ADVANCES IN DIFFERENTIAL GEOMETRY AND TOPOLOGY

Editor

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PREFACE

The aim of this volume is to offer a set of highly qualified contributions on recent advances in Differential Geometry and Topology, with some emphasis on their application in Physics.

A broad range of themes is covered in the book, including convex sets, Kaehler manifolds and moment map, combinatorial Morse theory and 3-manifolds, knot theory and statistical mechanics.

The motivation for publishing the book, thus giving the reader an opportunity to be introduced in attractive areas of current research, was originated by the success of a workshop held in Torino in June 1987 at Villa Gualino.

This meeting has been possible by the financial aid of the Italian Minister of Public Instruction, under the project "Geometry of Manifolds".

Its success was undoubtedly due both to the contributions of the distinguished Mathematicians who accepted our invitation and to the incomparable support offered by the Institute of Scientific Interchange of Torino.

CONTENTS

CONVEX SETS AND KÄHLER MANIFOLDS

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0.1. *Brunn-Minkowski inequality*. Recall that the *Minkowski sum* X+Y of subsets X and Y in the Euclidean space $\mathbb{R}^n$ is the set of the sums $x+y \in \mathbb{R}^n$ for all $x \in X$ and $y \in Y$. An equivalent definition is

$$X+Y = \bigcup_{y \in Y} X+y$$

where X+y denotes the y-translate of X which is the same thing as the sum of X with the one-point set $\{y\}$. Note that
$X+Y = Y+X$ as $x+y = y+x$ in $\mathbb{R}^n$.

0.1 A. *Example*. Let X_ε be the ε-ball in $\mathbb{R}^n$ around the origin. Then, by the second definition, $X_\varepsilon+y$ equals the union of the ε-balls in $\mathbb{R}^n$ with centers in Y which is customary called the *ε-neighborhood* of Y.

0.1.B. **Brunn-Minkowski theorem**. The n-dimensional volume (i.e. Lebegue's measure) *of* X+Y *is bounded from below by*

$$[\mathrm{Vol}\,(X+Y)]^{1/n} \geq (\mathrm{Vol}\,X)^{1/n} + (\mathrm{Vol}\,Y)1/n\,. \qquad (*)$$

Remarks and corollaries.0.1 B_1. We are most interested here in the classical case of (*) where X and Y are *bounded convex* subsets in $\mathbb{R}^n$. Yet, (*) remains valid for arbitrary (measurable) subsets X and Y in $\mathbb{R}^n$ (see 3.1.).

0.1.B_2. Let X and Y be rectangular solids with mutually parallel edges of lengths $a_1,\ldots,$ a_n and $b_1,\ldots, b_n$. Say,

$$X = [o,a_1] \times \ldots \times [o,a_n],$$
$$Y = [o,b_1] \times \ldots \times [o,b_n].$$

Then

$$X+Y = [o,a_1+b_1] \times \ldots \times [o,a_n+b_n]$$

and (*) reduces to the following well-known algebraic inequality,

$$\left(\prod_{i=1}^{n} (a_i+b_i)\right)^{1/n} \geq \left(\prod_{i=1}^{n} a_i\right)^{1/n} + \left(\prod_{i=1}^{n} b_i\right)^{1/n}. \qquad (*)$$

0. 1 B_3. Let Y have *smooth* boundary ∂Y and take the ε-ball B_ε in r^n for X. Then, one easily sees (compare 0.1.A) that the (n-1)-dimensional volume of ∂Y satisfies

$$\mathrm{Vol}\, \partial Y = \lim_{\varepsilon \to 0} \varepsilon^{-(n-1)} (\mathrm{Vol}\, (Y+B_\varepsilon) - \mathrm{Vol}\, Y).$$

Thus, (*) yields the *Euclidean isoperimetric inequality*,

$$\mathrm{Vol}\, \partial Y \geq C_n (\mathrm{Vol}\, Y)^{\frac{n}{n-1}}$$

where C_n denotes the (n-1)-dimensional volume of the boundary sphere of the ball $B \subset \mathbb{R}^n$ normalized by the condition Vol B = 1.

