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Jacob Burbea & P Masani

Banach and Hilbert spaces of vector-valued functions

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Banach and Hilbert spaces of vector-valued functions

Their general theory and applications to holomorphy

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Preface

The idea of positive definiteness is playing an increasingly important role in complex, real, stochastic and abstract analysis. In the complex domain the idea began to fructify with the work of S. Bergman in the 1920s, and in the real domain it appeared with the Herglotz and Bochner theorems on Fourier-Stieltjes transforms. With the advent of the kernel theorems of Kolmorogov, Aronszajn and Pedrick in the 1940s and 1950s, the notion has been linked to the study of the geometry and kinematics of Hilbertian varieties. In the present monograph we bring the last viewpoint to bear on the first, our point of departure being Rovnyak's 1963 extension of Pick's 1916 theorem to Hilbert space-valued functions. Our work may be looked upon as a far-reaching ramification of Rovnyak's dissertation [26] and of the cognate work of Nagy and Foias [22,23].

A preliminary outline of our ideas was presented at the Second Conference on Probability Theory on Vector Spaces in Błazejewko, Poland, in 1979 (see [10]). The first author thanks the Swedish Royal Academy of Sciences for a Visiting Scholar award in 1981-82 at the Mittag Leffler Institute, Djursholm, Sweden, during which time a part of this research was completed. The second author sincerely thanks the Alexander von Humboldt Foundation for a Senior Visiting Scientist award in 1979-80, during which period a part of this research was carried out. It was in Germany that he had the benefit of conversations with Professor Rovnyak. Both of us would like to thank Professor Rovnyak as well as our Pittsburgh colleague Professor F. Beatrous for such conversations.

We are grateful to Professor F.F. Bonsall for suggesting the Pitman series as a venue for publication and for forwarding our manuscript to them, and we are indebted to Dr M. Dixon of Pitman Books for looking after the many details of the actual publication. Last but not least our sincere thanks go to Mrs D. Luppino-Grant for her excellent typing of an extremely complicated handwritten manuscript. For the final typing as it appears in the monograph we are much obliged to Ms Terri Moss.

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Part I
General theory

1 Introduction

Let $k(\cdot,\cdot)$ be a complex-valued positive-definite (PD) kernel on the Carterian product DxD of the open unit disk D in the complex plane[†] $\mathbb{C}$, let W_1, W_2 be complex Hilbert spaces and let $Y(\cdot)$ be a non-constant function on D the values of which are continuous linear operators on W_1 to W_2. Under what conditions on $k(\cdot,\cdot)$ and $Y(\cdot)$ will the W_2-to-W_2 linear operator-valued kernel $L_0(\cdot,\cdot)$, defined by

$$(1.1) \qquad L_0(z,\zeta) = k(z,\zeta)\{I_{W_2} - Y(\zeta)Y(z)^*\}, \quad z, \zeta \in D,$$

where I_{W_2} is the identity operator on W_2 and $Y(z)^*$ is the adjoint of $Y(z)$, be PD on DxD?

Now $L_0(z,\zeta) = k(z,\zeta)\cdot\Delta(z,\zeta)$, where $\Delta(z,\zeta) \underset{d}{=} I_{W_2} - Y(\zeta)Y(z)^*$. But to answer the question we cannot fall back on the simple result that the product of PD kernels is PD, for the kernel $\Delta(\cdot,\cdot)$ is not PD on DxD for non-constant $Y(\cdot)$. The question is in fact non-trivial. With $W_1 = W_2 = \mathbb{C}$ and with $k(\cdot,\cdot)$ equal to the Szegö kernel

$$(1.2) \qquad k(z,\zeta) \underset{d}{=} 1/(1 - \bar{z}\zeta), \quad z, \zeta \in D,$$

it goes back to Pick [25] in 1915. Pick showed that in this case $L_0(\cdot,\cdot)$ is PD when $Y(\cdot)$ is a holomorphic function on D into D, and conversely, if $L_0(\cdot,\cdot)$ is PD, then $Y(\cdot)$ either must be as just described or must be the function with the constant-value 1. He thus gave a complete answer to our question for non-constant $Y(\cdot)$, to wit, $L_0(\cdot,\cdot)$ is PD if and only if $Y(\cdot)$ is holomorphic on D to D.

