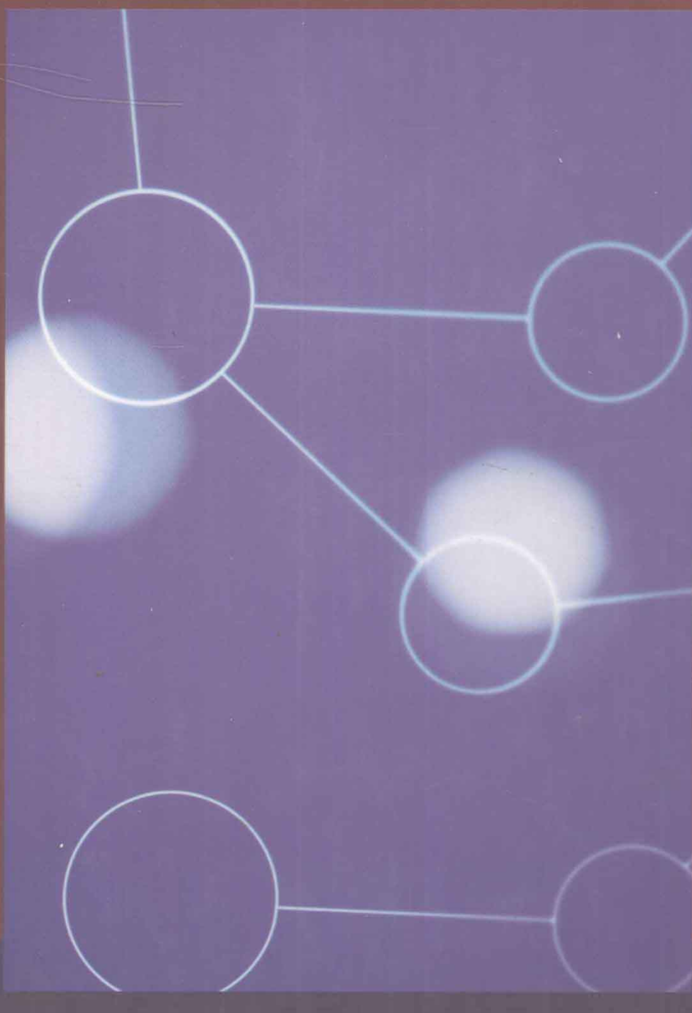
The Walter Rudin Student Series in Advanced Mathematics

Gary Chartrand Ping Zhang

Introduction to Graph Theory



Introduction to Graph Theory

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INTRODUCTION TO GRAPH THEORY

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PREFACE

Perhaps it's not so surprising that when we (the authors) were learning mathematics, we thought that we were being taught some well-known facts – facts that had been around forever. It wasn't until later that we started to understand that these facts (the word "theorem" was beginning to become part of our vocabulary) had not been around forever and that *people* had actually discovered these facts. Indeed, *names* of people were becoming part of the discussion.

Mathematics has existed for many centuries. In the ancient past, certain cultures developed their own mathematics. This was certainly the case with Egypt, Babylonia, Greece, China, India, and Japan. In recent centuries, there has become only one international mathematics. It has become more organized and has been divided into more clearly defined areas (even though there is significant overlap). While this was occurring, explanations (proofs) as to why mathematical statements are true were becoming more structured and clearly written.

The goal of this book is to introduce undergraduates (and perhaps some high school students as well) to the mathematical area called *graph theory*, which only came into existence during the first half of the 18th century. This area didn't start to develop into an organized branch of mathematics until the second half of the 19th century, and there wasn't even a book on the subject until the first half of the 20th century. Since the second half of the 20th century, however, the subject has exploded.

It is our intent to describe some of the major topics of this subject to you and to inform you of some of the people who helped develop and shape this area. In the beginning, most of these people were just like you – students who enjoyed mathematics but with a great sense of curiosity. As with everything else (though not as often talked about), mathematics has its non-serious side and we've described some of this as well. Even the most brilliant mathematicians don't know everything and we've presented some topics that have not been well-studied and in which the answers (and even the questions) are not known. This will give you the chance to do some creative thinking of your own. In fact, maybe the next person who will have an influence on this subject is you.

Part of what makes graph theory interesting is that graphs can be used to model situations that occur within certain kinds of problems. These problems can then be studied (and possibly solved) with the aid of graphs. Because of this, graph models occur frequently throughout this textbook. However, graph theory is an area of mathematics and consequently concerns the study of mathematical ideas – of concepts and their connections with each other. The topics and results we have included were chosen because we feel they are interesting, important and/or are representative of the subject.