0.2. *Hodge-Teissier-Hovanski inequality.* Consider the Cartesian product of two complex projective spaces $P_1 \times P_2$ with the standard metric and let V be a complex algebraic subvariety in $P_1 \times P_2$ of complex dimension n. (The reader unfamiliar with this terminology is addressed to section 3.3.). Denote by $V_1 \subset P_1$ and $V_2 \subset P_2$ the projections of V to P_1 and to P_2.

0.2.A. **Alqebraic Brunn-Minkowski.** *If* V *is irreducible* (see 3.3.), *then the* 2n-*dimensional volumes of* V, V_1 and V_2 *satisfy*

$$(\mathrm{Vol}\, V)^{1/n} \geq (\mathrm{Vol}\, V_1)^{1/n} + (\mathrm{Vol}\, V_2)^{1/n}. \qquad (+)$$

Remark and Corollaries. 0.2.A_1. If n=1, then (+) is trivial. In fact one has equality in this case.

0.2.A_2. If n=2 then (+) is equivalent to the *Hodge index theorem* (see 3.3.). Note that (+) may easily fail if V is reducible. For example, take

$$V = (V_1 \times v_2) \cup (v_2 \times V_2)$$

for $V_i \subset P_i$ and $V_i \subset P_i$ for i = 1,2. Then

$$(\mathrm{Vol}\ V)^{1/n} = (\mathrm{Vol}\ V_1 + \mathrm{Vol}\ V_2)^{1/n} < (\mathrm{Vol}\ V_1)^{1/n} + (\mathrm{Vol}\ V_2)^{1/n}.$$

0.2.A_3. The inequality (+) for $n \geq 3$ was discovered by Hovanski and Teissier. Their proof (see 3.3.) goes by induction on n = dim V which starts with n = 2, where the inductive step for $n \geq 3$ is realized by intersecting V with an appropriate hypersurface H in $P_1 \times P_2$, and where the irreducibility of the intersection $V \cap H$ (having the dimension by one less than V) is achieved with the *Bertini* theorem (see 3.3.). In fact, Teissier and Hovanski proved a refinement of (+) which is parallel to the *Alexandrov-Fenchel inequality* for convex sets (see 1 .6.). Alexandrov gave two proofs of his inequality. The first proof (see $[Al]_1$) is combinatorial and resembles the algebra-geometric argument by Hovanski and Teissier (instead of Hodge index theorem for n = 2 Alexandrov uses a corresponding geometric inequality of Minkowski). The second proof by Alexandrov (see $[Al]_2$) appeals to the elementary theory of second order elliptic operators. We shall see in §2 that a modern reedition of Alexandrov's proof (exterior products of differential forms instead of mixed discriminants of quadratic forms) yields the Hodge-Teisdsier-Hovanski inequality as readily (even faster) as it yields the Alexandrov-Fenchel inequality (for n = 2 Alexandrov's argument is essentially equivalent to Hodge's proof of his index theorem).

0.2.B. *Moment map, Legendre transform and the implication* (+) $\Rightarrow$ (*). A variety V is called *toral* if it admits an isometric (for the metric induced from $P_1 \times P_2 \supset V$) action of the torus T^n. Such an action induces what is called *the moment map* $M : V \Rightarrow \mathbb{R}^n$ which is defined with the induced *symplectic* (Kähler) *form* on V (see 3.2.). Similar (moment) maps, also denoted M, are defined for V_1 and V_2. One shows (see 3.2.) that M preserves volumes (up to a normalizing constant) and that the image M(V) is the Minkowski sum of the moment-images of V_1 and V_2,

$$M(V) = M(V_1) + M(V_2).$$

Thus (*) $\Rightarrow$ (+) for toral varieties V. On the other hand one knows (see 2.4. and 3.3) that for any pair of *convex polyhedra* X_1 and X_2 in $\mathbb{R}^n$ with vertices in the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, there exists a toral variety V such that $M(V_i) = X_i$ for i = 1,2. Using this along with an approximation of convex sets by polyhedra with rational vertices one derives (*) from (+) for all *convex* subsets in $\mathbb{R}^n$.