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† Throughout this monograph $\mathbb{C}$ is the complex number field, $\mathbb{R}$ the real number field, and $\mathbb{F}$ refers to any one of these. $\mathbb{Z}$ is the set of integers. $\mathbb{R}_+$, $\mathbb{N}_+$ and $\mathbb{R}_{0+}$, $\mathbb{N}_{0+}$ denote the subsets of positive elements and the subsets of non-negative elements of $\mathbb{R}$ and $\mathbb{N}$, respectively.

The consideration of the question above for complex Hilbert spaces W_1, W_2 and the Szegö kernel (1.2) began, it would seem, with the efforts of J. Rovnyak [26] in 1963 to extend the Pick theory to vector-valued functions. In the spirit of Pick, he proved that the kernel $L_0(\cdot,\cdot)$ is PD when $Y(\cdot)$ is a contractive (operator-valued) holomorphic function on D, i.e. is such that

$$Y(z) = \Sigma_0^\infty A_k z^k, \quad |Y(z)| \leq 1, \quad z \in D,$$

the Taylor coefficients A_k now being continuous linear operators on W_1 to W_2. Rovnyak also proved that these very conditions ensure the positive-definiteness of the related dual W_1-to-W_I linear operator-valued kernel $M_0(\cdot,\cdot)$ defined by

$$(1.3) \qquad M_0(z,\zeta) = \overline{k(z,\zeta)}\{I_{W_1} - Y(\zeta)^* Y(z)\}, \quad z,\zeta \in D.$$

(In the case $W_1 = W_2 = \mathbb{C}$, $M_0(\cdot,\cdot)$ simply becomes the complex-conjugate of $L_0(\cdot,\cdot)$, thereby losing interest and escaping notice.)

The proofs of these theorems for $L_0(\cdot,\cdot)$ and $M_0(\cdot,\cdot)$ with the Szegö kernel (1.2) given by Rovnyak [26] and subsequently by Nagy [22] and by Nagy and Foias [23: pp. 231-233] lean heavily on the analytic implications of these kernels and exploit the theory of the Hardy class H_2 on the disk D. Our point of departure came from reflection of an aspect of the Nagy-Foias proof: the appearance of the Hardy spaces $H_2(D,W_1)$, $H_2(D,W_2)$ of W_1- and W_2-vector-valued functions on D and of the *multiplication operator* M_Y from the first of these spaces to the second that $Y(\cdot)$ induces. This suggested to us the feasibility of an abstract generalization, from which analyticity would disappear, but which when applied to analytic contexts would not only subsume the Rovnyak results but considerably enlarge their scope with regard to both the choice of the domain D and of the scalar-valued kernel $k(\cdot,\cdot)$. A convenient way to see this is to pose the following question:

1.4 Question. *Let* Λ *be any non-void set,* $K_1(\cdot,\cdot)$, $K_2(\cdot,\cdot)$ *be* PD *kernels on* $\Lambda \times \Lambda$ *to*† $CL(W_1,W_1)$ *and* $CL(W_2,W_2)$, *respectively, and let* $Y(\cdot)$ *be a function on* Λ *to* $CL(W_1,W_2)$. *Under what conditions will the* W_2*-to-*W_2 *operator-valued*

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† $CL(X,Y)$ stands for the space of continuous linear operators on the Banach space X to the Banach space Y. I_X is the identity operator on X.

kernel $L(\cdot,\cdot)$ *defined by*

$$L(\lambda,\lambda') = K_2(\lambda,\lambda') - Y(\lambda')K_1(\lambda,\lambda')Y(\lambda)^*, \quad \lambda,\lambda' \in \Lambda,$$

be PD *on* $\Lambda \times \Lambda$ *to* $CL(W_2,W_2)$?