As we said, this text has been written for undergraduates. Keeping this in mind, we have included a proof of a theorem if we believe it is appropriate, the proof technique is informative, and if the proof is not excessively long. We would like to think that the material in this text will be useful and interesting for mathematics students as well as for other students whose areas of interest include graphs. This text is also appropriate for self-study.

We have included three appendixes. In Appendix 1, we review some important facts about sets and logic. Appendix 2 is devoted to equivalence relations and functions, while Appendix 3 describes methods of proof. Knowing that undergraduates are still in the process of mastering proofs, we have indicated at the beginning of each proof, the proof technique (or techniques) we are using. We understand how frustrating it is for students (or anyone!) who try to read a proof that is not reader-friendly and which leaves too many details for the reader to supply. Consequently, we have endeavored to give clear, well-written proofs.

Although this can very well be said about any area of mathematics or indeed about any scholarly activity, we feel that appreciation of graph theory is enhanced by being familiar with many of the people, past and present, who were or are responsible for its development. Consequently, we have included several remarks that we find interesting about some of the "people of graph theory". Since we believe that these people are part of the story that is graph theory, we have discussed them within the text and not simply as footnotes. We often fail to recognize that mathematics is a living subject. Graph theory was created by *people* and is a subject that is still evolving.

There are several sections that have been designated as "Excursion". These can be omitted with no negative effect if this text is being used for a course. In some cases, an Excursion is an area of graph theory we find interesting but which the instructor may choose not to discuss due to lack of time or because it's not one of his or her favorites. In other cases, an Excursion brings up a sidelight of graph theory that perhaps has little, if any, mathematical content but which we simply believe is interesting.

There are also sections that we have designated as "Exploration". These sections contain topics with which students can experiment and use their imagination. These give students opportunities to practice asking questions. In any case, we believe that this might be fun for some students.

As far as using this text for a course, we consider the first three chapters as introductory. Much of this could be covered quite quickly. Students could read these chapters on their own. It isn't necessary to cover connectivity and Menger's Theorem if the instructor chooses not to do so. Sections 8.3, 9.2, 10.3, and 11.2 could easily be omitted; while material from Chapters 12 and 13 can be covered according to the teacher's interest.

Solutions or hints for the odd-numbered exercises in the regular sections of the text, references, an index of mathematical terms, an index of people, and a list of symbols are provided at the end of the text.

It was because of discussions that we had with Robert Ross, Executive Editor at McGraw-Hill, that the idea to write this text was initiated. We thank him for this and for his encouragement. We also thank others at McGraw-Hill for their assistance, namely, Daniel Seibert, Editorial Assistant, and Vicki Krug, Senior

Project Manager. In addition, we are most grateful to the reviewers who did a tremendous job and who gave us numerous valuable suggestions: Jay Bagga, Ball State University; Richard Borie, University of Alabama; Anthony Evans, Wright State University; Mark Ginn, Appalachian State University; Mark Goldberg, Rensselaer Polytechnic Institute; Arthur Hobbs, Texas A&M University; Garth Isaak, Lehigh University; Daphne Liu, California State University, Los Angeles; Alan Mills, Tennessee Technological University; Dan Pritikin, Miami University; John Reay, Western Washington University; Yue Zhao, University of Central Florida.

Gary Chartrand and Ping Zhang

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Chapter 1

Introduction

1.1 Graphs and Graph Models

A major publishing company has ten editors (referred to by $1, 2, \ldots, 10$) in the scientific, technical, and computing areas. These ten editors have a standard meeting time during the first Friday of every month and have divided themselves into seven committees to meet later in the day to discuss specific topics of interest to the company, namely, advertising, securing reviewers, contacting new potential authors, finances, used copies and new editions, competing textbooks, and textbook representatives. This leads us to our first example.

Example 1.1 The ten editors have decided on the seven committees: $c_1 = \{1, 2, 3\}$, $c_2 = \{1, 3, 4, 5\}$, $c_3 = \{2, 5, 6, 7\}$, $c_4 = \{4, 7, 8, 9\}$, $c_5 = \{2, 6, 7\}$, $c_6 = \{8, 9, 10\}$, $c_7 = \{1, 3, 9, 10\}$. They have set aside three time periods for the seven committees to meet on those Fridays when all ten editors are present. Some pairs of committees cannot meet during the same period because one or two of the editors are on both committees. This situation can be modeled visually as shown in Figure 1.1.