0.2.B_1. *REMARK.* The correspondence between toral varieties and convex polyhedra goes back to Newton and Minding (see the discussion by Hovanşki in chapter 4 of [Bu-Za]). The relation between (+) and (*) was discovered by Teissier and Hovanski (see [T] and [B-z]). The approach using the moment map is due to Arnold and Atiyah (see $[At]_2$).

0.2.B_2. The action of T^n on V can be complexified to an action of $(\mathbb{C}^\times)^n = T^n \times (\mathbb{R}_+^\times)^n$ on V (see 3.2.). Then the restriction of the moment map to the $(\mathbb{R}_+^\times)^n$-orbits can be identified with the *Legendre transform* for the *Kähler potential* on V (see 3.2. and $[At]_2$). Note that this kind of Legendre transform is built in into Alexandrov's argument as it applies to *supporting functions* of the convex sets in question (see $[Al]_2$).

§1. *Legendre transform, mixed volumes and Kähler forms.* Consider a C^1-function f on a linear space L and let us interprete the differential of f as a map of L into the dual space L', say $D_f : L \to L'$ (If L is a Hilbert space one can use instead the *gradient* map $L \to L$ for $x \to \mathrm{grad}_x f$ that some people find more geometric).

Recall that a map $\varphi : L \to L'$ is called *monotone increasing* if

$$< \varphi(x_1) - \varphi(x_2), x_1 - x_2 > \geq 0$$

for all x_1 and x_2 in L. One calls φ strictly increasing if the above inequality is strict for all x_1 and $x_2 \neq x_1$. It is obviou that every strictly increasing map is one-to-one. In particular, if such a φ is continuous and L is finite dimensional, then φ is a homeomorphism. Also observe that the map $\varphi = D_f$ is (strictly) increasing if and only if f is (strictly) convex. Thus we obtain the

1.1. *Homeomorphism property. If is a strictly convex C^1-function on* $\mathbb{R}^n$ *then the map* $D_f : \mathbb{R}^n \to \mathbb{R}^n$ *is a homeomorphism* (*onto an open subset in* $\mathbb{R}^n$).

The following property is somewhat more exciting.

1.2. *Convexity theorem. If* f *is convex then the closure of the image* $D_f(\mathbb{R}^n) \subset \mathbb{R}^n$ *is convex.*

Proof. For an arbitrary function f on L denote by $L'_f \subset L'$ the set of those linear functions y on L for which

$$f - y \geq \text{const} > -\infty . \qquad (1)$$

This means that the graph $\Gamma_f \subset L \times \mathbb{R}$ lies above that of the function y-const, and so L'_f contains the image $D_f(L) \subset L'$ for convex functions f. In fact if $y = d_{x_0} f$ then the graph $H_y \subset L \times \mathbb{R}$ of the function $y(x) + f(x_0) - y(x_0)$ is tangent to the graph $\Gamma_f \subset L \times \mathbb{R}$ at the point $(x_0, f(x_0))$. Hence Γ_f lies above the hyperplane H_y for all $y \in D_f(L)$ in the case where f is convex.

