors for channel flows always depend strongly on fluid viscosity. If the viscosity is denoted by η, the friction factors in laminar hydrodynamic and hydromagnetic flows tend to vary as η and $\sqrt{\eta}$, respectively, while $1/\sqrt{f}$ for turbulent hydrodynamic flow tends to vary linearly with $\log \eta$.

Both Murgatroyd[12] and Hartmann and Lazarus[7] realized that the theoretical portions of their papers were at best incomplete, and indicated that they were carrying on further work that would be published in the future. These works have not yet appeared, however, and the phenomena exhibited in the published experiments have remained largely unexplained.

The objectives of the present study are somewhat more ambitious than those of the theoretical efforts just described, for we wish not only to provide a correlation scheme that will relate the two sets of experimental data available but also to use these data to deduce the internal structure of the flow and thereby gain a better understanding of the details of the interactions within the flow. The approach used here is based on the suppositions that turbulent hydrodynamic and hydromagnetic flows should be closely related, much as the corresponding laminar flows, and that the techniques that have proven most fruitful in the analysis of turbulent hydrodynamic flows might also permit a successful attack on the turbulent hydromagnetic flow. Thus, we are led to apply the "semiempirical" techniques of fluid mechanics to the present problem. These combine the use of the basic hydromagnetic equations, the available experimental data, and the logic of dimensional analysis, drawing from each all the information that can be obtained simply.

Because these techniques and the results obtained by their use in hydrodynamics are unfamiliar to many electrical engineers, we describe in Chapter 2 the basic method of analysis used, develop the accepted formulas for velocity profiles, velocity correlations, and friction factors in turbulent hydrodynamic flows, and discuss the agreement between these theoretical predictions and parts of the vast body of supporting experimental data that has accumulated in the past thirty years.

In Chapter 3, we turn to the basic equations governing incompressible hydromagnetic flows. This chapter treats three topics. In the first section, we write the basic equations, select a normalization scheme, and derive several useful forms for the normalized equations. The second part consists of two sections in which we develop the boundary conditions for the flow with a d-c transverse magnetic field and also a classification, which turns out to be valid for both the d-c and a-c flows, of all the interesting flow variables according to their even or odd spatial symmetry. The third part consists of the derivation of a partial solution in which we express the mean velocity and mean perturbation field in terms of the parameters of the system and definite integrals of two statistical correlations. The principal results of this part, however, are the conclusions that the normalized mean velocity can be approximated by a function of M, R^*, and ξ only, where M is the channel Hartmann number, R^* the Reynolds number, and ξ a normalized distance, and that the normalized perturbation field can be approximated by the product of R_v and another function of M, R^*, and ξ. Here R_v is the magnetic Reynolds number. The worker in magnetohydrodynamics will recognize these as approximations suited to simple flows with small magnetic Reynolds numbers in which the perturbation fields are proportional to R_v but so small that they have little effect on fluid motion.

In Chapter 4, we combine the conclusions of Chapter 3 with the analytical techniques of Chapter 2 to gain the simplification obtainable from the logic of dimensional analysis. In this process, we introduce as assumptions the general statements of two empirical laws of hydrodynamic channel flows. These are (1) that at points near the boundaries, the mean-velocity profile is independent of the channel span and (2) that at points near the center, the local structure of the mean flow is independent of fluid viscosity. The first assumption implies that the flow near the boundary looks like a semi-infinite flow past a single surface; the second implies that viscous dissipation in the central part of the channel is negligible. Both of these assumptions are valid for turbulent hydrodynamic flows and for laminar hydromagnetic flows with high Hartmann numbers. Their final justification here, however, rests with the end results of the study. The principal results of Chapter 4 are formulas giving the velocity profile and friction factor in terms of two unknown functions. One of these depends on M^2/R^* (which does not involve fluid viscosity) and gives the distortion in the velocity profile caused by hydromagnetic effects. The other depends on M/R^* (which does not involve the channel span) and represents, over the central part of the channel, a constant addition to the mean velocity caused by hydromagnetic effects.

