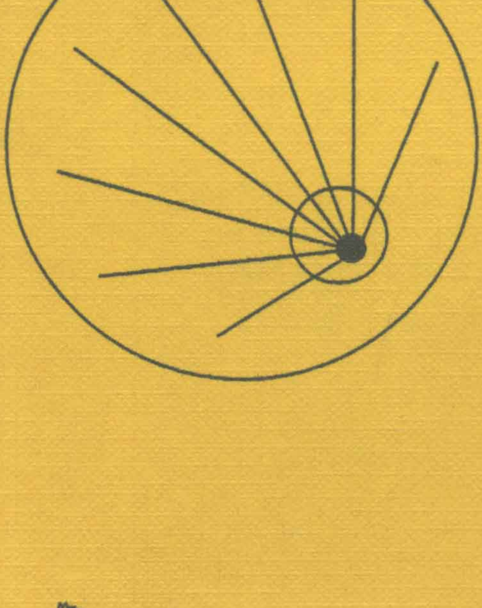
Lecture Notes in Mathematics 1490

Klaus Metsch

Linear Spaces with Few Lines



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FÜR GERTRUD

Franz kam sein Leben zwischen den Büchern unwirklich vor. Er sehnte sich nach dem wirklichen Leben, nach Berührung mit anderen Menschen, die an seiner Seite gingen, er sehnte sich nach ihrem Geschrei. Es war ihm nicht klar, daß gerade das, was ihm unwirklich schien (die Arbeit in der Einsamkeit von Studierzimmer und Bibliothek), sein wirkliches Leben war, während die Umzüge, die für ihn die Wirklichkeit darstellten, nur Theater waren, ein Tanz, ein Fest, mit anderen Worten ein Traum.

(Milan Kundera: Die unerträgliche Leichtigkeit des Seins)

INTRODUCTION

It is known since 40 years that a linear space has at least as many lines as points with equality only if it is a generalized projective plane. This result of de Bruijn and Erdös (1948) led to the conjecture that every linear space with "few lines" can be obtained from a certain projective plane by changing only a small part of its structure. It is surprising that it took more than 20 years until Bridges (1972) showed that every linear space with $b = v+1 \neq 6$ (b the number of lines, v the number of points) is a punctured projective plane. However, since then many results have been obtained. It is the main purpose of this paper to study systematically this embedding problem. In particular, we shall collect the old results and present quite a few new ones. We shall, however, also study linear spaces with few lines which have no natural embedding in a projective plane.

When studying finite linear spaces which have a chance to be embeddable in a projective plane of order n, it is sensible to suppose that $b \leq n^2+n+1$ (which is the number of lines in a projective plane of order n) and $v \geq (n-1)^2+(n-1)+2 = n^2-n+2$ (this is due to the fact that a projective plane of order n−1 has n^2-n+1 points and such a plane is certainly not embeddable in a projective plane of order n).

In the first chapter, we shall give the definitions and most of the notions needed later. Also the most important examples of linear spaces which we shall deal with are given. At the end of this chapter, the reader will find some basic properties of linear spaces and (n+1,1)-designs.

Chapter 2 begins with a new proof of the theorem of de Bruijn and Erdös. This proof uses an easy algebraic method, which will be also useful in the proof of Totten's theorem in chapter 8. The inequality $b \geq v$ has been improved by Erdös, Mullin, Sós, and Stinson (1985) for non-degenerate linear spaces as follows. If n is the unique integer satisfying $n^2-n+1 = (n-1)^2+(n-1)+1 < v \leq n^2+n+1$, then we have

$$b \geq B(v) := \begin{cases} n^2+n-1, & \text{if } v = n^2-n+2 \neq 4 \\ n^2+n, & \text{if } n^2-n+3 \leq v \leq n^2+1 \text{ or } v = 4 \\ n^2+n+1, & \text{if } n^2+2 \leq v. \end{cases}$$

We conclude chapter 2 with a new proof of this result.

In chapter 3, basic properties and results of $(n+1,1)$-designs can be found. We shall prove a theorem of Vanstone which says that any $(n+1,1)$-design with $4 \leq n^2 \leq v$ and $b = n^2+n+1$ can be embedded in a projective plane of order n, and we shall also show that $n^2-n+2 \leq v \leq b \leq n^2+n$ implies embeddability.

Now we are ready to study systematically non-degenerate linear spaces L with $v \geq n^2-n+2$ points and $b \leq n^2+n+1$ lines. In chapter 4 we consider the case that some point lies on at most n lines. We shall determine L except in the case where $v = n^2-n+2$ and $b = n^2+n+1$. It turns out that L is one of a few exceptional linear spaces or that L can be extended to a projective plane of order n. This result implies that every point of a non-degenerate linear space with n^2+n+1 lines and at least n^2-n+3 points lies on at least $n+1$ lines, a consequence which will be very helpful in the next chapters.