Next we observe the subset $L'_f \subset L$ is convex, as the inequalities

$$f - y_1 > -\infty \text{ and } f - y_2 > -\infty$$

obviously imply the same inequality for convex combinations of y_1 and y_2,

$$f - (ty_1 + (1-t)y_2) > -\infty .$$

To conclude the proof we must show that L'_f is *contained* in the closure of $D_f(\mathbb{R}^n)$. This is equivalent to

$$\inf_{x \in L} \|d_x g\| = 0$$

for the functions g = f-y satisfying the above (boundness away from $-\infty$) condition (1). In fact, if (2) is violated and
$\|d_x g\| > \varepsilon > 0$ for all $x \in L$, then $\inf g = -\infty$ as the following trivial lemma shows.

1 .2.A. *Let* X *be a complete metric space and* $g : X \to \mathbb{R}$ *a continuous function, such that for every* $x \in X$ *there exists* $x' \in x$ *different from* x, such that

$$g(x) - g(x') \geq \varepsilon \, \text{dist}(x, x'),$$

where ε is a fixed positive number. Then

$$\inf_X g(x) = -\infty .$$

1.2.B. *Remark.* The above discussion presents a tiny piece of the convex duality theory going back to Legendre whose name is attached to the transform of f from L to $D_f(L) \subset L'$ by the map D_f,

$$f^{\cdot}(y) = f(D_f^{-1}).$$

The Legendre transform $f^{\cdot}$ of f is correctly defined for strictly convex functions f as D_f is one-to-one. In this case $f^{\cdot}$ also is strictly convex and satisfies *Legendre duality relation* $D_{f^{\cdot}} = D_f^{-1}$ under an appropriate (reflexivity) condition on L (which is obviously satisfied for $L = \mathbb{R}^n$).

1.3. *Minkowski additivity* of L'_f and $D_f(L)$. If $y_1 \in L_{f_1}$ and $y_2 \in L_{f_2}$ (see (1) above), then, obviously) $\inf_L(f_1+f_2-y_1-y_2) > -\infty$, that is y_1+y_2 is contained in the Minkowski sum of L_{f_1} and L_{f_2}.

In other words

$$L'_{f_1} + L'_{f_2} \subset L'_{f_1+f_2} . \tag{2}$$

It is equally obvious that

$$D_{f_1}(L) + D_{f_2}(L) \supset D_{f_1+f_2}(L), \tag{3}$$

as $d(f_1 + f_2) = df_1 + df_2$.

Thus we obtain the following

1.3.A. **Additivity.** *If* f_1 *and* f_2 *are strictly convex functions on* $\mathbb{R}^n$, *then*

$$D_{f_1+f_2}(\mathbb{R}^n) = D_{f_1}(\mathbb{R}^n) + D_{f_2}(\mathbb{R}^n). \tag{4}$$

1.4. *Brunn-Minkowski theorem for convex functions* f. Let $[D^2f]^n$ denote the determinant of the Hessian D^2f of f ,

$$[D^2f]^n = \mathrm{Det}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

and note that $[D^2f]^n$ equals the Jacobian of the map $D_f : \mathbb{R}^n \to \mathbb{R}^n$.
Therefore

$$\mathrm{Vol}\, D_f(\mathbb{R}^n) = \int_{\mathbb{R}^n} [D^2 f]^n \qquad (5)$$

for all strictly convex C^2 -functions f on $\mathbb{R}^n$.

1.4.A. *Remark.* For an arbitrary (non-smooth) ccnvex function f one can define $[D^2f]^n$ as a measure on $\mathbb{R}^n$ and show that

$$\mathrm{Vol}\, L' = \int_{R^n} [D^2 f]^n .$$

Now we apply (Brunn-Minkowski) inequality (*) in 0.1.B. to $D_{f_1}(\mathbb{R}^n)$ i = 1,2 and obtain the following

1 .4.B. **Theorem.** *Every two strictly convex* C^2*-functions* f_1 *and* f2 on $\mathbb{R}^n$ *satisfy*

$$\left(\int_{R^n} \left[D^2(f_1 + f_2)\right]^n\right)^{1/n} \geq \left(\int_{R^n} \left[D^2 f_1\right]^n\right)^{1/n} + \left(\int_{R^n} \left[D^2 f_2\right]^n\right)^{1/n} . \quad (**)$$

1.4.B_1. *Remark.* This inequality remains valid for *all* convex functions on $\mathbb{R}^n$. This can be derived from (**) by a simple approximation argument or proved more directly using (*) and 1.4.A.

