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The Bondi-Mass Type Estimate on a Closed CR 3-Manifold with Nonnegative Paneitz Operator*

Shu-Cheng Chang † Chin-Tung Wu ‡

Abstract

In this paper, we show the CR analogue of Bondi-mass type estimate on a closed CR 3-manifold with nonnegative CR Paneitz operator. With its applications, by combining Harnack-type estimate, Hamilton rescaling method and Gromov compactness theorem, one obtains the long time existence and asymptotic convergence of solutions of the Calabi flow on a closed CR 3-manifold with nonnegative CR Paneitz operator. As a consequence, we have an affirmative answer to the CR Yamabe problem on a closed CR 3-manifold with the positive Yamabe constant and nonnegative CR Paneitz operator. In particular, there exists a contact form of positive constant Tanaka-Webster curvature on a closed CR 3-manifold with positive Yamabe constant and vanishing pseudohermitian torsion.

Keywords and Phrases: CR Calabi flow, CR Böchner formula, Tanaka-Webster curvature, Pseudoharmonic manifold, CR Paneitz operator, Sub-Laplacian, CR Yamabe constant, Positive mass theorem.

1 Introduction

Let (M^{2n+1}, J) denote a closed $(2n+1)$-dimensional CR manifold. The CR Yamabe problem is to find a contact form on M with constant Tanaka-Webster curvature. In serial papers ([JL1], [JL2], [JL3]), D. Jerison and J. Lee initiated the study of this problem . They confirmed the CR Yamabe problem in case of the CR Yamabe constant $\lambda(M)$ is less than the one $\lambda(\mathbf{S}^{2n+1})$ for a standard CR sphere $(\mathbf{S}^{2n+1}, \widehat{J})$. Their methods can be compared to the partial completion of the proof of Riemannian Yamabe problem by T. Aubin ([?]). By the pseudohermitian analogue, the remaining cases should be solved using a CR analogue of the positive mass conjecture which is not available at this present stage of research.

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However in the present paper, due to the previous works on the so-called Bondi-mass type estimate of the Calabi flow on a Riemannian manifold ([Chru], [Ch6]), we do have the CR analogue of Bondi-mass type estimate of the CR Calabi flow on a closed CR 3-manifold with nonnegative CR Paneitz operator.

With its application of the CR Bondi-mass type estimate, we have an affirmative answer to the CR Yamabe problem on a closed CR 3-manifold with $\lambda(M) > 0$ and nonnegative CR Paneitz operator P_0. In particular, there exists a contact form of positive constant Tanaka-Webster curvature on a closed CR 3-manifold with positive Yamabe constant and vanishing pseudohermitian torsion.

Let $\widehat{\theta}$ be a fixed background contact form and $[\widehat{\theta}]$ be the fixed contact class. Write $\theta = e^{2\lambda}\widehat{\theta}$, we can associate the so-called Tanaka-Webster curvature W. Let $d\mu = \theta \wedge d\theta$ denote the volume form. We define the energy functional $\mathcal{E}(\theta)$ associated with θ as follows:

$$\mathcal{E}(\theta) = \int_M W^2 d\mu,$$

for $\theta = e^{2\lambda}\widehat{\theta}$.

We consider the negative gradient flow of $\mathcal{E}(\theta)$ as follows:

$$\frac{\partial\theta}{\partial t} = 2(\Delta_b W)\theta.$$

Now if $\theta = e^{2\lambda}\widehat{\theta}$ for some real (smooth) function λ on $M \times [0, T)$. Then we can express the so-called Calabi flow on a CR 3-manifold $(M, [\widehat{\theta}])$:

$$\begin{cases} \frac{\partial\lambda}{\partial t} = \Delta_b W, \\ \theta = e^{2\lambda}\widehat{\theta}, \ \lambda_0(p) = \lambda(p, 0), \\ \int_M e^{4\lambda_0} d\widehat{\mu} = \int_M d\widehat{\mu}. \end{cases} \tag{1.1}$$

Here Δ_b denotes the sub-Laplacian operator with respect to θ. Also it is easy to see that the volume

$$\int_M d\mu = \int_M \theta \wedge d\theta = \int_M e^{4\lambda} d\widehat{\mu}\ , \quad \text{where} \quad d\widehat{\mu} = \widehat{\theta} \wedge d\widehat{\theta}.$$

is preserved under the flow (1.1) by the divergence formula in pseudohermitian geometry ([L2]).