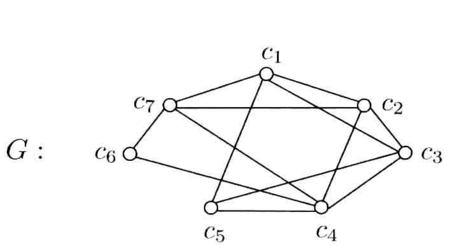

Figure 1.1: A graph

In this figure, there are seven small circles, representing the seven committees and a straight line segment is drawn between two circles if the committees they represent have at least one committee member in common. In other words, a straight line segment between two small circles (committees) tells us that these two committees should not be scheduled to meet at the same time. This gives us a picture or a "model" of the committees and the overlapping nature of their membership. ♢

What we have drawn in Figure 1.1 is called a graph. Formally, a **graph** G consists of a finite nonempty set V of objects called **vertices** (the singular is **vertex**) and a set E of 2-element subsets of V called **edges**. The sets V and E are the **vertex set** and **edge set** of G, respectively. So a graph G is a pair (actually an *ordered* pair) of two sets V and E. For this reason, some write $G = (V, E)$. At times, it is useful to write $V(G)$ and $E(G)$ rather than V and E to emphasize that these are the vertex and edge sets of a particular graph G. Although G is the common symbol to use for a graph, we also use F and H as well as G', G'' and G_1, G_2, etc. Vertices are sometimes called **points** or **nodes** and edges are sometimes called **lines**. Indeed, there are some who use the term **simple graph** for what we call a graph. Two graphs G and H are **equal** if $V(G) = V(H)$ and $E(G) = E(H)$, in which case we write $G = H$.

It is common to represent a graph by a diagram in the plane (as we did in Figure 1.1) where the vertices are represented by points (actually small circles – open or solid) and whose edges are indicated by the presence of a line segment or curve between the two points in the plane corresponding to the appropriate vertices. The diagram itself is then referred to as a graph. For the graph G of Figure 1.1 then, the vertex set of G is $V(G) = \{c_1, c_2, \cdots, c_7\}$ and the edge set of G is

$$\begin{aligned} E(G) \quad = \quad & \{\{c_1, c_2\}, \{c_1, c_3\}, \{c_1, c_5\}, \{c_1, c_7\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_2, c_7\}, \\ & \{c_3, c_4\}, \{c_3, c_5\}, \{c_4, c_5\}, \{c_4, c_6\}, \{c_4, c_7\}, \{c_6, c_7\}\}. \end{aligned}$$

Let's consider another situation. Have you ever encountered this sequence of integers before?

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Every integer in the sequence is the sum of the two integers immediately preceding it (except for the first two integers of course). These numbers are well-known in mathematics and are called the **Fibonacci numbers**. In fact, these integers occur so often that there is a journal (*The Fibonacci Quarterly*, frequently published *five* times a year!) devoted to the study of their properties. Our second example concerns these numbers.

Example 1.2 Consider the set $S = \{2, 3, 5, 8, 13, 21\}$ of six specific Fibonacci numbers. There are some pairs of distinct integers belonging to S whose sum or difference (in absolute value) also belongs to S, namely, $\{2, 3\}$, $\{2, 5\}$, $\{3, 5\}$, $\{3, 8\}$, $\{5, 8\}$, $\{5, 13\}$, $\{8, 13\}$, $\{8, 21\}$, and $\{13, 21\}$. There is a more visual way of identifying these pairs, namely, by the graph H of Figure 1.2. In this case, $V(H) = \{2, 3, 5, 8, 13, 21\}$ and

$$E(H) = \{\{2,3\}, \{2,5\}, \{3,5\}, \{3,8\}, \{5,8\}, \{5,13\}, \{8,13\}, \{8,21\}, \{13,21\}\}. \diamondsuit$$