1.4.C. *Implication* (**) ⇒ (*) *for convex sets in* $\mathbb{R}^n$. Let Y be a convex bounded open subset in $L' = \mathbb{R}^n$ and define f (x) on $L = \mathbb{R}^n$ by

$$f(x) = \log \int_R \exp\langle x,y\rangle dy .$$

One checks by a straightforward computation that f is real analytic and strictly convex, and that the map $D_f : \mathbb{R}^n \to \mathbb{R}^n$ sends each x to the center of gravity of the measure $\exp\langle x,y\rangle dy$ on Y. It follows that

$$D_f(\mathbb{R}^n) \subset Y.$$

To show that $D_f(\mathbb{R}^n) \supset Y$ take a point x_0 such that the linear function $\langle x_0,y\rangle$ on $L' \supset Y$ has *only one* maximum point, say y_0, in the closure $\mathrm{Cl}Y \subset L'$ of Y. Note that these points y_0 are exactly the extremal points of ClY. Now we look at the measures $\exp(\lambda x_0,y)\, dy$ on Y and see that these concentrate at y_0 as $\lambda \to \infty$. Hence, the closure of the image of D_f contains *all* extremal points y_0 of Cl Y. Since the image $D_f(\mathbb{R}^n) \subset Y$ is convex, it necessarily equals Y.

1.4.C_1. **Conclusion.** *Every convex bounded open subset* Y *in* $\mathbb{R}^n$ *admits a surjective diffeomorphism* $D_f : \mathbb{R}^n \to Y$ *for some strictly convex* C^2*-function* f *on* $\mathbb{R}^n$. *Thus* (**) $\Rightarrow$ (*) *for convex bounded open subsets. This trivially implies that* (**) $\Rightarrow$ (*) *for all convex subsets in* $\mathbb{R}^n$.

1.4.C_2. *Remark.* There are many convex functions f with $D_f(\mathbb{R}^n) = Y$. For example, instead of the Lebegue's measure dy on Y one can take an arbitrary measure $d\mu$ on Y, such that the convex hull of the support of μ equals the closure of Y. Then one sees as earlier that $D_f(\mathbb{R}^n) = Y$ for

$$f(x) = \log \int_{\mathbb{R}^n} \exp\langle x,y\rangle d\mu .$$

However, for every compact convex subset Y, there is a distinguished convex function f_o (which is *non-smooth* and not *strictly* convex), called the support function of Y, such that $L'_{f_0} = Y$. This function is characterized by the homogeneity, $f_o(\alpha x) = \alpha f_o(x)$ for all $\alpha \geq 0$ (as well as by convexity and the relation $L'_{f_0} = Y$). It is easy to see that f_o equals the infimum of the convex functions f, such that $L'_f \supset Y$). Usually, one defines f_o as the infimum of the *linear* functions y (x) on L over all $y \in L' \setminus Y$.

Our choice of $f = \log \int \exp$ is motivated by the Kähler geometry in $\mathbb{C}P^n$ (see 2.4.).