Note that the linearization of the right side of (1.1) is a fourth-order subelliptic operator (the leading term basically is the negative square sublaplacian) So the short time solution and uniqueness follows from a standard argument parallel to the parabolic (or elliptic) case. The analogous situation on a Riemannian 4-dimension manifold has been studied by the first author ([Ch6]). On the other hand, the Calabi flow on a CR 3-manifold is corresponding to the Calabi flow on a Riemannian 4-manifold.

Since (1.1) is a fourth-order flow, there doesn't seem to be any proper maximum principle. Therefore we have to invoke some kind of integral estimate. The idea is to get the so-called Bondi-mass loss formula which comes from the work of P. T. Chruściel ([Chru]). He studied the 2-dimensional Calabi flow on a closed surface Σ , known as the Robinson-Trautman equation. The corresponding integral

$$\int_{\Sigma} e^{3\lambda} d\hat{\mu}$$

is known to be the so-called Bondi-mass in the theory of general relativity. Moreover, the first author generalized this type estimate to the higher dimensional case ([Ch2], [Ch4], [Ch6], [CW3]).

Before state the main theorems, we recall several definitions as below.

Definition 1.1. *The CR Yamabe constant Q on $(M, J, [\widehat{\theta}])$ is defined by*

$$Q(M, J) = \inf_{\varphi \neq 0} \frac{E_{\theta}(\varphi)}{(\int |\varphi|^4 d\mu)^{\frac{1}{2}}},$$

where $d\mu = \theta \wedge d\theta$ is the volume form and E_θ is

$$E_{\theta}(\varphi) = \int |\nabla_b \varphi|_h^2 \, d\mu + \int W \varphi^2 \, d\mu.$$

Note that $E_{\widehat{\theta}}(\varphi) = E_{\theta}(u\varphi)$ for $\widehat{\theta} = u^2\theta$. This implies that $Q(M, J)$ is a CR invariant. Next we define

$$P\varphi = (\varphi_{\bar{1}}{}^{\bar{1}}{}_{1} + iA_{11}\varphi^1)\theta^1 = P\varphi = (P_1\varphi)\theta^1,$$

which is an operator that characterizes CR-pluriharmonic functions. Here $P_1\varphi = \varphi_{\bar{1}}{}^{\bar{1}}{}_{1} + iA_{11}\varphi^1$ and $\overline{P}\varphi = (\overline{P}_1)\theta^{\bar{1}}$, the conjugate of P.

Definition 1.2. *The CR Paneitz operator P_0 is defined by*

$$P_0\varphi = 4\left(\delta_b(P\varphi) + \overline{\delta}_b(\overline{P}\varphi)\right), \tag{1.2}$$

where δ_b is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_1\theta^1) = \sigma_{1,}{}^{1}$, and similarly, $\overline{\delta}_b(\sigma_{\bar{1}}\theta^{\bar{1}}) = \sigma_{\bar{1},}{}^{\bar{1}}$.

We observe that

$$\int \langle P\varphi + \overline{P}\varphi, d_b\varphi \rangle_{L^*_\theta} \, d\mu = -\frac{1}{4} \int P_0\varphi \cdot \varphi \, d\mu \tag{1.3}$$

with $d\mu = \theta \wedge d\theta$. One can check that P_0 is self-adjoint, that is, $\langle P_0\varphi, \psi \rangle = \langle \varphi, P_0\psi \rangle$ for all smooth functions φ and ψ. For the details about these operators, the reader can make reference to [GL], [L1], [GG] and [FH].

Definition 1.3. *On a complete pseudohermitian 3-manifold (M, J, θ), we call the CR Paneitz operator P_0 with respect to (J, θ) nonnegative if*

$$\int_M P\varphi \cdot \varphi d\mu \geqslant 0$$

for all real C^∞ smooth functions.

