CONCEPTS IN DISCRETE MATHEMATICS

SARTAJ SAHNI

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PREFACE

This book contains a collection of mathematical topics that are of immense value to everyone who is pursuing a course of study in science or engineering. While a variety of mathematical tools are needed to successfully complete a course of study in these fields, most science and engineering curricula include mathematical courses in calculus and algebra only. Important concepts such as proof methods; difference equations; combinatorics; graph theory; etc. are often omitted and the student is expected to pick up these concepts along the way (somehow).

In this text, I have made an attempt to include those mathematical topics whose understanding is essential to science and engineering, but which are not covered in mathematical courses traditionally required of students in these disciplines. The topics covered are: logic; sets; relations; functions and computability; analysis of algorithms; recurrence equations; combinatorics; discrete probability; graphs; and algebra. While many of these topics are the subject of individual courses offered in traditional mathematical curricula, mathematics departments seldom have a one or two course sequence that covers all of them. The depth of coverage of each of the topics included in this text is about what is needed to successfully complete courses typically found in science and engineering curricula.

There is a bias towards computer science in this text. Such a bias can hardly be avoided today given the rapid growth in the use of computers and the permeation of computer science courses in virtually all curricula. The material in this text is illustrated by a large number of examples that have been carefully and completely worked out. There are over two hundred exercises that have been designed to enhance one's understanding of the material.

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Sartaj Sahni
Minneapolis
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CHAPTER 1

LOGIC

1.1 PROPOSITIONS AND WELL FORMED FORMULAS

Logical reasoning is the stuff proofs are made of and proofs are what scientific and other knowledge rests upon. The importance of proofs and thus of logical reasoning and logic cannot be overstated. Our faith in the thousands of theories postulated by scientists, mathematicians, etc. would be considerably less if there did not exist strong (and often conclusive) logical arguments in favor of these theories. What would be the status of the following statements if it were not for the existence of mathematically acceptable proofs establishing their validity?

(a) If A is a right angled triangle with sides of length a, b, and c, then $a^2 + b^2 = c^2$ where c is the length of the hypotenuse.
(b) The sum of the angles in any triangle is 180 degrees.
(c) The derivative of x^3 is $3x^2$.
(d) $\sum_{i=1}^{n} i = n(n+1)/2$.
(e) The area of a square of side d is d^2.

Given the importance of logical reasoning to mathematics, science, engineering, etc., it is appropriate that we begin our study of mathematical concepts with the study of the principles of reasoning (i.e., logic). First, we introduce some terms.

A *declarative sentence* is any sentence that can possibly be true or false. Some examples of declarative sentences are:

(a) The voltage across a resistor is the product of the current and the resistance ($V = IR$).
(b) There exist intelligent life forms on planets other than earth.
(c) Tom dislikes the discrete structures course.
(d) This text is fantastically clear.
(e) Mary had a little lamb.

It is quite meaningful for us to consider whether each of the above declarative sentences is true or false. Every electrical engineer knows that

(a) is true for ideal resistors. Tom knows whether (c) is true or not, and most three year olds have reasons to believe that (e) is true. One could have considerable debate over the truth of (b) and (d).

A *proposition* is a declarative sentence that must be either true or false but not both. Each of the five declarative sentences listed above is a proposition. We already know that each is either true or false. It is not too difficult to see that none of these five sentences can be both true and false. For example, this text is either fantastically clear or it is not. It cannot be both fantastically clear and not fantastically clear.

The use of the word either, in English, is often ambiguous. For example, consider the sentence:

> Tom is either guilty or innocent.

This sentence is readily seen to be equivalent to the sentence:

> Tom is either guilty or innocent, but not both.

On the other hand, the sentence:

> Good performance on either the exams or the assignments is sufficient to pass the course.

is not equivalent to:

> Good performance on either the exams or the assignments, but not on both, is sufficient to pass the course.

Rather, it is equivalent to:

> Good performance on either the exams or the assignments, or on both, is sufficient to pass the course.