1 .5. *Kähler formulation of* (**). Let us identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way,

$$\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^n \oplus \sqrt{-1}\ \mathbb{R}^n = \mathbb{C}^n,$$

and let us denote by $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the operator corresponding to the multiplication by $\sqrt{-1}$ in $\mathbb{C}^n$. We also denote by J the induced operator on vector fields and differential forms on $\mathbb{R}^{2n}$. We call an exterior 2-form ω on $(\mathbb{R}^{2n}, J)$ *positive* if $\omega(\partial, J\partial) \geq 0$ for all vector fields ∂ on $\mathbb{R}^{2n}$ and call ω *strictly positive* is if $\omega(\partial, J\partial) > 0$ for all non-vanishing fields ∂. Say that ω is a (1,1)-*form* if $J\omega = \omega$, that is $\omega(J\partial_1, J\partial_2) = \omega(\partial_1, \partial_2)$. Since $J^2 = -\mathrm{Id}$ and ω is antisymmetric, this is equivalent to the symmetry of the form h defined by $h(\partial_1, \partial_2) = \omega(\partial_1, J\partial_2)$. Note that such a ω is positive if and only if the (quadratic) form h is positive semidefinite.

Recall that the *second differential* $H = D^2f$ of a function f is the quadratic form defined by the formula

$$H(\partial_1,\partial_1) = \partial_1(\partial_2 f)$$

for all *translation invariant* (parallel) vector fields ∂_1 and ∂_2, where ∂f stands for the (Lie) derivative of f along ∂.

Another useful second order differential operator, now from functions to exterior 2-forms, is

$$f \to \omega = dJdf,$$

where d is the exterior differential first applied to f and then to the 1-form Jdf.

A straightforward computation expresses dJd in terms of D^2 by

$$h = H + JH \tag{6}$$

where $H = D^2f$ and h is defined along with $\omega = dJdf$ by $h(\partial_1,\partial_2) = \omega(\partial_1, J\partial_2)$. Note that the definition of h on the left hand side of identity (6) uses only J while the definition of H via D^2 needs the *affine structure* of $\mathbb{R}^{2n}$.

Since the form H is symmetric, the form h also is symmetric as well as J-invariant. Hence, the *form* $\omega = dJdf$ *is* (1,1) *for all functions* f *on* $\mathbb{R}^{2n}$.

Let us divide $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \sqrt{-1}\ \mathbb{R}^n$ by the n-dimensional lattice $\sqrt{-1}\ \mathbb{Z}^n$ and denote by $V = \mathbb{R}^n \times T^n$ the resulting manifold for the torus

$$T^n = \sqrt{-1}\,\mathbb{R}^n / \sqrt{-1}\,\mathbb{Z}^n.$$

The notions of a (1,1)-form and of positivity descend from $\mathbb{R}^{2n}$ to V along with the operator J which acts on the tangent bundle of V. Now we can formulate the following

1.5.A. Brunn-Minkowski inequality for T^n-invariant forms on V.
Let ω_1 and ω_2 be exact positive (1,1)-forms on V which are invariant under the natural T^n-action on V. Then the top-dimensional exterior power $(\omega_1+\omega_2)^n$ satisfies,

$$\left(\int_V (\omega_1+\omega_2)^n\right)^{1/n} \geq \left(\int_V \omega_1^n\right)^{1/n} + \left(\int_V \omega_2^n\right)^{1/n}.$$

Let us show that (***) is equivalent to (**) in 1.4.B. First we prove the implication (***) $\rightarrow$ (**) by relating to each function f on $\mathbb{R}^n$ the form ω on V which is the djd $\tilde{f}$ of the pull-back $\tilde{f}$ of f to V for the projection $V \rightarrow \mathbb{R}^n = V/T^n$. It is clear that

$$\int_V \omega^n = \int_{R^n} [D^2 f]^n$$

for all f on $\mathbb{R}^n$ and that ω is positive if and only if f is convex. Since $\omega = dJd\tilde{f}$ it is an *exact* (1,1)*-form* and so (***) $\rightarrow$ (**).