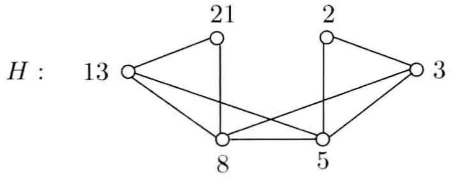

Figure 1.2: Another graph

When dealing with graphs, it is customary and simpler to write a 2-element set $\{u, v\}$ as uv (or vu). If uv is an edge of G, then u and v are said to be **adjacent** in G. The number of vertices in G is often called the **order** of G, while the number of edges is its **size**. Since the vertex set of every graph is nonempty, the order of every graph is at least 1. A graph with exactly one vertex is called a **trivial graph**, implying that the order of a **nontrivial graph** is at least 2. The graph G of Figure 1.1 has order 7 and size 13, while the graph H of Figure 1.2 has order 6 and size 9. We often use n and m for the order and size, respectively, of a graph. So, for the graph G of Figure 1.1, $n = 7$ and $m = 13$; while for the graph H of Figure 1.2, $n = 6$ and $m = 9$.

A graph G with $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{uv, uw, vw, vx, wx, xy\}$ is shown in Figure 1.3(a). There are occasions when we are interested in the structure of a graph and not in what the vertices are called. In this case, a graph is drawn without labeling its vertices. For this reason, the graph G of Figure 1.3(a) is a **labeled graph** and Figure 1.3(b) represents an **unlabeled graph**.

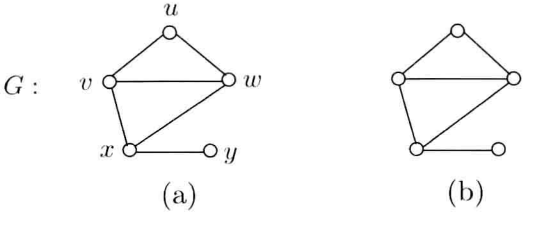

Figure 1.3: A labeled graph and an unlabeled graph

Let us now turn to yet another situation.

Example 1.3 Suppose that we have two coins, one silver and one gold, placed on two of the four squares of a 2×2 checkerboard. There are twelve such configurations, shown in Figure 1.4, where the shaded coin is the gold coin.

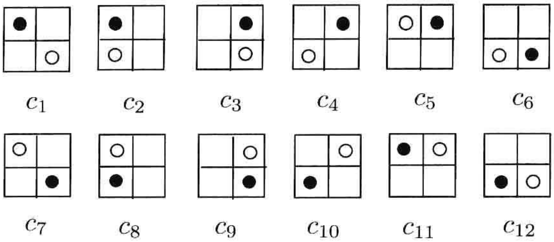

Figure 1.4: Twelve configurations

A configuration can be transformed into other configurations according to certain rules. Specifically, we say that the configuration c_i can be transformed into a configuration c_j $(1 \leq i, j \leq 12, i \neq j)$ if c_j can be obtained from c_i by performing exactly one of the following two steps:

(1) moving one of the coins in c_i horizontally or vertically to an unoccupied square;

(2) interchanging the two coins in c_i.

Necessarily, if c_i can be transformed into c_j, then c_j can be transformed into c_i. For example, c_2 can be transformed into (i) c_1 by shifting the silver coin in c_2 to the right, (ii) c_4 by shifting the gold coin to the right, or (iii) c_8 by interchanging the two coins (see Figure 1.5).

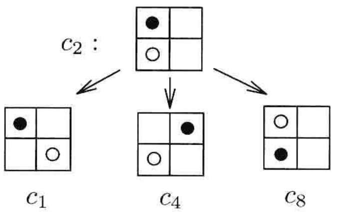

Figure 1.5: Transformations of the configuration c_2

Now consider the twelve configurations shown in Figure 1.4. Some pairs c_i, c_j of these configurations, where $1 \leq i, j \leq 12$, $i \neq j$, can be transformed into each other, and some pairs cannot. This situation can also be represented by a graph, say by a graph F where $V(F) = \{c_1, c_2, \ldots, c_{12}\}$ and $c_i c_j$ is an edge of F if c_i and c_j can be transformed into each other. This graph F is shown in Figure 1.6. ◊

Let's look at a somewhat related example.

Example 1.4. Suppose that we have a collection of 3-letter English words, say

ACT, AIM, ARC, ARM, ART, CAR, CAT, OAR, OAT, RAT, TAR.

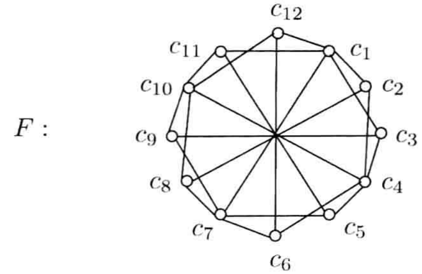

Figure 1.6: Modeling transformations of twelve configurations

We say that a word W_1 can be transformed into a word W_2 if W_2 can be obtained from W_1 by performing exactly one of the following two steps:

(1) interchanging two letters of W_1;

(2) replacing a letter in W_1 by another letter.