The results obtained up to here will now be used to determine all linear spaces with the minimal possible number B(v) of lines. If n is the integer with $n^2-n+2 \leq v \leq n^2+n+1$, then every non-degenerate linear L space with v points and B(v) lines can be embedded in a projective plane of order n unless L is one exceptional linear space with 8 points. This will be shown in chapter 5. We shall also study linear spaces with n^2-n+2 points and n^2+n lines and obtain a classification of all linear spaces with $n^2-n+2 \leq v < b \leq n^2+n$. Only two such linear spaces can not be embedded in a projective plane of order n and both have 8 points.

In chapter 5 we will also determine all linear spaces with n^2-n+1 points and n^2+n lines for which every point has degree at least $n+1$ and some point has degree at least $n+2$, $n \neq 4,9$. It turns out that these spaces are related to a complement of a Baer-subplane in an affine plane.

In the following two chapters we consider linear spaces with n^2+n+1 lines and $v \geq n^2-n+2$ points in which every point lies on at least $n+1$ lines.

We start with the very difficult case that every point lies on exactly n+1 lines. In chapter 3 we have already proved that $v \geq n^2$ implies embeddability. In the last 15 years this bound has been improved many times, for example by Bose and Shrikhande (1973), McCarthy and Vanstone (1977), and Dow (1982, 1983). We shall improve all these bounds again by showing that as well $v \geq n^2 - ½n+6$ as $v \geq n^2 - ½(\sqrt{5} - 1)n + 17\sqrt{(n/5)}$ implies embeddability.

In chapter 7, we assume that some point lies on more than n+1 lines. Because such linear spaces can not be embedded in a projective plane of order n, it has been conjectured that they do not exist. However, we shall construct an infinite class of counterexamples. Such a counterexample has at most $n^2+1-\sqrt{n}$ points with equality if it is the closed complement of a Baer-subplane in a projective plane of order n. In chapter 7 we shall show that these are essentially the only examples for $v > n^2+1-½n$. Since they can not be embedded in a projective plane of order n, we obtain the optimal bound for the embedding of linear spaces in projective planes of order n whenever n is the order of a projective plane having a Baer-subplane.

Chapter 8 starts with a proof of Totten's classification of restricted linear spaces (1976) (linear spaces satisfying $(b-v)^2 \leq v$). We shall improve this result slightly by determining all linear spaces satisfying $(b-v)^2 \leq b$. As corollaries we obtain the classifications of linear spaces with $b = v+1$ of Bridges (1972), $b = v+2$ of de Witte (1976), and $b = v+3$ of Totten (1976). A consequence of the Theorem of Totten is that every linear space satisfying $v \leq n^2+n+1 < b \leq v+n$ is an inflated affine plane of order n, that is an affine plane **A** of order n together with a linear space with at most n+1 points which is imposed on some of the infinite points of **A**. It seems likely that this results remains true if one weakens the upper bound $b \leq v+n$ for the number of lines. In the case that the number of points is n^2+n+1, Blokhuis, Schmidt, and Wilbrink (1988) showed that the condition $b \leq n^2+3n+1-4\sqrt{n}$ is strong enough. We shall sligthly improve this result in chapter 9.

In chapter 10, we introduce a class of structures which we call L(n,d). By definition an L(n,d), $1 \leq d \leq n-1$ is an (n+1,1)-design with n^2-d points in which every point lies on n lines of degree n and on a unique line of degree n-d. We shall see that this implies that there is an integer z such that $z(n-d) = d(d-1)$ and $b = n^2+n+z$. It is easy to see that an $L(n,d)$ is a punctured affine plane if $z = 0$ and the complement of a Baer-subplane in a projective plane if $z = 1$. For $z \geq 2$ no examples are known and we give non-existence criteria. However even for $z \geq 2$ these (hypothetical) linear spaces are interesting from many points of view. For example, if $z \geq 2$, then we obtain the following characterization of an $L(n,d)$ with $z(n-d) = d(d-1)$: Suppose **L** is a linear space with maximal point degree $n+1$ and $b = n^2+n+z$. Then $v \leq n^2-d$ and equality implies that **L** is an $L(n,d)$.

Each $L(n,d)$ gives rise to a *closed* $L(n,d)$, which is a linear space with n^2+n+z lines and n^2-d+1 points, and to a *reduced* $L(n,d)$, which is a linear space with n^2-n+2 points, $b = n^2+n+z-1$ lines, and a point of degree n. For $z = 2$, these structures will play a crucial role in the next two chapters. Suppose that n is a positive integer and let d be the positive number defined by $2n = d(d-1)$. In chapter 11, we prove that every linear space with n^2+n+2 lines has at most n^2-d+1 points with equality if and only if it is a closed $L(n,d)$, which implies that d is an integer. In the next chapter we shall show that every linear space with n^2-n+2 points and n^2+n+1 lines which has a point of degree n is a reduced $L(n,d)$ (which implies again that d is an integer) or one of a few exceptional linear spaces. This result solves the unsettled case of chapter 4.