To prove that (**) $\rightarrow$ (***) we start with an exact (1,1)-form ω on V. The exactness obviously implies that ω vanishes on every T^n-orbit in V. It follows that the associated quadratic form h admits a unique decomposition

$$h = \tilde{H} + J\tilde{H}$$

where $\tilde{H}$ is induced from a quadratic form H on $\mathbb{R}^n$. The equality $d\omega = 0$ implies (by a straightforward computation) that

$$\partial_1 H(\partial_2, \partial_3) = \partial_2 H(\partial_1, \partial_3)$$

for all parallel fields ∂_1, ∂_2 and ∂_3 on $\mathbb{R}^n$. Hence $H = D^2 f$ for some function on $\mathbb{R}^n$ which is convex as ω is positive.
Hence, (***) does follow from (**).

1.6. *Mixed volumes.* Let us use notation y^I for the monomial $y_1^{i_1} \dots y_k^{i_k}$, where $y = (y_1, \dots, y_k)$ is a string of variables and $I = (i_1, \dots, i_k) \in Z_+^k$ denotes the multiindex with non-negative integer entries. We denote by $|I| = i_1 + \dots + i_k$ the *degree* of y^I and observe that the monomials $\{y^I\}_{|I|=n}$ constitute a basis in the space of polynomials in $y_1, \dots y_k$ of degree n. Next, we write $Iy = i_1 y_1 + \dots + i_k y_k$ and observe that (by a trivial argument) the polynomials $\{(Iy)^n\}_{|I|=n}$ also constitute a basis in the space of polynomials. In other words every monomial is a linear combination of some Iy with universal coefficients. For example,

$$y_1 y_2 = 1/2((y_1+y_2)^2 - y_1^2 - y_2^2).$$

Similarly, for strings of differential 2-forms, $\Omega = (\omega_1, \dots, \omega_k)$ we write

$$\Omega^I = \omega_1^{i_1} \wedge \dots \wedge \omega^{i_k}$$

and we are interested in the integrals $\int_V \Omega^I$, where $\dim V = 2n = 2|I|$.

We note that every such integral is a linear combination of the integrals

$$\int (I\Omega)^n \text{ for } I\Omega = i_1\omega_1 + \dots + i_k\omega_k,$$

with the above universal coefficients.

Now, let $V = \mathbb{R}^n \times T^n$ and $\omega_1, \dots, \omega_k$ correspond to convex bodies $Y_1, \dots, Y_k$. Namely $\omega_j = dJd\tilde{f}_j$ for $j = 1, \dots, k$ where $\tilde{f}$ is a smooth T^k-invariant function on V, such that the corresponding function on $\mathbb{R}^n$ is convex and $D_{f_j}(\mathbb{R}^n) = Y_j$ for $j = 1, \dots, k$.

1.6.A. **Proposition-Definition.** *The integral* $\int_V \Omega^I$, *where* $|I|=n$ *only depends on* $Y = (Y_1, \dots, Y_k)$ *but not on a choice of the functions* f_j. *This integral is called the* I^{th} *mixed volume of* $Y_1, \dots, Y_k$, *and denoted* $[y^I] = \left[Y_1^{i_1} \dots Y_k^{i_k}\right]$.

Proof. By the previous discussion each mixed volume is a universal linear combination of the volumes of the Minkowski sums $IY = i_1Y_1 + \dots + i_kY_k$ where iX denotes $\underbrace{X+X+\dots+X}_{i}$.

1.6.A_1. *Remark.* As it is clear from this definition, the volume Vol(IY) expands in the usual way into the sum of mixed volumes. For example,

$$\mathrm{Vol}(Y_1+Y_2) = [(Y_1+Y_2)^n] = \sum_{i=0}^{n} b_i \left[Y_1^i Y_2^{n-i}\right],$$

where $b_i = \frac{n!}{i!\,(n-i)!}$.