To see the relevance of this question to the problem under discussion and to discern the sort of answer that may be expected, two observations are in order:

(i) On letting $\Lambda = D$ and $K_j(z,\zeta) = k(z,\zeta)I_{W_j}$, $j = 1,2$, the kernel $L(\cdot,\cdot)$ reduces to the $L_0(\cdot,\cdot)$ of (1.1), and Question 1.4 reduces to our initial question. Thus an answer to Question 1.4 would automatically answer the initial question.

(ii) From the fundamental work of Kolmogorov, Aronszajn and Pedrick [18, 2, 24], it follows that given a Hilbert space W and a W-to-W operator-valued PD kernel $K(\cdot,\cdot)$ on $\Lambda \times \Lambda$, there exists a Hilbert space $F = F_{\Lambda,W}$ comprising functions on Λ to W for which the *evaluation operators*, $E_F(\lambda)$, $\lambda \in \Lambda$, on F to W, defined by

$$E_F(\lambda)(f) = f(\lambda), \quad f \in F, \ \lambda \in \Lambda$$

are continuous and satisfy the equalities

$$E_F(\lambda')\cdot E_F(\lambda)^* = K(\lambda,\lambda'), \quad \lambda,\lambda' \in \Lambda.$$

Further inquiry reveals that when $\Lambda = D$ and $K(\cdot,\cdot) = k(\cdot,\cdot)I_W$, where $k(\cdot,\cdot)$ is the Szegö kernel (1.2), we have $F = H_2(D,W)$, which for $W = W_1$ and $W = W_2$ are the very spaces appearing in the Nagy-Foias proof. It is therefore reasonable to expect that the answer to Question 1.4 will also involve the multiplication operator M_Y induced by $Y(\cdot)$ on F_{Λ,W_1} to F_{Λ,W_2}, these being the function Hilbert spaces corresponding to the kernels $K_1(\cdot,\cdot)$, $K_2(\cdot,\cdot)$, respectively.

We see from this discussion that Question 1.4 captures in the abstract the essential core of our initial question. With this clarification, we can conveniently turn to outlining the main new results obtained in this monograph.

(1) We settle Question 1.4 by proving that the kernel $L(\cdot,\cdot)$ in it is PD on $\Lambda \times \Lambda$ to $CL(W_2,W_2)$ when the multiplication operator M_Y is a contraction on

F_{Λ,W_1} to F_{Λ,W_2}, these being the function Hilbert spaces for the kernels $K_1(\cdot,\cdot)$, $K_2(\cdot,\cdot)$ in Question 1.4, cf. Corollary 3.5.

(2) With the notation of Question 1.4, let J_1, J_2 be conjugations on the Hilbert spaces W_1, W_2, let

$$\hat{Y}(\cdot) \underset{d}{=} J_1 Y(\cdot)^* J_2,\ \hat{K}_j(\cdot,\cdot) \underset{d}{=} J_j K_j(\cdot,\cdot) J_j,\quad j = 1,2,$$

and define the W_1-to-W_1 operator-valued kernel $M(\cdot,\cdot)$ by

$$M(\lambda,\lambda') \underset{d}{=} \hat{K}_1(\lambda,\lambda') - Y(\lambda')^* \hat{K}_2(\lambda,\lambda')Y(\lambda),\quad \lambda,\lambda' \in \Lambda.$$

Then we show that $M(\cdot,\cdot)$ is PD on $\Lambda \times \Lambda$ to $CL(W_1,W_1)$, provided that the multiplication operator $M_{\hat{Y}}$ is a contraction on F_{Λ,W_2} to F_{Λ,W_1}, these spaces being as in (1), cf. Corollary 4.7.

(3) With the notation of Question 1.4, let * be an involution on Λ, and let

$$\tilde{Y}(\lambda) \underset{d}{=} Y(\lambda^*)^*,\ \tilde{K}_j(\lambda,\lambda') \underset{d}{=} K_j(\lambda^*,\lambda'^*),\quad \lambda,\lambda' \in \Lambda,\ j = 1,2,$$

and define the W_1-to-W_1 operator valued kernel $N(\cdot,\cdot)$ by

$$N(\lambda,\lambda') \underset{d}{=} \tilde{K}_1(\lambda,\lambda') - Y(\lambda')^* \tilde{K}_2(\lambda,\lambda')Y(\lambda),\quad \lambda,\lambda' \in \Lambda.$$

Then we show that $N(\cdot,\cdot)$ is PD on $\Lambda \times \Lambda$ to $CL(W_1,W_1)$, provided that the multiplication operator $M_{\tilde{Y}}$ is a contraction on F_{Λ,W_2} to F_{Λ,W_1}, these again being as in (1), cf. Corollary 5.5.