Remark 1.4. 1. *The nonnegativity of P_0 is a CR invariant in the sense that it is independent of the choice of the contact form θ.*

2. *Let (M, J, θ) be a closed pseudohermitian 3-manifold with free torsion. Then the corresponding CR Paneitz operator is nonnegative ([CCC]).*

Now we are ready to state our main theorem on the long-time existence of solutions of (1.1).

Theorem 1.5. *Let $(M, [\widehat{\theta}])$ be a closed CR 3-manifold with nonnegative CR Paneitz operator. Then there exist solutions $\{e^{2\lambda}g_0\}$ of (1.1) on $M \times [0, \infty)$.*

For the asymptotic behavior of solutions of (1.1), we have

Theorem 1.6. *Let $(M, [\widehat{\theta}])$ be a closed CR 3-manifold with nonnegative CR Paneitz operator. If $Q > 0$ with $\widehat{W} \leqslant 0$ in a ball $B(p_0, \rho_0)$, there exists a subsequence of solutions $\{e^{2\lambda}g_0\}$ of (1.1) on $M \times [0, \infty)$ which converges smoothly to a positive constant Tanaka-Webster curvature metric g_∞.*

Remark 1.7. *1. For any closed CR 3-manifold $(M, \widehat{\theta})$ with $Q > 0$, there always exists a metric θ_1 of positive Tanaka-Webster curvature which is pointwise conformal to $\widehat{\theta}$. Due to CR analogue of [KW, Lemma 6.2] and [CH], there is a metric θ_2 which is CR equivalent to θ_1 with $\widehat{W} \leqslant 0$ in a ball $B(p_0, \rho_0)$ with $Q([\theta_2]) > 0$.*

2. Note that the nonnegativity of P_0 is preserved with respect to θ_2 due to Remark 1.4.

From the previous remark, one obtains the following Yamabe problem on a closed CR 3-manifold with positive Yamabe constant.

Corollary 1.8. *There exists a contact form of positive constant Tanaka-Webster curvature on a closed CR 3-manifold with positive Yamabe constant and nonnegative CR Paneitz operator.*

In particular, from Remark 1.4, one obtains

Corollary 1.9. *There exists a contact form of positive constant Tanaka-Webster curvature on a closed CR 3-manifold with positive Yamabe constant and vanishing pseudohermitian torsion.*

We also have the corresponding results on a closed CR 3-manifold with non-positive Yamabe constant Q and nonnegative CR Paneitz operator P_0.

Theorem 1.10. *Let $(M, [\widehat{\theta}])$ be a closed CR 3-manifold with nonnegative CR Paneitz operator. If $Q < 0$ or $Q = 0$, there exists a subsequence of solutions $\{e^{2\lambda}g_0\}$ of (1.1) on $M \times [0, \infty)$ which converges smoothly to a constant Tanaka-Webster curvature metric g_∞.*

In section 2, we drive the CR analogue of Bondi-mass type estimate by the CR Bochner formula. In section 3, based on [CY], [Ch2], [Chru], we obtain the C^0-bound and the higher order $S^{k,2}$-estimates of the solution for (1.1) if we have the L^p-bound of the Tanaka-Webster curvature, $p > 2$. Then the long-time existence of solutions of (1.1) follows easily under the curvature assumption.

In section 4, based on the CR Bondi-mass type estimate, we first obtain the Harnack-type estimate for the equation (1.1). That is, one can have the uniformly bound on $\int_M e^{5\lambda}d\widehat{\mu}$ if we have the uniformly lower bound for solution $\lambda(t)$ of (1.1) plus the L^p-bound of the Tanaka-Webster curvature. Second, we are able to get the uniformly lower bound for solution $\lambda(t)$ of (1.1) if we have the L^p-bound of the Tanaka-Webster curvature. Then we have the asymptotic convergence of solutions of (1.1) under the curvature assumption. In section 5, our aim is to characterize the asymptotic behavior of solutions of the Calabi flow as $t \to T$. Moreprecisely, we need to analysis the behavior of the type I and type II singularities. If the singularity occurs as $t \to T \leqslant \infty$, one can blow up the solution by using rescaling techniques ([H3]). This will lead to a contradiction with a result of [Yau]. Then we are done.

