PLANE TRIGONOMETRY

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WITH TABLES

PLANE TRIGONOMETRY

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FIRST EDITION

NEW YORK TORONTO LONDON
McGRAW-HILL BOOK COMPANY, INC.
1950

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PREFACE

This text has been prepared as a first course in trigonometry for college students who need a sufficient mastery of the subject for its employment in other fields. As a first consideration in the interest of the student, careful thought has been given to his reading ability. In avoiding discussions and explanations of a too brief, too condensed nature, a serious effort has been made to present each new topic in complete, clear, and simple language.

Careful attention has also been given to the arrangement of the material. Instead of presenting analytical trigonometry and computational trigonometry as somewhat separate units, these two aspects of the subject alternate in a logical, natural way; that is, applications involving computations are made at intervals following the development of the necessary theory. By this plan of continuity the numerical problems reveal interesting uses of trigonometry and at the same time provide welcome pauses in the study of the more difficult but extremely important analytical side. Also, students who do not compute with facility will profit by not having numerical trigonometry in an unbroken series of lessons.

Other features of the book include the following:

1. The first chapter contains a development of the introductory material leading to the definitions of the trigonometric functions. This material, already largely familiar to the student, provides a gradual entry into the study of trigonometry. Here the degree and the radian are defined, and these two systems of angular measure are continued throughout the book.

2. The trigonometric functions of the general angle are first defined. Then these definitions are specialized to the acute angles of a right triangle.

3. The solution of right triangles is first considered without the use of logarithms. This permits the new problem of triangle

solving to be attacked without the added complication of new methods of computing and at the same time brings into use the table of natural trigonometric functions.

4. The treatment of the theory and applications of logarithms is adequate and does not assume their previous study on the part of the student. One simple rule giving the relation of the decimal point of a number and the characteristic of its logarithm replaces the usual multiple rules.

5. More than casual attention has been given to approximate numbers and significant figures. The results of the worked-out examples and the answers to problems are expressed in keeping with the accuracy of the given data.

6. Full discussion is given to line values of the trigonometric functions. The reduction of functions to acute angles, following logically their line segment representation, is treated in a simple, easy-to-understand way.

7. Vectors are introduced in connection with right triangles and are considered also in problems involving oblique triangles.

8. New topics are amply illustrated with worked-out examples. These problems have been selected with the view of bringing to light the usual troublesome points. The exercises, occurring at short intervals, begin with easy problems which call for the use of a new principle in a simple way. By gradually increasing in difficulty the problems challenge and stimulate the interest of the student. Answers are given to the odd-numbered problems.

The author is deeply indebted to Professor Ernest Williams of Alabama Polytechnic Institute for studying the manuscript and suggesting numerous improvements.

GORDON FULLER

AUBURN, ALA.
December, 1949

THE GREEK ALPHABET

Alpha (a)	A α or α
Beta (b)	B β or β
Gamma (g)	Γ γ
Delta (d)	Δ δ or ∂
Epsilon (e)	E ϵ
Zeta (z)	Z ζ
Eta (h)	H η
Theta (th)	Θ θ or ϑ
Iota (i)	I ι
Kappa (k)	K κ or $\varkappa$
Lambda (l)	Λ λ
Mu (m)	M μ
Nu (n)	N ν
Xi (x)	Ξ ξ
Omicron (o)	O o
Pi (p)	Π π
Rho (r)	P ρ
Sigma (s)	Σ σ or ς
Tau (t)	T τ
Upsilon (u)	Υ υ
Phi (ph)	Φ φ or ϕ
Chi (ch)	X χ
Psi (ps)	Ψ ψ
Omega (o)	Ω ω

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CHAPTER 4. FUNDAMENTAL IDENTITIES

CHAPTER 5. LOGARITHMS

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CHAPTER 13. COMPLEX NUMBERS

Chapter 1

ANGLES

1. Introduction

Trigonometry had its beginnings many years ago. As early as the second century B.C., Hipparchus, a Greek astronomer, made a creditable advance toward the founding of this science by collecting and extending some of the basic ideas. Since then, and particularly in modern times, the knowledge and applications of trigonometry have increased tremendously. The earlier use of trigonometry was that of solving triangles; that is, it was used to compute unknown sides and angles of triangles. The basic feature of triangle solving involves quantities which are determined from the angles. These quantities are called ***trigonometric functions,*** and are defined in Sec. 10. Trigonometry, in the nature of triangle solving, has had practical applications through the centuries and is today used extensively in astronomy, land measurement, navigation, and in numerous other ways. But this phase of trigonometry has been surpassed in importance by the development of other aspects of the subject. The theory of the trigonometric functions has contributed to the advancement of mathematics in its several branches. In turn, the physical and engineering sciences, leaning heavily on mathematics, have profited immeasurably by the employment of the trigonometric functions, apart from their triangle-solving uses. Hence these functions, based on the concept of angle, are vital in the structure and applications of mathematics.

2. Angles

The figure made by two half lines drawn from a point (that is, each line extends in only one direction from the point) is called an ***angle.*** In elementary geometry positive angles not exceeding

360° are studied. We shall extend the idea of angle to include both positive and negative angles of any size or magnitude.