(4) Applying the result in (1) to the case in which Λ is an open set $\Omega \subseteq \mathbb{C}^q$, $q \in \mathbb{N}_+$, and $K_j(\cdot,\cdot) = k(\cdot,\cdot)I_{W_j}$, $j = 1,2$, where $k(\cdot,\cdot)$ is a weighted Bergman kernel for Ω, we deduce that the kernel $L_0(\cdot,\cdot)$ defined by

$$L_0(z,\zeta) = k(z,\zeta)\{I_{W_2} - Y(\zeta)Y(z)^*\},\quad z,\zeta \in \Omega$$

is PD on $\Omega \times \Omega$ to $CL(W_2,W_2)$, provided that $Y(\cdot)$ is a holomorphic function on Ω to $CL(W_1,W_2)$ and $|Y(z)| \le 1$ for $z \in \Omega$, cf. Theorem 10.9. This extends the Rovnyak result for $L_0(\cdot,\cdot)$ with the Szegö kernel (1.2) for the disk D to the Bergman kernel for an arbitrary open set Ω in the space of several complex variables.

(5) Again applying the result in (1) with Λ a bounded open set $\Omega \subset \mathbb{C}^q$ with a sufficiently smooth boundary $\partial\Omega$, and $K_j(\cdot,\cdot) = k(\cdot,\cdot)I_{W_1}$, $1 = 1,2$ where now $k(\cdot,\cdot)$ is the Szegö kernel for Ω, we deduce that the kernel $L_0(\cdot,\cdot)$ defined by the formula in (4) is PD on $\Omega \times \Omega$ to $CL(W_2,W_2)$, provided that $Y(\cdot)$ is again a holomorphic function on Ω to $CL(W_1,W_2)$ and $|Y(z)| \leq 1$ for $z \in \Omega$, cf. Theorem 11.14. This extends the Rovnyak result for $L_0(\cdot,\cdot)$ with the Szegö kernel (1.2) for the disk D to the Szegö kernel for any smooth domain Ω in $\mathbb{C}^q$.

(6) Applying the results in (2) and (3) with Λ an open set $\Omega \subseteq \mathbb{C}^q$, $q \in \mathbb{N}_+$, and $K_j(\cdot,\cdot) = k(\cdot,\cdot)I_{W_1}$, $j = 1,2$, where $k(\cdot,\cdot)$ is the Bergman kernel for Ω, we deduce that if $Y(\cdot)$ is holomorphic on Ω to $CL(W_1,W_2)$ and $|Y(z)| \leq 1$ for $z \in \Omega$, then the kernel $M_0(\cdot,\cdot)$ defined by

$$M_0(z,\zeta) = \overline{k(z,\zeta)}\{I_{W_1} - Y(\zeta)^*Y(z)\}, \quad z,\zeta \in \Omega$$

is PD on $\Omega \times \Omega$ to $CL(W_1,W_1)$, and in case $\Omega = \bar{\Omega} \underset{d}{=} \{\bar{z}: z \in \Omega\}$, so too is the kernel $N_0(\cdot,\cdot)$ defined by

$$N_0(z,\zeta) = k(\bar{z},\bar{\zeta})\{I_{W_1} - Y(\zeta)^*Y(z)\}, \quad z,\zeta \in \Omega$$

cf. Theorem 10.9. These results extend the Rovnyak result for $M_0(\cdot,\cdot)$ in (1.3) with the Szegö kernel (1.2) for D to the Bergman kernel for arbitrary open sets $\Omega \subset \mathbb{C}^q$, even when $\Omega \neq \bar{\Omega}$.