Note that due to another proof of Riemannian Yamabe problem by A. Bahri, recently N. Gamara and Y. Yacoub ([GY]) gereneralized it to CR category.

2 Bondi-mass estimates and CR Paneitz operator

We first give a brief introduction to pseudohermitian geometry (see [L1], [L2] for more details). Let M be a closed 3-manifold with an oriented contact structure ξ. There always exists a global contact form θ, obtained by patching together local ones with a partition of unity. The characteristic vector field of θ is the unique vector field T such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$. A CR-structure compatible with ξ is a (smooth) endomorphism $J : \xi \to \xi$ such that $J^2 = -identity$. A pseudohermitian structure compatible with ξ is a CR-structure J compatible with ξ together with a global contact form θ.

Given a pseudohermitian structure (J, θ), we can choose a complex vector field Z_1, an eigenvector of J with eigenvalue i, and a complex 1-form θ^1 such that $\{\theta, \theta^1, \theta^{\bar{1}}\}$ is dual to $\{T, Z_1, Z_{\bar{1}}\}$. It follows that $d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$ for some nonzero real function $h_{1\bar{1}}$. If $h_{1\bar{1}}$ is positive, we call such a pseudohermitian structure (J, θ) positive, and we can choose a Z_1 (hence θ^1) such that $h_{1\bar{1}} = 1$. That is to say

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}}. \tag{2.1}$$

We'll always assume our pseudohermitian structure (J, θ) is positive and $h_{1\bar{1}} = 1$ throughout the paper. The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes C$ (and extended to tensors) given by

$$\nabla Z_1 = {\omega_1}^1 \otimes Z_1, \nabla Z_{\bar{1}} = {\omega_{\bar{1}}}^{\bar{1}} \otimes Z_{\bar{1}}, \nabla T = 0$$

in which the 1-form ${\omega_1}^1$ is uniquely determined by the following equation with a normalization condition:

$$\begin{aligned} d\theta^1 &= \theta^1 \wedge {\omega_1}^1 + {A^1}_{\bar{1}}\theta \wedge \theta^{\bar{1}} \\ {\omega_1}^1 &+ {\omega_{\bar{1}}}^{\bar{1}} = 0. \end{aligned} \tag{2.2}$$

The coefficient $A^1{}_{\bar{1}}$ is called the (pseudohermitian) torsion. Since $h_{1\bar{1}} = 1$, $A_{\bar{1}\bar{1}} = h_{1\bar{1}} A^1{}_{\bar{1}} = A^1{}_{\bar{1}}$. And A_{11} is just the complex conjugate of $A_{\bar{1}\bar{1}}$. Differentiating $\omega_1{}^1$ gives

$$d\omega_1{}^1 = W\theta^1 \wedge \theta^{\bar{1}} + 2iIm(A_{11,\bar{1}}\theta^1 \wedge \theta)$$

where W is the Tanaka-Webster curvature.

We can define the covariant differentiations with respect to the pseudohermitian connection. For instance, $f_1 = Z_1 f$, $f_{1\bar{1}} = Z_{\bar{1}} Z_1 f - \omega_1{}^1(Z_{\bar{1}}) Z_1 f$, $f_0 = Tf$ for a (smooth) function f. We define the subgradient operator ∇_b and the sublaplacian operator Δ_b by

$$\nabla_b f = f_{\bar{1}} Z_1 + f_1 Z_{\bar{1}},$$
$$\Delta_b f = f_{1\bar{1}} + f_{\bar{1}1},$$

respectively. We also define the Levi form $<,>_\theta$ by

$$< V, U >_\theta = 2d\theta(V, JU) = v_1 u_{\bar{1}} + v_{\bar{1}} u_1$$

for $V = v_1 Z_{\bar{1}} + v_{\bar{1}} Z_1$, $U = u_1 Z_{\bar{1}} + u_{\bar{1}} Z_1$ in ξ. The associated norm is defined as usual: $|V|_\theta^2 =< V, V >_\theta$. Next we define the divergence operator δ_b by

$$\delta_b(V) = v_1{}^1 + v_{\bar{1}}{}^{\bar{1}}$$

for $V = v_1 Z_{\bar{1}} + v_{\bar{1}} Z_1$ in ξ.