Therefore, if W_1 can be transformed into W_2, then W_2 can be transformed into W_1. This situation can be modeled by a graph G, where the given words are the vertices of G and two vertices are adjacent in G if the corresponding words can be transformed into each other. This graph is called **the word graph of the set of words**. For the 11 words above, its word graph G is shown in Figure 1.7.

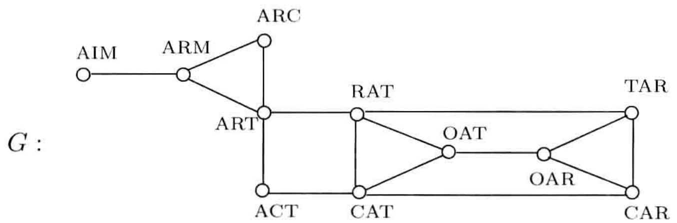

Figure 1.7: The word graph of a set of 11 words

In this case, a graph G is called **a word graph** if G is the word graph of some set S of 3-letter words. For example, the (unlabeled) graph G of Figure 1.8(a) is a word graph because it is the word graph of the set $S =$ {BAT, BIT, BUT, BAD, BAR, CAT, HAT}, as shown in Figure 1.8(b). (This idea is related to the concept of "isomorphic graphs", which will be discussed in detail in Chapter 3.) ◇

We conclude this section with one last example.

Example 1.5 Figure 1.9 shows the traffic lanes at the intersection of two busy streets. When a vehicle approaches this intersection, it could be in one of the nine lanes: L1, L2, ..., L9.

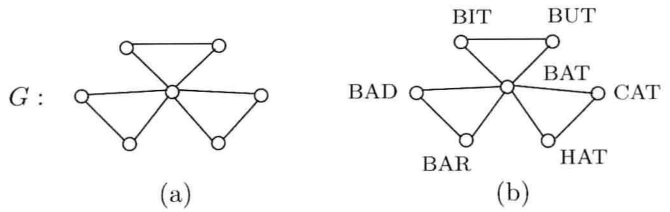

Figure 1.8: A word graph

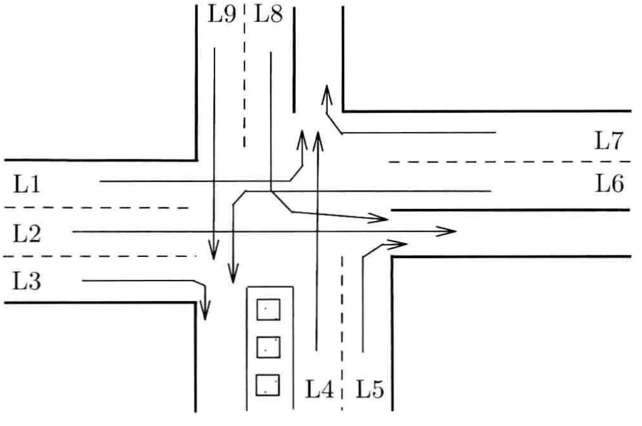

Figure 1.9: Traffic lanes at street intersections

This intersection has a traffic light that informs drivers in vehicles in the various lanes when they are permitted to proceed through the intersection. To be sure, there are pairs of lanes containing vehicles that should not enter the intersection at the same time, such as L1 and L7. However, there would be no difficulty for vehicles in L1 and L5 to drive through this intersection at the same time. This situation can be represented by the graph G of Figure 1.10, where $V(G) = \{\text{L1, L2, \ldots, L9}\}$ and two vertices (lanes) are joined by an edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there would be a possibility of an accident. ◇

What we have just seen is how five different situations can be represented by graphs. Actually, in each case, there is a set involved: (1) a set of committees; (2) a set of integers, (3) a set of configurations consisting of two coins on a 2×2 checkerboard, (4) a set of 3-letter words, and (5) a set of traffic lanes at a street intersection. Certain pairs of elements in each set are related in some manner: (1) two committees have a member in common; (2) the sum or difference (in absolute value) of two integers in the set also belongs to the set; (3) two configurations can be transformed into each other according to some rule; (4) two 3-letter words can be transformed into each other by certain movements of letters; and (5) cars