Suppose that a half line OX, drawn from O, is turned about O in a plane to another position OP. Then the angle XOP is said to be *generated.* OX is called the ***initial side,*** and OP the ***generating,*** or ***terminal, side,*** and O is the ***vertex*** of the angle. If OX is turned counterclockwise, the generated angle is defined as *positive,* and a clockwise rotation makes a *negative* angle. The magnitude of the angle is determined by the amount of turning made in

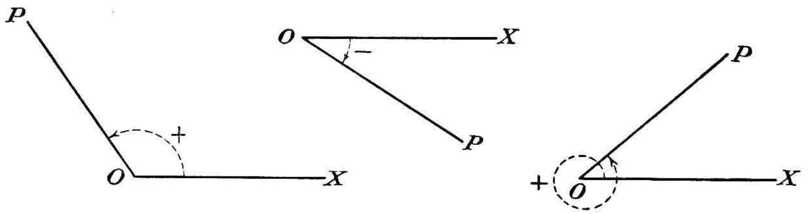

Figure 1

generating the angle. Obviously, the generating side may make any number of revolutions and any part of a revolution; hence an angle of any magnitude may be formed. In the figure the curved arrow in each case points in the direction of rotation, and the sign of the angle is indicated by + or − .

3. Units of Measurement

If the angle formed by the generating side in making one revolution is divided into 360 equal parts, then each part is called an angle of one ***degree*** (written 1°). One of the 60 equal parts of a degree is called a ***minute*** (written 1′). The angle of one minute, in turn, is divided into 60 equal parts, each part being called a ***second*** (written 1″). The degree, minute, and second are the units of the ***sexagesimal,*** or ***degree, system*** of angular measure. This system is used in astronomy, surveying, and engineering.

A unit of angular measure, called the radian, is more convenient than other units in certain theoretical work, particularly in the study of calculus. An angle with its vertex at the center of a circle is called a central angle. A central angle which intercepts an arc of the circle equal in length to the radius of the circle is defined as having a measure of 1 ***radian.*** Radian measure is frequently called ***circular*** measure.

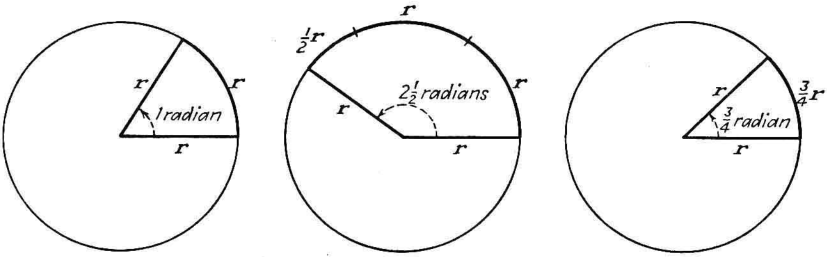

Figure 2

From geometry we know that the circumference of any circle is 2π times its radius; that is,

$$c = 2\pi r.$$

Then a length of 1 radius can be measured off along the circumference 2π times. This means that the entire circumference subtends a central angle of 2π, or, approximately, 6.2832 radians. It should be noticed that 2π radians are obtained regardless of the length of the radius; hence the radian is an angle of fixed size just as the degree is a fixed angle.

4. The Relation between Degrees and Radians

Since the entire circumference of a circle subtends a central angle of 2π units in terms of radians, and 360 units in terms of degrees,

$$2\pi \text{ radians} = 360°,$$

and

$$\pi \textbf{ radians} = \mathbf{180°}.$$

By this relation, a radian may be expressed in terms of degrees, and also a degree may be expressed in radian measure. That is,

$$1 \text{ radian} = \frac{180°}{\pi} = 57.296° = 57°\ 17.7', \text{ approx.},$$

and

$$1° = \frac{\pi}{180} \text{ radian} = 0.017453 \text{ radian, approx.}$$

In expressing an angle in terms of radians, frequently the word "radian" is omitted. For example, using π radians and $\pi/4$ radian in this way, we write $\pi = 180°$ and $\pi/4 = 45°$.

***Example* 1:** Express $5\pi/6$ in terms of degrees.

Solution: Using the relation $\pi = 180°$, we have

$$\frac{5\pi}{6} = \frac{5}{6}(180°) = 150°.$$

***Example* 2:** Express 2.3 radians in degrees.

Solution: Using the approximation 1 radian = 57.3°, we obtain

$$2.3 \text{ radians} = 2.3(57.3°) = 131.8°.$$

***Example* 3:** Express 160° in radians.

Solution:
$$160° = 160\frac{\pi}{180} = \frac{8\pi}{9}.$$

***Example* 4:** Change 31° 21′ to radian measure.

Solution: We first express 21′ as a decimal part of a degree and then use the approximation 1° = .01745 radian. Thus,

$$31°\,21' = 31.35° = 31.35(.01745) = .5471 \text{ radian}.$$

EXERCISE 1

Express, in terms of degrees and minutes, the angles which have the following radian measure:

1. $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}$.