We recall what the Folland-Stein space $S^{k,p}$ is. Define

$$(V, U)_\theta = \int_M < V, U >_\theta \theta \wedge d\theta.$$

For a vector $X \in \xi$, we define $|X|^2 \equiv< X, X >_\theta$. It follows that $|\nabla_b f|^2 = 2 f_1 f_{\bar{1}}$ for a real valued (smooth) function f. Also the square modulus of the sub-Hessian $\nabla_b^2 f$ of f reads $|\nabla_b^2 f|^2 = 2 f_{11} f_{\bar{1}\bar{1}} + 2 f_{1\bar{1}} f_{\bar{1}1}$. Let D denote a differential operator acting on functions. We say D has weight m, denoted $w(D) = m$, if m is the smallest integer such that D can be locally expressed as a polynomial of degree m in vector fields tangent to the contact bundle ξ. We define the Folland-Stein space $S^{k,p}$ of functions on M by

$$S^{k,p} = \{f \in L^p : Df \in L^p \text{ whenever } w(D) \leqslant k\}.$$

We define the L^p norm of $\nabla_b f$, $\nabla_b^2 f$, ... to be $(\int |\nabla_b f|^p \theta \wedge d\theta)^{1/p}$, $(\int |\nabla_b^2 f|^p \theta \wedge d\theta)^{1/p}$, ..., respectively, as usual. So it is natural to define the $S^{k,p}$ norm of $f \in S^{k,p}$ as follows:

$$||f||_{S^{k,p}} \equiv (\sum_{0 \leqslant j \leqslant k} ||\nabla_b^j f||_{L^p}^p)^{1/p}.$$

The function space $S^{k,p}$ with the above norm is a Banach space for $k \geqslant 0$, $1 < p < \infty$. There are also embedding theorems of Sobolev type. For instance, $S^{2,2} \subset S^{1,4}$ (for $\dim M = 3$). We refer the reader to, for instance, [FS] and [Fo] for more discussions on these spaces.

Let $(M,\ J,\ \widehat{\theta})$ be a closed pseudohermitian 3-manifold. Let $\{\widehat{\theta},\widehat{\theta}^1,\widehat{\theta}^{\bar{1}}\}$ satisfy (2.1). Now consider the change of contact form: $\theta = e^{2\lambda}\widehat{\theta}$. Choose $\theta^1 = e^{\lambda}(\widehat{\theta}^1 + 2i\lambda_1\widehat{\theta})$ such that $h_{1\bar{1}} = \widehat{h}_{1\bar{1}}$ (= 1 by assumption). One checks easily that $\{\theta,\theta^1,\theta^{\bar{1}}\}$ also satisfies (2.1). Then the associated Tanaka-Webster curvature W and the sublaplacian operator Δ_b transform as follow:

$$W = e^{-2\lambda}(\widehat{W} - 4\widehat{\Delta}_b\lambda - 4|\widehat{\nabla}_b\lambda|^2), \tag{2.3}$$

$$\Delta_b W = e^{-2\lambda}(\widehat{\Delta}_b W + 2 < \widehat{\nabla}_b W, \widehat{\nabla}_b\lambda >), \tag{2.4}$$

where the operators or quantities with "hat" are with respect to the coframe $\{\widehat{\theta},\widehat{\theta}^1,\widehat{\theta}^{\bar{1}}\}$, and so are the covariant derivatives of λ. Define $Tor(\nabla_b f, \nabla_b f) = iA_{\bar{1}\bar{1}}f_1 f_1 - iA_{11}f_{\bar{1}}f_{\bar{1}}$.

Now we can state the CR Bochner formula.