Generally, the context in which 'either a or b' is used determines whether 'either a or b or both' or 'either a or b but not both' is meant. To avoid possible confusion resulting from the use of the word 'either', we shall usually state explicitly which interpretation is intended. When no interpretation is provided, we shall always mean 'either or both'(in this text).

Not all sentences are declarative sentences. For example:

(a) Pass me the butter.
(b) Has flight 201 from New York arrived?
(c) Can't you do anything right?

Furthermore, not all declarative sentences are propositions. For example, the sentence:

This statement is false.

can be neither true nor false. If the statement is true, then it asserts that it is false. If it is false, then it must be true. All propositions obey the following law:

Propositional Calculus Axiom: Every proposition is either true or false (but not both).

Observe that the above axiom includes the famous *law of contradiction* which states that no proposition is both true and false.

In algebra, symbols are used to denote numbers. For example, in the arithmetic expression $x + y$, the symbols x and y denote variables and the expression $x + y$ has value 10 when x is assigned the value 8 and y the value 2 or when $x = 6$ and $y = 4$, etc. In logic, we use capital letters (A, B, ..., Z) as variables (called propositional variables). These variables can be assigned propositions as values. For instance, P could denote any of the following propositions:

(a) It rains in Minneapolis.
(b) Stan is a democrat.
(c) Minnesota does not have a computer science department.
etc.

The *truth value* of a propositional variable P is true if the proposition assigned to it is true. It is false otherwise. If P denotes either of propositions (a) and (b) above, it is true. If P denotes proposition (c), then the truth value of P is false. Clearly, any propositional variable (i.e., A, B, ..., Z) can have a *truth value* either true or false depending upon which proposition it denotes. Propositional variables can be combined together using logical operators to get well formed formulas (wffs). This is similar to the use of +, −, /, *, etc. to combine together arithmetic variables to obtain arithmetic expressions. The logical operators we shall be dealing with are: $\neg$(not), $\vee$ (or), $\wedge$ (and), $\Longrightarrow$ (implies), and $\Longleftrightarrow$ (if and only if).

NOT ($\neg$)

The operator $\neg$ denotes negation. If P is a proposition then $\neg P$ (also written as $\overline{P}$) is its negation. The *negation* of a proposition P is another proposition that is true whenever P is false and is false whenever P is true. This can be stated in terms of a truth table (Figure 1.1(a)). In a truth table, the truth values true and false are abbreviated T and F respectively. The truth table

P	$\neg P$
T	F
F	T

(a) $\neg P$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) $P \vee Q$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(c) $P \wedge Q$

P	Q	$P \Longrightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(d) $P \Longrightarrow Q$

P	Q	$P \Longleftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

(e) $P \Longleftrightarrow Q$

Figure 1.1 Truth tables for logical operators.

for $\neg P$ has one column for P and one for $\neg P$. In the column for P we list the two possible truth values of P. The column for $\neg P$ gives the corresponding truth values for $\neg P$. Hence, from the truth value of P and the truth table of Figure 1.1(a) one can determine the value of $\neg P$. Consider the proposition:

This pie is good.

Its negation is:

This pie is not good.

Which is equivalent to:

It is not the case that this pie is good.

OR ($\vee$)

The operator $\vee$ obtains the disjunction of two propositions. The disjunction of the propositions P and Q is written $P \vee Q$ and read as "P or Q". Figure 1.1(b) gives the truth table for $P \vee Q$. The truth table for $P \vee Q$ has three columns. One for each of P, Q, and $P \vee Q$. There is one row for each combination of truth values of P and Q. The entry in the column for $P \vee Q$ in any row of the truth table gives the truth value of $P \vee Q$ when P and Q have the truth values given in that row. Note that $P \vee Q$ is true iff (if and only if) at least one of P and Q is true.

AND ($\wedge$)

The conjunction of two propositions P and Q is obtained by using the operator $\wedge$. It is denoted $P \wedge Q$ and read as "P and Q". Figure 1.1(c) gives the truth table for $P \wedge Q$. Observe that the truth value of $P \wedge Q$ is true iff both P and Q are true.