2. $\frac{3\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{12}$.

3. $\frac{2\pi}{9}, \frac{-5\pi}{3}, \frac{5\pi}{12}, \frac{-\pi}{24}$.

4. $\frac{-5\pi}{6}, \frac{5\pi}{4}, \frac{-8\pi}{9}, \frac{-7\pi}{15}$.

5. 3, −1.5, 2.7, −0.6.

6. 2, −1.7, −1.8, 0.03.

Express the following angles in radians:

7. 90°, 60°, 45°, 120°.

8. 30°, 75°, 180°, 270°.

9. −180°, 135°, 210°, −300°.

10. −150°, 240°, −225°, 330°.

11. 720°, 450°, 540°, 840°.

12. 900°, 630°, 510°, 585°.

13. 17°, 47°, 12° 39′, 8° 12′.

14. 41°, 67°, 19° 42′, 7° 24′.

15. 83° 18′, 106° 27′, 253° 54′.

16. 47° 30′, 114° 36′, 316° 51′.

5. Length of Arc

Let us suppose that an angle of θ radians is constructed with its vertex at the center of a circle of radius r, and that this angle intercepts an arc of length s along the circumference of the circle. Since a central angle of 1 radian intercepts an arc equal in length to the radius of the circle, a central angle of θ radians intercepts an arc of length θ times the radius. That is,

$$s = r\theta.$$

This relation among the quantities s, r, and θ may be used to compute any one of them if the other two are known. It is essential that the radius r and the arc s be measured in the same linear units and that θ be in radians.

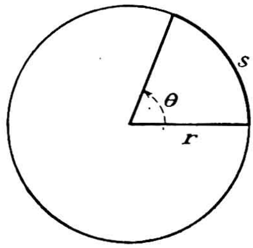

Figure 3

Example 1: Find the central angle subtended by an arc of length 5 if the radius is 2.

Solution: Substituting in the relation $s = r\theta$, we obtain $5 = 2\theta$. Hence, $\theta = 2.5$ radians.

Example 2: A central angle of 38° intercepts an arc of 3 feet. Find the radius of the circle.

Solution: In order to use the relation $s = r\theta$, we first express 38° in terms of radians.

$$38° = 38(.01745) = 0.6631 \text{ radian.}$$

Hence,

$$3 = 0.6631r, \qquad \text{and} \qquad r = 4.524 \text{ feet.}$$

EXERCISE 2

1. Find the length of the arc of a circle of radius 10 inches which subtends a central angle of (*a*) 1.7 radians; (*b*) 60°; (*c*) 200°.

2. An arc of 7 inches subtends central angles of 3.8 radians, 120°, and 160°, respectively, in each of three circles. Find the radii of the circles.

3. Find the central angle which intercepts an arc of 3 feet if the radius of the circle is (*a*) 5 feet; (*b*) 20 inches; (*c*) 2 yards.

4. Two points on the earth's equator differ in longitude by 3°. Find the distance between the points, figuring the radius of the earth as 4000 miles.

5. A town is 120 miles due north of another town. Find the difference of their latitudes if the earth's radius is 4000 miles.

6. Find the acute angle in radians made by the hands of a clock at 8 A.M.

7. The pendulum of a clock, 25 inches long, swings through an arc of 18°. Find the distance which the tip travels in one swing.

6. Area of a Sector

The figure made by an arc of a circle and the radii drawn to its extremities is called a ***sector*** of the circle. The area of a sector may be readily expressed in terms of its angle and the radius. If θ is the angle of the sector in radians, the area A of the sector is a fractional part $(\theta/2\pi)$ of the area of the circle (πr^2); that is,

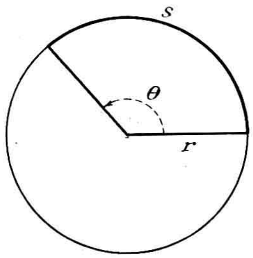

Figure 4

$$A = \frac{\theta}{2\pi}(\pi r^2).$$

Hence,

$$A = \tfrac{1}{2}r^2\theta.$$

This relation may be used to solve for A, r, or θ if the other two are known.

Example: Find the area of a sector which has a central angle of 40° and a radius of 3 inches.

Solution: We express the central angle in radians and then substitute in $A = \frac{1}{2}r^2\theta$. Thus,

$$40° = 40(.01745) = .6980 \text{ radian},$$

and

$$A = \tfrac{1}{2}(3)^2(.6980) = 3.141 \text{ square inches}.$$

7. Circular Motion

Radian measure is helpful in the study of circular motion. Suppose that a particle moves with constant speed along the circumference of a circle, and in one unit of time traverses a length of arc which subtends a central angle ω. The angle ω is called the ***angular velocity.*** If ω is expressed in radian measure, then the distance which the particle travels in one unit of time is $r\omega$. This is the ***linear velocity*** of the particle. Denoting the linear velocity by v, we have the simple but useful formula

$$v = r\omega.$$

***Example* 1:** A particle moves with a constant speed of 2 feet per second along the circumference of a circle of radius 3 feet. Find the angular velocity and the total angle through which the particle moves in 10 seconds.