

*Unified
Calculus and Analytic
Geometry*

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Geometry

BY EARL D. RAINVILLE

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*Unified Calculus
and Analytic Geometry*

By Earl D. Rainville

ELEMENTARY DIFFERENTIAL EQUATIONS, *Second Edition*

A SHORT COURSE IN DIFFERENTIAL EQUATIONS, *Second Edition*

SPECIAL FUNCTIONS

UNIFIED CALCULUS AND ANALYTIC GEOMETRY

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DIFFERENTIAL AND INTEGRAL CALCULUS, *Fifth Edition* (with
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INTERMEDIATE DIFFERENTIAL EQUATIONS (*published by John
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Preface

This book is intended as a text for courses which furnish the backbone of the student's collegiate mathematics. Before going into the course, the student should know the basic principles and formulas of elementary trigonometry and advanced high school, or beginning college, algebra. At the end of the course, he should be prepared for advanced calculus, or advanced mathematics for engineers, for more thorough courses in differential equations, and even for the milder versions of first courses in the Laplace transform, Fourier series, complex variables, matrices, theory of equations, theory of numbers, linear vector spaces, etc.

Throughout the book I have attempted to effect a balance between the desire for rigor and the cold fact of the probable state of mathematical maturity of the student. If we proceed to teach without rigor, what we teach is not mathematics. If, at this stage, we teach entirely rigorously, we gain much stature for the course and lose—the student.

I have tried to state definitions and theorems with care, to prove some things rigorously, to present others with discussions aimed only at making the facts seem plausible. By means of what I hope are illuminating discussions in the text, I attempt to increase both the student's maturity and his knowledge of the subject. There are in this book not only a large number of illustrative examples but also well over five thousand exercises for the student. The exercises have been constructed carefully. They are supposed not only to develop gradually a considerable manipulative skill but also in numerous instances to add appreciably to the student's basic knowledge. The instructor will notice many exercises clearly intended to prepare the student for specific problems which he may encounter in more advanced courses.

I have included a few topics which are not often discussed at this stage in the student's training. For example, the error function is studied in § 349 to show the student how easily some so-called nonelementary functions may be handled.

When calculus and analytic geometry are combined, there is a strong tendency to neglect the geometry so much that at times the student is at a loss as to what he can use his calculus upon. I feel that there is, for instance, little point to teaching a student how to obtain the area bounded by two plane curves before he can sketch the curves without time-consuming point plotting. The instructor will find in this book a thorough treatment of the

analytic geometry needed. Solid analytic geometry, in particular, receives much more attention than is usually accorded it in combined courses.

A shorter course may be taught from this book by omitting all or any of the following: §§ 121–123, 156–161, 171–172, Chapters 21–22; §§ 253–254, Chapter 31; §§ 271–272, 278, 290, Chapter 36; §§ 327, 338–339, 348–349, Chapter 42; and all or part of the five chapters on differential equations. I strongly advise against omitting Newton's method, Chapter 21, but it, like the other sections listed above, may be omitted without interrupting the continuity of the course. Sections explicitly concerned with applications may also be omitted safely, if such omission seems desirable.

By explicit agreement with the interested parties, I have been allowed to incorporate in this book portions of three other Macmillan books, my *Elementary Differential Equations* (2nd edition, 1958), the Love and Rainville *Differential and Integral Calculus* (5th edition, 1954), and the Love and Rainville *Analytic Geometry* (5th edition, 1955). I have particularly taken from those books large numbers of exercises.

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EARL D. RAINVILLE

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