IMPLICATION ($\Longrightarrow$)

$P \Longrightarrow$ Q is read as "if P then Q" or as "P implies Q". P is the *antecedent* of "$\Longrightarrow$" and Q is its *consequent*. The truth table for $P \Longrightarrow Q$ is given in Figure 1.1(d). This truth table merits further discussion. The statement if P then Q essentially says that Q is true whenever P is true. It does not say anything about the truth value of Q when P is false. So, when P is false, Q can be either true or false. Hence the entries corresponding to $P =$ F and $Q = T$ or F are T. The only time the statement $P \Longrightarrow Q$ is false is when P is true and Q is false.

To understand the preceding discussion better, consider the proposition R:

If it rains, the ground will get wet.

Let P denote "it rains" and let Q denote "the ground will get wet". The proposition R is then equivalent to $P \Longrightarrow Q$. If it rains and the ground doesn't get wet, then R is false. So, the truth table entry for P true and Q false is F. Now, suppose it doesn't rain. It is still possible for the ground to get wet (someone may throw a bucket of dirty water on the ground). But, the fact that the ground has gotten wet despite the fact that it hasn't rained does not contradict R. This agrees with the truth table entry corresponding to P false and Q true. Similarly, if it doesn't rain and the ground isn't wet then P and Q are both false. Once again, this does not contradict the statement R and R remains true. The important point is that a statement of the type if P, then Q (written $P \Longrightarrow Q$) is false only if it is the case that Q is false when P is true. For $P \Longrightarrow Q$ to be true Q must be true whenever P is true. Q can take on any truth value when P is false.

IF AND ONLY IF ($\Longleftrightarrow$, iff)

$P \Longleftrightarrow Q$ (P iff Q) has the truth table given in Figure 1.1(e). $P \Longleftrightarrow Q$ is equivalent to the statement P implies Q and Q implies P. So, the truth table for $P \Longleftrightarrow Q$ must correspond to that for $(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$. One may easily verify that this is so.

Other logical operators such as exclusive or (XOR), not and (NAND), and not or (NOR) are defined in the exercises.

A *well formed formula* (wff) is defined recursively as below:

(a) All propositional variables and the constants true and false are wffs.
(b) If α and β are wffs, then $\neg\alpha$, $\bar{\alpha}$, (α), $[\alpha]$, $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \Longrightarrow \beta)$, $(\alpha \Longleftrightarrow \beta)$, $[\alpha \vee \beta]$, $[\alpha \wedge \beta]$, $[\alpha \Longrightarrow \beta]$, and $[\alpha \Longleftrightarrow \beta]$ are all wffs.
(c) Nothing else is a wff.

Some examples of well formed formulas are:

(a) $(P \vee Q)$
(b) $(P \Longrightarrow Q)$
(c) $[P \wedge Q]$
(d) $(((P \wedge Q) \vee R) \Longrightarrow (A \wedge \bar{A}))$
(e) $(((P \Longleftrightarrow Q) \wedge (R \Longleftrightarrow S)) \vee (T \Longrightarrow S))$

We shall often eliminate many of the parentheses that arise in wffs. This, of course, will be done only when there is no confusion about the meaning of the wff. So, for example, the five wffs given above can also be written as:

(a) $P \vee Q$
(b) $P \Longrightarrow Q$
(c) $P \wedge Q$
(d) $(P \wedge Q) \vee R \Longrightarrow A \wedge \bar{A}$
(e) $((P \Longleftrightarrow Q) \wedge (R \Longleftrightarrow S)) \vee (T \Longrightarrow S)$

The logical operators may be assigned priorities, P, as below:

$$P(\neg) = 5;\ P(\wedge) = 4;\ P(\vee) = 3;\ P(\Longrightarrow) = 2;\ \text{and}\ P(\Longleftrightarrow) = 1$$

These may be used to resolve ambiguities when parentheses have been dropped. Thus, if the sequence aQb appears in a wff (where a and b are logical operators), then Q is the right operand of a iff $P(a) \geqslant P(b)$. If Q is not the right operand of a, then it is the left operand of b.

Let us look at a few examples of translations of English statements into wffs. Consider the statement:

> If Tom fails the discrete structures final, he will have to retake the final or be placed on probation.