(7) Applying the results in (2) and (3) with Λ a smoothly bordered bounded open set Ω in $\mathbb{C}^q$ and $K_j(\cdot,\cdot) = k(\cdot,\cdot)I_{W_j}$, $j = 1,2$, where now $k(\cdot,\cdot)$ is the Szegö kernel for Ω, we deduce the precise analogues of the results in (6), cf. 11.14. In case $\Omega = D$, these reduce to Rovnyak's result for $M_0(\cdot,\cdot)$ in (1.3) with the Szegö kernel (1.2).

Thus from the abstract and general theorems mentioned in (1), (2), (3) we are able to get the analytical results in (4), (5), (6), (7), which are substantial extensions of the Pick-Rovnyak results for the disk. And since the category of complex-valued kernels $k(\cdot,\cdot)$ on $\Omega \times \Omega$, where Ω is an open set in $\mathbb{C}^q$, to which the results in (1)-(4) are applicable contain several which are neither ordinary Bergman nor Szegö,† the scope of our abstract approach extends well beyond the results in (4)-(7).

In the course of proving the results in (1)-(7), we have had to cover some hitherto unexplored ancillary ground. Among the new concepts and results emerging from our investigations in this ancillary realm, the following may be cited.

(8) The kernel theorem emerging from the work of Kolmogorov, Aronszajn and Pedrick to which we alluded in (ii) above and which is crucial to our abstract treatment, has not been formally enunciated in the literature. Kolmogorov and Aronszajn dealt with scalar-valued kernels exclusively, and Pedrick's important extension to kernels whose values are operators from W* to W, where W is a locally convex topological vector space and W* is its adjoint, is available only in the unpublished report [25]. A version of the kernel theorem for W-to-W*-valued PD kernels, where W is a Banach space, has been established by one of us and put to considerable use in the propagator theory of Hilbertian varieties [21: Thm. 2.10]. In this, however, the auxiliary Hilbert space F comprises scalar-valued functions on $\Lambda \times W$, and not W-valued functions on Λ as required in (ii) above, cf. [21: App. C]. Indeed the very concepts of W-*vector-valued function Hilbert space*, where W is a Hilbert space, and of its *'reproducing' kernel* do not seem to appear in the printed literature. In this monograph we demarcate these concepts after introducing first the more basic concept of a W-*vector-valued function Banach space* (Chapter 2), and establish the Kernel Theorem 2.12, by deducing it from the lemmas employed in the paper [21].

(9) The answer to Question 1.4 cited in (1), though it suffices for the analytic applications (4), (5), is not the best possible. In our Main Theorem 3.4, we give a better answer which allows the multiplication operator M_Y to be defined merely on a proper subset of F_{Λ,W_1}, not necessarily everywhere dense in F_{Λ,W_1}, but places a restraint on the domain of $M_Y \cdot (M_Y)^*$. For this we first show that M_Y is always closed, and then appeal to our general lemma that if T is *any* closed operator from a Hilbert space H_1 to a Hilbert

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† For instance, the kernel $k(\cdot,\cdot)$ defined by $k(z,\zeta) = 1/(1-\bar{z}\zeta)^r$, $z,\zeta \in D$, where $r \in [1,\infty)$.

space H_2, then (although T* can be many-valued) TT* is single-valued and self-adjoint from H_2 to H_2, cf. Appendix A.2. This last is a slight improvement of the classical von Neumann result for closed, *densely-defined* operators from H_1 to H_2.

(10) 'Inflated' function Hilbert spaces, $F_{\Lambda,W}$ i.e. ones for which the kernel $K(\cdot,\cdot)$ has the simple structure $k(\cdot,\cdot)I_W$, where $k(\cdot,\cdot)$ is a scalar-valued kernel, are conspicuous in all our analytical applications (4)-(7). But the elegant properties of such spaces are not recorded in the literature. We do so in this monograph (Chapter 7) and show that such a space $F_{\Lambda,W}$ is isometrically isomorphic to the tensor product $W \otimes F_{\Lambda,\mathbb{C}}$, i.e. to the Hilbert space of Hilbert-Schmidt operators on W to $F_{\Lambda,\mathbb{C}}$, the last being the function Hilbert space with the scalar-valued kernel $k(\cdot,\cdot)$, Theorem 7.8.