The theoretical analyses presented in Chapters 3 and 4 reduce the problem of finding the normalized mean-velocity profile from the determination of a single function of four variables, M, R^*, R_v, and ξ, to the determination of two functions, each dependent on a single variable. We must turn to the experimental data for further simplification. The two sets of experimental data used here are described in some detail in Chapter 5. Chapter 6 is devoted to the correlation of theory and experiment. Fortunately, the common characteristics exhibited by the data give good reason for neglecting one of the unknown functions and provide an empirical determination of the other. The close fit, when plotted properly, of points derived from all the experimental data for highly turbulent flows to a single function of M^2/R^* represents an experimental justification of the preceding theory. The data also suggest a method for computing an analytic approximation to this unknown function. The assumptions on which this approximation are based, however, are not valid at low values of M, and the result consequently differs from the empirical curve by a nearly constant value for moderate and large M^2/R^*. This approximation does serve, nevertheless, as a check on the internal consistency of our results.

There is some difficulty, discussed fully in Chapter 6, in the correlation with theory of the data obtained from flows that were almost laminar. In effect, we are forced to admit that there is no sharp boundary between laminar and turbulent hydromagnetic flows, and that our simplifying assumptions do not always conform to the complex reality of flows that are neither laminar nor very turbulent. Hartmann[6] long ago provided a good analysis of the laminar flow; the present study appears to provide a satisfactory treatment of flows that are highly turbulent. But there still exists a small range of conditions near transition where neither analysis is really valid. This situation is not surprising, however, for the same condition still exists in the analysis of the simpler hydrodynamic flows.

Once all the unknown functions are eliminated from the analysis, we can easily compute mean-velocity profiles, approximate current distributions and velocity correlations, and, if we wished, the mean perturbation fields from the mathematical formulas in Chapters 3 and 4. Chapter 7 contains results of these calculations together with a discussion of some interesting trends occurring at high values of M^2/R^*. These curves are the essential results of the study.

Chapters 8 and 9 constitute a short discussion of laminar and turbulent induction-driven flows. They consist primarily of mathematical analyses based on the hydromagnetic equations set forth in Chapter 3. The major qualitative conclusion derived from these chapters is that, under reasonable operating conditions, the velocity and electromagnetic-field distributions in the induction-driven flows are analogous to those in the corresponding d-c flows. By "reasonable operating conditions," we refer here to situations in which the electrical slip frequencies seen by the moving fluid are sufficiently low that the electrical skin effect is negligible, yet sufficiently high that the zero-average double-frequency components of forces acting on the fluid are effectively filtered out by fluid inertia. When these conditions are satisfied, the transverse magnetic field is uniform across the channel, and the distribution of effective forces acting on the fluid is similar to that in the corresponding d-c flows. Then, the variations in mechanical quantities, such as mean velocity and wall shear stress, in the a-c flow are similar to those found for the corresponding d-c flow, and variations in electrical quantities, such as magnetic flux density, electric field, and current density, differ from those found for the d-c flows mainly by a superposed traveling-wave modulation.

In the induction-driven flows, the magnetic Reynolds number always is a very important variable; the operation of every induction machine depends on the interaction of the fields associated with the electrical excitation and the reaction fields associated with the currents in the moving material. The latter, of course, are the analog of the perturbation fields in the d-c flow, except that they cannot be small if there is significant energy conversion in the flow. Though often large, the effect of magnetic Reynolds number is quite simple, and it appears in our analysis as a variation, with average slip and channel reactance-to-resistance ratio, of the total transverse magnetic field and, therefore, of the effective Hartmann number for the flow.

The analysis in Chapter 8 generally follows lines laid down by Blake in a paper[3] on the design of an induction pump, except that it discusses more fully the mechanical aspects of the flow. The last section of this chapter is a digression in which several well-known analogies between the operation of induction pumps and conventional induction motors are related to the present analysis.

Chapter 9 is similar in content to Chapter 3. Here we show that the equation determining the mean-velocity profile in the induction-driven flow takes a form nearly identical to that governing the velocity profile in the d-c flow. The apparent implication of this result is that the curves for d-c flows presented in Chapter 7 can be adapted, by minor modifications, for application to a turbulent induction-driven flow.

The last chapter, Chapter 10, contains a summary of conclusions derived from this study and several suggestions for interesting future work.