In order to state the Alexandrov Fenchel inequality concerning mixed volumes we need the following notion of convexity for real functions on the *discrete simplex*

$$\Delta_n^{k-1} = \{I \in Z_+^k \mid |I| = n\} \subset \mathbb{R}^k.$$

In other words, Δ_n^{k-1} is the set of multiindices $(i_1,\dots,i_k)$ with $i_1+\dots+i_k = n$. We say that a function $l(I)$ is *1-concave* on Δ_n^{k-1} if the restriction of l to every line parallel to one of the edges of Δ_n^{k-1} is concave. For example, if k-1 = 1, then this is the usual concavity,

$$l\left(\sum_\nu a_\nu I_\nu\right) \geq \sum_\nu a_\nu l(I_\nu)$$

for $I_\nu = (\nu, n-\nu) \in \Delta_n^1$ and all those convex combinations, where $\sum_\nu a_\nu I_\nu$ lies in Δ_n^1 (i.e. is integral).

In general, Δ_n^{k-1} has $\frac{k(k-1)}{2}$ edges. A line parallel to an edge is given by fixing k-2 (out of k) coordinates $(i_1,\dots,i_k)$. For example, a line parallel to the "first" edge is given by fixing the last (k-2) coordinates $i_3, i_4, \dots i_k$. If $i_3+i_4+\dots+i_k = m \leq n$, then this line is $\{\nu, n-m-\nu, i_3,\dots,i_k\}$ for $\nu = 0,1,\dots,n-m$, and the 1-concavity condition on this line amounts to the above (7).

1.6.B. *Alexandrov-Fenchel theorem.* Let $Y = (Y_1,\dots Y_k)$ *be a sequence of convex bounded open subsets in* $\mathbb{R}^n$. *Then the mixed volumes* $[Y^I]$ *for* $|I|=n$ *are positive and the function*

$$l_Y(I) = \log\,[Y^I]$$

is 1-concave.

Remark and corollaries. 1 .6. B_1. *The mixed volume* $[Y^I]$ *for* $I = (i_1,...,i_k)$ *is bounded from below by the following weighted product of the volumes of* $Y_1,...,Y_k$,

$$[Y^I] \geq (\mathrm{Vol}\, Y_1)^{\frac{i_1}{n}},\ldots,(\mathrm{Vol}\, Y_k)^{\frac{i_k}{n}}. \qquad (8)$$

In fact, every 1-concave function $l(I)$ (obviously) satisfies $l(I) \geq \frac{i_1}{n} l(n, 0,\ldots,0) + \frac{i_2}{n} l(0, n, 0,\ldots,0) + \frac{i_k}{n} l(0,\ldots,0,n)$.

1.6.B_2. Inequality (8) for k=2 reads

$$\left[Y_1^i, Y_2^{n-i}\right] \geq (\mathrm{Vol} Y_1)^{\frac{i}{n}} (\mathrm{Vol} Y_2)^{\frac{n-i}{n}}$$

which implies the Brunn-Minkowski inequality as

$$\mathrm{Vol}\,(Y_1 + Y_2) = \sum_i b_i \left[Y_1^i Y_2^{n-i}\right].$$

1.6.B_3. I do not know if $l Y(I)$ is a concave function in I.

1.6.B_4. We shall prove the 1-concavity of $\log[K^I]$ along with the following

1.6.C. *Alexandrov-Fenchel inequality on compact manifolds*. Recall that the mixed volume $[Y^I]$ is defined as the integral $\int_V \Omega^I$, where the string of 2-forms, $\Omega = (\omega_1,\ldots,\omega_k)$ on $V = \mathbb{R}^n \times T^n$, corresponds to convex sets $Y_1,\ldots,Y_k$ in $\mathbb{R}^n$.
Thus the Alexandrov-Fenchel theorem amounts to 1-concavity of $\log \int \Omega^I$ for exact positive (1,1)-forms $\omega_1,\ldots,\omega_k$ on V. This is proven in §2 where we start with the following result concerning *compact* manifolds V.

1.6.C_1. **Theorem**. *Let* V *be a compact complex manifold. Then for every sequence of strictly positive closed* (1,1)*-forms* $\Omega = (\omega_1,\ldots,\omega_k)$ *on* V *the function*