Proposition 2.1. ([CC2]) *For a (smooth) real function φ on M, we have*

$$\begin{aligned}\frac{1}{2}\Delta_b|\nabla_b\varphi|^2 = |\nabla_b^2\varphi|^2 + &< \nabla_b\varphi, \nabla_b\Delta_b\varphi >_\theta + W|\nabla_b\varphi|^2 \\ &+ Tor(\nabla_b\varphi,\nabla_b\varphi) - 2i(\varphi_1\varphi_{0\bar{1}} - \varphi_{\bar{1}}\varphi_{01})\end{aligned} \tag{2.5}$$

Proof. First it is easy to see that

$$\frac{1}{2}\Delta_b|\nabla_b\varphi|^2 = |\nabla_b^2\varphi|^2 + \varphi_1(\varphi_{\bar{1}1\bar{1}} + \varphi_{\bar{1}\bar{1}1}) + \varphi_{\bar{1}}(\varphi_{11\bar{1}} + \varphi_{1\bar{1}1}). \tag{2.6}$$

By the commutation relations, we compute

$$\begin{aligned}&\varphi_{\bar{1}1\bar{1}} = \varphi_{1\bar{1}\bar{1}} - i\varphi_{0\bar{1}},\\ &\varphi_{\bar{1}\bar{1}1} = \varphi_{\bar{1}1\bar{1}} - i\varphi_{\bar{1}0} + W\varphi_{\bar{1}} = \varphi_{\bar{1}1\bar{1}} - i\varphi_{0\bar{1}} + iA_{\bar{1}\bar{1}}\varphi_1 + W\varphi_{\bar{1}}.\end{aligned}$$

Substituting the above equation and its complex conjugate into (2.6) and observing that

$$< \nabla_b\varphi, \nabla_b\Delta_b\varphi >_\theta = \varphi_1(\varphi_{1\bar{1}\bar{1}} + \varphi_{\bar{1}1\bar{1}}) + \varphi_{\bar{1}}(\varphi_{\bar{1}11} + \varphi_{1\bar{1}1}),$$

thus we obtain the identity (2.5).

We also have the following identity from the equation (3.5) in [CC2]. □

Proposition 2.2. ([CC2]) *For a (smooth) real function φ on M, we have*

$$\int_M(\varphi_0)^2d\mu = \int_M(\Delta_b\varphi)^2d\mu + \int_M Tor(\nabla_b\varphi,\nabla_b\varphi)d\mu - \frac{1}{2}\int_M P_0\varphi\cdot\varphi d\mu, \tag{2.7}$$

where P_0 is the CR Paneitz operator.

In this section, we will drive the key estimates of equation (1.1) from the Bochner formula (2.6) and the identity (2.7). This is the so-called Bondi-mass type estimates as in ([Ch2]).

Lemma 2.3. *Under the flow (1.1), one has*

$$\frac{\partial}{\partial t}W = -2W\Delta_b W - 4\Delta_b^2 W,\ \frac{\partial}{\partial t}d\mu = 4\Delta_b W d\mu \tag{2.8}$$

and

$$\int_M d\mu = \int_M e^{4\lambda}d\widehat{\mu} = \int_M e^{4\lambda_0}d\widehat{\mu} = \int_M d\widehat{\mu}. \tag{2.9}$$

Proof. The evolution equation (2.8) follows from (1.1) and (2.3) by a straightforward computation. While (2.9) comes from integrating the second identity of (2.8) over M. Since $\int_M \Delta_b W d\mu = 0$, by the divergence formula in pseudohermitian geometry. □

Lemma 2.4. *Under the flow (1.1), we have*

$$\frac{d}{dt}\int_M W^2 d\mu = -8\int_M (\Delta_b W)^2 d\mu.$$

As a consequence,

$$\int_M W^2 d\mu \leqslant C(\widehat{W}, \lambda_0),$$

for all t.