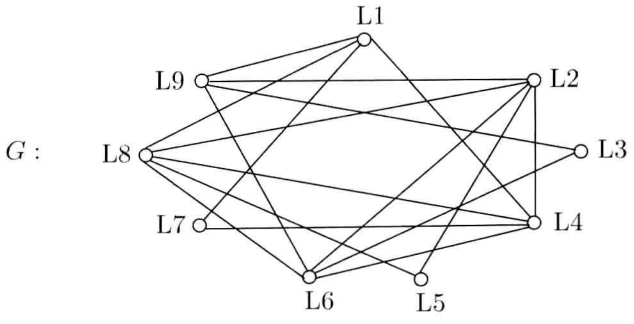

Figure 1.10: The graph G in Example 1.5

in certain pairs of traffic lanes cannot enter the intersection at the same time. In each case, a graph G is defined whose vertices are the elements of the set and two vertices of G are adjacent if they are related as described above. The graph G then *models* the given situation. Often questions concerning the situations described above arise and can be analyzed by studying the graphs that model them.

Exercises for Section 1.1

1.1 What is a logical question to ask in Example 1.1? Answer this question.

1.2 Create an example of your own similar to Example 1.1 with nine editors and eight committees and then draw the corresponding graph.

1.3 Let $S = \{2, 3, 4, 7, 11, 13\}$. Draw the graph G whose vertex set is S and such that $ij \in E(G)$ for $i, j \in S$ if $i + j \in S$ or $|i - j| \in S$.

1.4 Let $S = \{-6, -3, 0, 3, 6\}$. Draw the graph G whose vertex set is S and such that $ij \in E(G)$ for $i, j \in S$ if $i + j \in S$ or $|i - j| \in S$.

1.5 Create your own set S of integers and draw the graph G whose vertex set is S and such that $ij \in E(G)$ if i and j are related by some rule imposed on i and j.

1.6 Consider the twelve configurations $c_1, c_2, \ldots, c_{12}$ in Figure 1.4. For every two configurations c_i and c_j, where $1 \leq i, j \leq 12$, $i \neq j$, it may be possible to obtain c_j from c_i by first shifting one of the coins in c_i horizontally or vertically *and* then interchanging the two coins. Model this by a graph F such that $V(F) = \{c_1, c_2, \ldots, c_{12}\}$ and c_ic_j is an edge of F if c_i and c_j can be transformed into each other by this 2-step process.

1.7 Following Example 1.4,

(a) give an example of ten 3-letter words, none of which are mentioned in Example 1.4, and whose corresponding word graph has at least six edges. Draw this graph.

(b) give a set of five 3-letter words whose word graph is shown in Figure 1.11 (with the vertices appropriately labeled).

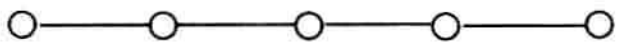

Figure 1.11: The graph in Exercise 1.7(b)

(c) give a set of five 3-letter words whose word graph is shown in Figure 1.12 (with the vertices appropriately labeled).

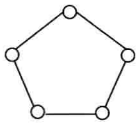

Figure 1.12: The graph in Exercise 1.7(c)

1.8 Let S be a finite set of 3-letter and/or 4-letter words. In this case, the word graph $G(S)$ of S is that graph whose vertex set is S and such that two vertices (words) w_1 and w_2 are adjacent if either (1) or (2) below occurs:

(1) one of the words can be obtained from the other by replacing one letter by another letter,

(2) w_1 is a 3-letter word and w_2 is a 4-letter word, and w_2 can be obtained from w_1 by the insertion of a single letter (anywhere, including the beginning or the end) into w_1.

(a) Find six sets $S_1, S_2, \ldots, S_6$ of 3-letter and/or 4-letter words so that for each integer i ($1 \le i \le 6$) the graph G_i of Figure 1.13 is the word graph of S_i.

(b) For another graph H (of your choice), determine whether H is a word graph of some set.

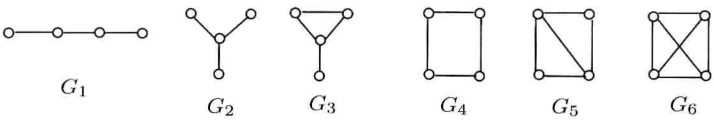

Figure 1.13: The graphs for Exercise 1.8(a)