Chapter 2

TURBULENCE IN HYDRODYNAMIC CHANNEL FLOWS

A Semiempirical Analysis

The most fruitful attack on the analysis of turbulence in common hydrodynamic channel flows has been one based on two empirical laws and the theory of dimensional analysis. The principal results of this analysis are predictions of a functional form, containing two undetermined constants, for the variation of friction factor with Reynolds number and a similar prediction for the variation of mean (with respect to time) velocity with distance from the channel wall and Reynolds number. The friction-factor relation has been found, from experiments performed by Nikuradse,[13] Laufer,[9] and other investigators, to be quite accurate over more than three decades of variation in the Reynolds number. The predicted velocity distribution does not satisfy either the boundary condition of zero velocity at the wall or the symmetry requirement of zero velocity gradient at the center of the channel; however, at points not too close to the boundary or the center, it also agrees well with experimental data.

Actually, there are two analyses of the type described here, one for flow in channels with very smooth walls and the other for flow in rough-walled channels. Both of these were developed by Millikan[11] in 1939. These two analyses establish bounds within which any practical situation must fall. Concern here is limited to the case of smooth-walled channels. The aim of this chapter is to outline Millikan's analysis for hydrodynamic flow in a smooth channel; later we shall adapt the reasoning used to the problem of magnetohydrodynamic turbulence in a smooth channel.

Consider a hydrodynamic flow in the x-direction between two smooth planes $y = \pm L$ spaced apart a distance 2L as shown in Fig. 2.1. The following notation is used:

ρ = fluid density
η = fluid viscosity
L = channel half-width
$\bar{u}$ = mean (time) flow velocity
$\bar{U}$ = maximum (space) mean flow velocity
$\bar{V}$ = average (space) mean flow velocity
τ_0 = flow shear stress at wall

The mean velocity $\bar{u}$ is assumed to vary only with y. The average pressure gradient $\nabla\bar{p}$ then has only an x-component, which is a constant. For steady flow, the pressure gradient $\partial\bar{p}/\partial x$ and the wall shear stress τ_0 are related by

$$A\left|\frac{\partial\bar{p}}{\partial x}\right| = S\tau_0 \qquad (2.1)$$

where S and A are, respectively, the wetted-wall perimeter and the cross-sectional area of the channel.

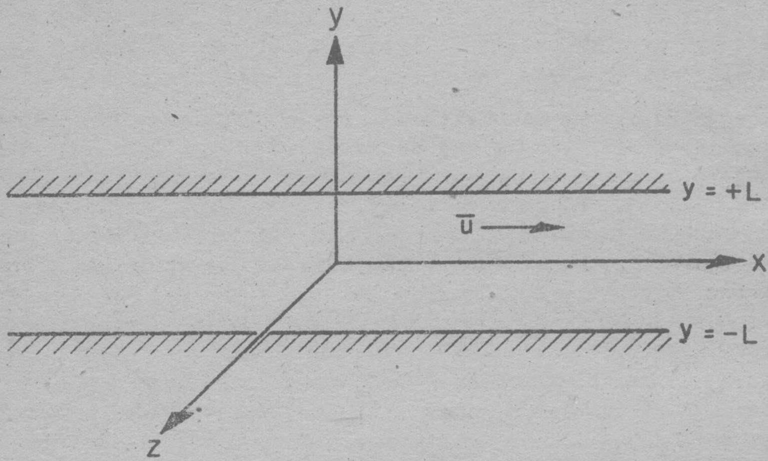

Fig. 2.1. Geometry for hydrodynamic flow between two parallel planes

The quantity

$$D_H = 4A/S \qquad (2.2)$$

is known as the "hydraulic diameter" of the channel. For a circular channel, the hydraulic diameter equals the tube diameter. For a flow between parallel planes, $D_H = 4L$. Equation 2.1 can be written

$$\left|\frac{\partial\bar{p}}{\partial\bar{x}}\right| = \frac{4\tau_0}{D_H} \qquad (2.3)$$

The physical situation is determined by the two relevant physical characteristics ρ and η of the fluid, the characteristic length L for the channel, and the intensity of the friction stress at the wall τ_0. Since $\bar{u}$ is known to vary with distance across the channel, we try to express $\bar{u}$ as

$$\bar{u} = f_1(\rho,\ \eta,\ L,\ \tau_0,\ y) \qquad (2.4)$$

where the right side of Eq. 2.4 represents an arbitrary function of the five arguments. Then if the "friction velocity" $\bar{u}^*$ and the "friction Reynolds number" R^* are defined by

$$\bar{u}^* = \sqrt{\tau_0/\rho} \qquad (2.5)$$

and

$$R^* = \frac{\rho \bar{u}^* L}{\eta} \tag{2.6}$$

and a dimensionless displacement from the wall ξ is defined by

$$\xi = \frac{L - y}{L} \tag{2.7}$$

application of the "Π theorem" of dimensional analysis to Eq. 2.4 yields

$$\frac{\bar{u}}{\bar{u}^*} = f_2(R^*, \xi) \tag{2.8}$$

where f_2 represents another arbitrary function of its two arguments.