Proof. From (2.8) and (2.9),

$$\begin{aligned}&-\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_M W^2 d\mu \\ &= 2\textstyle\int_M W^2\Delta_b W d\mu + 2\int_M(-W^2\Delta_b W + 2W\Delta_b^2 W)d\mu \\ &= 4\textstyle\int_M (\Delta_b W)^2 d\mu.\end{aligned}$$

Thus

$$\frac{d}{dt}\int_M W^2 d\mu \leqslant 0.$$ □

Now we have the followings integral estimates:

Theorem 2.5. *Suppose the Paneitz operator associated with* $\widehat{\theta}$ *is positive and* λ *satisfy* (1.1) *on* $M \times [0,T)$ *with*

$$\int_M |W|^p d\mu \leqslant K, \text{ for some } p > 2 \tag{$*$}$$

for a positive constant K *which is independent of* t. *Then there exists a subsequence* $\{t_j\}$ *as in* (4.3) *such that*

$$\frac{d}{dt}\int_M e^{\alpha\lambda}d\widehat{\mu} \leqslant I_1 + I_2\int_M e^{\alpha\lambda}d\widehat{\mu} - I_3\int_M e^{(\alpha-4)\lambda}|\widehat{\nabla}_b\lambda|^4 d\widehat{\mu},\ \ 6 < \alpha < 46/7$$

where the constants I_i *are independent of* t.

Proof. From (2.3) and (2.4), for $\alpha > 4$, we compute

$$\begin{aligned}
&\frac{1}{\alpha}\frac{d}{dt}\int_M e^{\alpha\lambda}d\widehat{\mu}\\
&= \int e^{\alpha\lambda}(\tfrac{\partial\lambda}{\partial t})d\widehat{\mu}\\
&= \int e^{\alpha\lambda}(\Delta_b W)d\widehat{\mu}\\
&= \int e^{(\alpha-2)\lambda}(\widehat{\Delta}_b W + 2 < \widehat{\nabla}_b W, \widehat{\nabla}_b\lambda >)d\widehat{\mu}\\
&= (\alpha-4)\int e^{(\alpha-2)\lambda}W[\widehat{\Delta}_b\lambda + (\alpha-2)|\widehat{\nabla}_b\lambda|^2]d\widehat{\mu}\\
&= (\alpha-4)\int e^{(\alpha-4)\lambda}[\widehat{W}\widehat{\Delta}_b\lambda + (\alpha-2)\widehat{W}|\widehat{\nabla}_b\lambda|^2 - 4(\widehat{\Delta}_b\lambda)^2\\
&\quad - 4(\alpha-1)\widehat{\Delta}_b\lambda|\widehat{\nabla}_b\lambda|^2 - 4(\alpha-2)|\widehat{\nabla}_b\lambda|^4]d\widehat{\mu}
\end{aligned}$$

Now let $f = e^{(4-\alpha)\lambda}$. Then

$$\begin{aligned}
|\widehat{\nabla}_b\lambda|^2 &= (\alpha-4)^{-2}f^{-2}|\widehat{\nabla}_b f|^2\\
\widehat{\Delta}_b\lambda &= (\alpha-4)^{-1}f^{-2}|\widehat{\nabla}_b f|^2 - (\alpha-4)^{-1}f^{-1}\widehat{\Delta}_b f.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{\alpha}\frac{d}{dt}\int e^{\alpha\lambda}d\widehat{\mu}\\
&= 2(\alpha-4)\int e^{(\alpha-4)\lambda}\widehat{W}|\widehat{\nabla}_b\lambda|^2 d\widehat{\mu} - (\alpha-4)\int e^{(\alpha-4)\lambda} < \widehat{\nabla}_b\widehat{W}, \widehat{\nabla}_b\lambda > d\widehat{\mu}\\
&\quad - 4(\alpha-4)^{-1}\int f^{-3}(\widehat{\Delta}_b f)^2 d\widehat{\mu} + 12(\alpha-4)^{-2}(\alpha-3)\int f^{-4}\widehat{\Delta}_b f|\widehat{\nabla}_b f|^2 d\widehat{\mu}\\
&\quad - 8(\alpha-4)^{-3}(\alpha-3)^2\int f^{-5}|\widehat{\nabla}_b f|^4 d\widehat{\mu}.
\end{aligned}$$