(11) The question whether function Hilbert spaces $F_1 = F_{\Lambda,W_1}$, $F_2 = F_{\Lambda,W_2}$, with the same Λ, are 'norm-related', i.e., whether the implication

$$f_1 \in F_1,\ f_2 \in F_2 \text{ and } |f_1(\cdot)|_{W_1} \le |f_2(\cdot)|_{W_2} \text{ on } \Lambda$$
$$\Rightarrow |f_1|_{F_1} \le |f_2|_{F_2}$$

prevails, loomed large in the initial phases of our research. Although, as we soon realized, this question can be bypassed, it has an interest of its own. It would be interesting, for instance, to characterize the $\mathbb{C}$-valued kernels $k(\cdot,\cdot)$ on $\Lambda \times \Lambda$ for which the function Hilbert space $F_{\Lambda,\mathbb{C}}$ is norm-related to itself. We broach some of these questions in Chapter 8.

The Bergman spaces $B_p(\Omega,\mathbb{C})$, $(p \ge 1)$ of $\mathbb{C}$-valued functions on an open set $\Omega \subseteq \mathbb{C}^q$, the Hardy spaces $H_p(\Omega,\mathbb{C})$, where Ω is a polydisk or some variant thereof, and the Hardy space $H_2(D,W)$ of functions on the disk D in $\mathbb{C}$ to a Hilbert space W appear extensively in the literature, but, so far the vector-valued function spaces $B_p(\Omega,W)$, $H_p(\Omega,W)$, for Banach spaces W and $p \ge 1$, have not been defined in their full generality, nor of course studied. Indeed the underlying concept of a W-*valued holomorphic function on an arbitrary set* $\Omega \subseteq \mathbb{C}^q$ has been systematically studied only in the cases $q = 1,\infty$.† In this

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† cf. Hille-Phillips [15]; they study the replacement of $\mathbb{C}^q$ by a Banach space and the problems so created.

monograph we have filled in these omissions in the interests of securing a coherent treatment free of ambiguity (Chapter 9). From among the results we obtain in this area the following bear mention.

(12) We state and prove a general 'Hartogs Theorem' for functions on an open set $\Omega \subseteq \mathbb{C}^q$ with values in a Banach space W, cf. Theorem 9.4, and study it more specifically for W equal to the space $CL(W_1,W_2)$ of continuous linear operators on a Banach space W_1 to a Banach space W_2.

(13) For an open set $\Omega \subseteq \mathbb{C}^q$ and a Hilbert space W we obtain the necessary and sufficient conditions that a PD kernel $K(\cdot,\cdot)$ on $\Omega \times \Omega$ to CL(W,W) must fulfil in order that the associated function Hilbert space $F_{\Lambda,W}$ comprise holomorphic functions on Ω to W exclusively, cf. Theorem 9.17.

(14) For a large class of reasonable non-negative measures μ on the family of Borel subsets of an open set $\Omega \subseteq \mathbb{C}^q$, for all Banach spaces W and for all $p > 0$, we define the W-ranged p,μ Bergman space $B_{p,\mu}(\Omega,W)$ and show that for $p \geq 1$ it is a Ω,W function Banach space, and that for a Hilbert space W, $B_{2,\mu}(\Omega,W)$ is an inflated function Hilbert space (Theorems 10.4, 10.5). Likewise we define the W-ranged p Hardy space $H_p(\Omega,W)$ for a large class of smoothly bordered domains Ω in $\mathbb{C}^q$, and show that for $p \geq 1$ it is a Ω,W Banach function space, and that for a Hilbert space W, $H_2(\Omega,W)$ is an inflated function Hilbert space (11.7, 11.9, 11.10).

Some cognate work by one of us on the operator extensions of converse Pick results for Bergman and Hardy spaces, and for Hardy spaces over domains Ω in $\mathbb{C}$, which are not smooth, but for which 'generalized Szegö' kernels are definable by means of Ahlfors functions, will appear elsewhere [6, 7, 8].