Using the symbolism:

P: Tom fails the discrete structures final
Q: Tom will have to retake the final
R: Tom will be placed on probation

the above statement may be written as:

$$P \Longrightarrow Q \vee R$$

As another example, consider:

> Mary can write her program in Pascal or Fortran or not write it at all. If she does not write her program she will get a zero and fail the course. If she fails the course she will be put on probation and if she gets a zero her boyfriend will desert her. If Mary writes her program in Fortran, she will fail the course but if she writes it in Pascal, she will pass.

Let us use the symbolism:

P: Mary writes her program in Pascal
Q: Mary writes her program in Fortran
R: Mary does not write her program
S: Mary gets a zero
T: Mary fails
U: Mary is put on probation
V: Mary's boyfriend deserts her

One might be tempted to write the first sentence concerning Mary as:

$$P \vee Q \vee R$$

Observe that it is not possible for P, Q, and R to all be simultaneously true. Assuming that it is possible for Mary to write her program in both Pascal and Fortran, the first sentence takes the symbolic form:

$$(P \vee Q \vee R) \wedge (P \vee Q \Longrightarrow \bar{R})$$

The wff corresponding to the set of statements about Mary is:

$$(P \vee Q \vee R) \wedge (P \vee Q \Longrightarrow \bar{R}) \wedge (R \Longrightarrow S \wedge T) \wedge (T \Longrightarrow U) \wedge (S \Longrightarrow V) \wedge (Q \Longrightarrow T) \wedge (P \Longrightarrow \bar{T})$$

Given a truth value for each of the propositional variables appearing in a wff one can determine the truth value of the wff. A wff that evaluates to true for all possible truth assignments to its variables is a *tautology* (a tautology is also called a *theorem*). A *contradiction* is a wff that evaluates to false for all possible truth assignments to its variables. A wff that is not a contradiction is said to be *satisfiable*. Note that a wff is satisfiable iff there is at least one set of truth assignments to its variables under which the wff evaluates to true.

One way to determine if a wff is a tautology, is satisfiable, or is a contradiction is to use the truth table method used earlier. By examining the columns for $\neg P$, $P \vee Q$, $P \wedge Q$, $P \Longrightarrow Q$ and $P \Longleftrightarrow Q$ in Figure 1.1, we can conclude that neither of these wffs is a tautology. For example, $P \vee Q$ is false when both P and Q are false. Also, neither of these is a contradiction. Each of these wffs is satisfiable. Figure 1.2 gives truth tables for several other wffs. Each of the wffs considered in Figure 1.2 is important. $P \wedge \bar{P}$ is the negation of the law of contradiction. $P \vee \bar{P}$ is the law of the excluded middle. As expected, $P \wedge \bar{P}$ is a contradiction and $P \vee \bar{P}$ is a tautology. The wffs of Figures 1.2(c) to (e) are all tautologies. The tautology $(\bar{P} \vee Q) \Longleftrightarrow (P \Longrightarrow Q)$ implies that we can do away with the operator " $\Longrightarrow$ ". As stated earlier, $P \Longleftrightarrow Q$ is equivalent to $(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$. So, the operator " $\Longleftrightarrow$ " can also be eliminated and we need only consider the three operators $\wedge$, $\vee$, and $\neg$. It is, however, often more convenient to use $\Longrightarrow$ and $\Longleftrightarrow$ in wffs rather than their equivalent forms. The tautologies of Figures 1.2(d) and (e) are known as DeMorgan's Laws.

Since each variable in a wff can be assigned one of two possible values (T or F), the number of rows in the truth table for a wff with r variables is 2^r. When r=3 the number of rows is 8 and when r=6, the number of rows is 64. The truth table method is therefore suitable only for wffs with a small number of variables. In subsequent sections, we shall examine alternate methods to determine if a wff is a tautology, is satisfiable or is a contradiction.

1.2 NORMAL FORMS AND BOOLEAN ALGEBRA

We have seen how to obtain a truth table for any given wff. Suppose we are given a truth table. How can we obtain a wff corresponding to this table?