Again let $F = f^r$, for some r to be chosen later. Then

$$\begin{aligned}
|\widehat{\nabla}_b f|^2 &= r^{-2}F^{\frac{2}{r}-2}|\widehat{\nabla}_b F|^2\\
\widehat{\Delta}_b f &= r^{-1}F^{\frac{1}{r}-1}\widehat{\Delta}_b F - (r-1)r^{-2}F^{\frac{1}{r}-2}|\widehat{\nabla}_b F|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{\alpha}\frac{d}{dt}\int e^{\alpha\lambda}d\widehat{\mu}\\
&= 2(\alpha-4)\int e^{(\alpha-4)\lambda}\widehat{W}|\widehat{\nabla}_b\lambda|^2 d\widehat{\mu} - (\alpha-4)\int e^{(\alpha-4)\lambda} < \widehat{\nabla}_b\widehat{W}, \widehat{\nabla}_b\lambda > d\widehat{\mu}\\
&\quad - 4(\alpha-4)^{-1}r^{-2}\int F^{-\frac{1}{r}-2}(\widehat{\Delta}_b F)^2 d\widehat{\mu}\\
&\quad + 4(\alpha-4)^{-2}r^{-3}[(2r+1)\alpha - (8r+1)]\int F^{-\frac{1}{r}-3}\widehat{\Delta}_b F|\widehat{\nabla}_b F|^2 d\widehat{\mu}\\
&\quad - 4(\alpha-4)^{-3}r^{-4}[r(r+1)\alpha^2\\
&\quad - (8r^2+5r-1)\alpha + 16r^2 + 4r - 2]\int F^{-\frac{1}{r}-4}|\widehat{\nabla}_b F|^4 d\widehat{\mu}.
\end{aligned}$$

In order to deal with the term $\int F^{-\frac{1}{r}-3}\widehat{\Delta}_b F|\widehat{\nabla}_b F|^2 d\widehat{\mu}$. We need the following two identities.

$$\begin{aligned}
0 &= \int_M \delta_b(F^{-\frac{1}{r}-2}\widehat{\Delta}_b F\widehat{\nabla}_b F)d\widehat{\mu}\\
&= \int F^{-\frac{1}{r}-2}(\widehat{\Delta}_b F)^2 d\widehat{\mu} + \int < \widehat{\nabla}_b(F^{-\frac{1}{r}-2}\widehat{\Delta}_b F), \widehat{\nabla}_b F > d\widehat{\mu}\\
&= \int F^{-\frac{1}{r}-2}(\widehat{\Delta}_b F)^2 d\widehat{\mu} - (\tfrac{1}{r}+2)\int F^{-\frac{1}{r}-3}\widehat{\Delta}_b F|\widehat{\nabla}_b F|^2 d\widehat{\mu}\\
&\quad + \int F^{-\frac{1}{r}-2} < \widehat{\nabla}_b\widehat{\Delta}_b F, \widehat{\nabla}_b F > d\widehat{\mu},
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
0 &= \int_M \delta_b(F^{-\frac{1}{r}-2}\widehat{\nabla}_b|\widehat{\nabla}_b F|^2)d\widehat{\mu}\\
&= \int F^{-\frac{1}{r}-2}\widehat{\Delta}_b|\widehat{\nabla}_b F|^2 d\widehat{\mu} - (\tfrac{1}{r}+2)\int F^{-\frac{1}{r}-3} < \widehat{\nabla}_b F, \widehat{\nabla}_b|\widehat{\nabla}_b F|^2 > d\widehat{\mu}\\
&= \int F^{-\frac{1}{r}-2}\widehat{\Delta}_b|\widehat{\nabla}_b F|^2 d\widehat{\mu} + (\tfrac{1}{r}+2)\int F^{-\frac{1}{r}-3}\widehat{\Delta}_b F|\widehat{\nabla}_b F|^2 d\widehat{\mu}\\
&\quad - (\tfrac{1}{r}+2)(\tfrac{1}{r}+3)\int F^{-\frac{1}{r}-4}|\widehat{\nabla}_b F|^4 d\widehat{\mu}.
\end{aligned} \tag{2.11}$$