We end this introduction by briefly describing the organization of this monograph. Chapter 2 is devoted to the basics of Λ,W function Hilbert spaces $F_{\Lambda,W}$, and to proving the Kernel Theorem 2.11. In Chapter 3 we bring in the multiplication operator $M_{Y(\cdot)}$ from F_{Λ,W_1} to F_{Λ,W_2} that a function $Y(\cdot)$ on Λ to $CL(W_1,W_2)$ induces, and prove our main theorem and Corollary (3.4, 3.5). Chapters 4 and 5 are concerned with the dualities emerging from the presence of conjugations on the Hilbert spaces W_1, W_2 and from possible involutions on the set Λ itself. In Chapter 6 we recapitulate the simplifications which accrue when $W = \mathbb{F}$ (i.e. $W = \mathbb{R}$ or $\mathbb{C}$), primarily in order to deal in Chapter 7 with inflated function Hilbert spaces. In Chapter 8 we broach the question of norm-related function Hilbert spaces. This exhausts our treatment of the abstract theory, and ends Part I of the monograph.

In Part II we turn to the analytic applications of the abstract theory. In Chapter 9 we study the concept of W-valued holomorphic functions on Ω, where W is a Banach space over $\mathbb{C}$, and Ω is an open set in $\mathbb{C}^q$, $q \geq 1$. In Chapters 10 and 11 we turn to the Bergman and Hardy spaces $B_{p,\mu}(\Omega,W)$ and $H_p(\Omega,W)$, paying special attention to the case $p = 2$ and W is a Hilbert space. Finally, in the Appendix we cover the operator-theoretical material that is required in the monograph, but is omitted from the main text in order to avoid digression.

2 Hilbert spaces of functions with values in a Hilbert space

To define Banach and Hilbert spaces of vector-valued functions we need a notation for evaluation operators:

2.1 Notation For non-void sets Λ, W, we let

$$W^{\Lambda} \underset{d}{=} \{f\colon f \text{ is a function on } \Lambda \text{ to } W\}$$

and

$$\forall\lambda \in \Lambda \quad\text{and}\quad \forall f \in W^{\Lambda}, \quad E_{\lambda}(f) \underset{d}{=} f(\lambda) \in W.$$

We call E_{λ} the *evaluation operator at* λ.

2.2 Definition Let Λ be a non-void set and W be a Banach space over $\mathbb{F}$. We say that F is a Λ,W *function Banach (or Hilbert) space*, iff (i) $F \subseteq W^{\Lambda}$, (ii) F is a Banach space (or Hilbert) space over $\mathbb{F}$, (iii)† $\forall\lambda \in \Lambda$, $E_F(\lambda) \underset{d}{=} \text{Rstr.}_F E_{\lambda} \in CL(F,W)$. A fuller notation for F is $F_{\Lambda,W}$.

Examples. Function Banach spaces are easy to exhibit. Thus, the Banach space $F = C(\Lambda,W)$ of continuous functions on a compact Hausdorff space Λ to a Banach space W under the sup-norm is a Λ,W function Banach space, since obviously $F \subseteq W^{\Lambda}$ and

$$\forall f \in F \text{ and } \forall\lambda \in \Lambda, \quad |E_{\lambda}(f)|_W = |f(\lambda)|_W \le \sup_{\lambda' \in \Lambda} |f(\lambda')|_W.$$

Likewise the space $F = \ell_2(\Lambda,W)$ of functions f on an arbitrary set Λ to a Banach space W for which

$$|f|_2 \underset{d}{=} \sqrt{\sum_{\lambda\in\Lambda} |f(\lambda)|_W^2} < \infty$$

is obviously a Λ,W function Banach space. In case W is a Hilbert space, this F becomes a Λ,W Hilbert space, under the inner-product

$(f,g) \underset{d}{=} \sum_{\lambda\in\Lambda} (f(\lambda), g(\lambda))_W$, $f,g \in F$.

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† $\text{Rstr.}_S F$ means the restriction of the function F to the set S.