The representation attempted in Eq. 2.4 and, therefore, in Eq. 2.8 is not determined uniquely by the physical situation. For example, the variable τ_0 in Eq. 2.4 logically might be replaced by $\bar{U}$, $\bar{V}$, or $(\partial\bar{p}/\partial x)$. In choosing this particular formulation, Millikan was guided by the results of approximately fifteen years of intensive experimental and theoretical efforts, mostly those[1] conducted by Prandtl, von Karman, Fritsch, Nikuradse, and their associates at Göttingen and Aachen. The virtue of this formulation lies in the simplicity of the results obtained and the sparseness of the necessary assumptions. Millikan's results, however, had been obtained earlier by Prandtl and/or von Karman from reasoning based on "mixing-length" theories which postulate a definite mechanism for turbulence analogous to molecular momentum transport.

The first of the empirical laws used by Millikan is known as the Prandtl wall-velocity law. It states that in the region near the wall of the channel, Eq. 2.8 reduces to

$$\frac{\bar{u}}{\bar{u}^*} = \chi(R^*\xi) \tag{2.9}$$

that is, the mean-velocity profile becomes independent of the channel size L. Equation 2.9 is obviously true in the laminar film right next to the wall, for here the velocity $\bar{u}$ varies approximately linearly with a slope given by

$$\frac{d\bar{u}}{d\xi} = \frac{L}{\eta}\tau_0$$

Use of Eqs. 2.5 and 2.6 then gives

$$\left[\frac{d(\bar{u}/\bar{u}^*)}{d\xi}\right]_{\text{wall}} = R^* \tag{2.9a}$$

The importance of Eq. 2.9 in the present analysis, however, lies in its applicability beyond the laminar-film region in the outer extremity of the turbulent core.

The other empirical result needed is von Karman's velocity-defect law. It states that, except for points too close to the wall, the difference (or defect) between the maximum mean velocity $\overline{U}$ and that at the point in question $\overline{u}$ can be written

$$\frac{\overline{U} - \overline{u}}{\overline{u}^*} = \Psi(\xi) \tag{2.10}$$

Thus the local structure of the turbulent core is independent of the fluid viscosity. Unlike Eqs. 2.8 and 2.9, which hold in the forms given only for situations involving smooth walls, Eq. 2.10 is valid regardless of the roughness of the channel walls and thus has something of a universal character.

The independence of the core structure from the fluid viscosity seems at first a very curious phenomenon and deserves some comment. We might expect that:

1. Since all the energy expended in driving a channel flow is dissipated by the fluid viscosity, the viscosity coefficient describing the dissipation mechanism should be an important parameter determining the structure of the flow.
2. As the viscosity is decreased, the flow structure should approach that for a frictionless fluid.

The second item can be dismissed almost immediately because it receives no support from either mathematical theory or physical experiments. The condition $\eta = 0$ reduces the order of the space dependence of the Navier-Stokes equations from second to first and thus changes the character of permissible boundary conditions at solid surfaces. Because the equations are nonlinear, this effect can result in a substantial change in the types of permissible solutions as the fluid viscosity is varied from zero to some small but nonzero value; the solutions need not vary continuously as the coefficients of the Navier-Stokes equations are varied. In engineering practice we encounter essentially incompressible turbulent flows of fluids ranging in properties from those of light oils to those of gases such as air. For these fluids the viscosity coefficient varies by approximately four decades. Experience has shown that the fluid property determining the character, laminar or turbulent, of the flow is not the viscosity but the ratio of viscosity to density (the kinematic viscosity).

There is no real conflict between the first item and Eq. 2.10. All that is required to keep the result expressed by Eq. 2.10 consistent with the engineer's intuition is a variable rate of energy dissipation across the channel. This situation occurs even in simple laminar flows where (for flow between two planes) the power dissipation per unit volume is proportional to the fluid velocity. Here the dissipation rate varies parabolically across the channel, taking a maximum at the center and reaching zero at the wall. In turbulent flow the dis-