In chapter 13, we consider two particular cases. First, we shall prove that every linear space with $13 < v \leq b \leq 21$ is a near-pencil or can be embedded in the projective plane of order 5. Then we show that every linear space satisfying $31 < v \leq b \leq 43$ is a near-pencil. This includes a new proof for the non-existence of a projective plane of order 6, and the pseudo-complement of an oval in a projective plane of order 6. Our proof does not use any graph theoretical results as did the original proof of de Witte (1977).

An interesting property of linear spaces was discovered by Hanani (1954/55): Every line of maximal degree of a linear space with v points meets at least $v-1$ other lines. In chapter 15 we prove the following generalisation. Suppose that c is a non-negative integer. Then all but a finite number of linear spaces in which every line of maximal degree meets at most $v-1+c$ other lines can be obtained from a projective plane by removing at most c points or from an affine pane by removing at most c−1 points. The proof applies a very general embedding result, which we give in chapter 14.

Another interesting property of linear spaces was conjectured by Dowling and Wilson (see Erdös, Fowler, Sós and Wilson (1985)): If (p,L) is a non-incident point-line pair of a linear space and if t is the number of lines through p which miss L, then $b \geq v+t$. This generalization of the de Bruijn-Erdös Theorem was proved in (Metsch, 1991c) and we shall present this proof in chapter 16.

In the last chapter we will be concerned with the uniqueness of embeddings. It is conceivable that a linear space may be embedded in two different ways in the same projective plane, or even that it can be embedded in two non-isomorphic planes. We shall, moreover, show that all embeddings considered in this paper are uniquely determined.

The first part of this paper was written while I was visiting the university of Florence, Italy. I would like to thank Prof. A. Barlotti very much for his kind hospitality.

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C O N T E N T S

1. Definition and basic properties of linear spaces

In this section, we shall give the definitions of the geometrical structures used in this paper. We shall also give examples and basic properties of linear spaces. The most general structure we shall use is the incidence structure.

An *incidence structure* is a triple $I = (p, L, I)$ of a set p of *points*, a set L of *lines*, and a set I of *incidences* satisfying

$$p \cap L = \emptyset \qquad \text{and} \qquad I \subseteq p \times L.$$

If $(p,L) \in I$, then we say that the point p lies on the line L or that L passes through p. If $(p,L) \notin I$, then we say p does not lie on L or p is not a point of L. Further geometric expressions explaining themselves are used.

We only consider incidence structures with a finite number of points and lines. The number of points is denoted by v and the number of lines by b. The number k_L of points on a line L is called the *degree* of L and the number r_p of lines passing through a point p is called the *degree* of p. A *k-line* is a line of degree k. The *parameters* of I are the number of points and lines, the degrees $r_1,\dots,r_v$ of the points, and the degrees $k_1,\dots,k_b$ of the lines.

Two lines L and H are called *parallel* if $L \neq H$ and if L and H do not intersect. A *parallel class* is a set π of lines such that every point lies on exactly one line of π.

The incidence structure $I_d = (L, p, \{(L,p) \mid (p,L) \in I\})$ is called *dual* to I.

An incidence structure $I' = (p', L', I')$ is said to be *embedded* in I, if $p' \subseteq p$, $L' \subseteq L$, and $I' \subseteq I \cap (p' \times L')$. An *isomorphism* from I onto an incidence structure $I'' = (p'', L'', I'')$ is a bijection α from $p \cup L$ onto $p'' \cup L''$ mapping points onto points and lines onto lines such that $(p,L) \in I$ if and only if $(\alpha(p), \alpha(L)) \in I''$.

If M is a set of mutually parallel lines of $I = (p, L, I)$ and if ∞ is a new symbol, then $I \infty M$ denotes the incidence structure $(p \cup \{\infty\}, L, I \cup \{(\infty, L) \mid L \in M\})$. We say that $I \infty M$ is the structure obtained from I, if we let the lines of M intersect in an infinite point ∞.

A v x b matrix $C = (c_{ij})$ is called an *incidence matrix* of I if there are orders $p_1,\dots,p_v$ and $L_1,\dots,L_b$ of the points and lines such that $c_{ij} = 1$ if p_i is a point of L_j and $c_{ij} = 0$ if not.

An incidence structure is called a *partial linear space* if any two distinct points lie on at most one common line, every line has at least two points, and there are at least two lines. A *linear space* is a partial linear space in which for any two distinct points p and q there is a line pq through p and q. A linear space which has a line passing through all but one of its points is called a *near-pencil*, a *degenerate* linear space, or a *degenerate projective plane*. A linear space which is not a near-pencil is called *non-degenerate*. A linear space with $(b-v)^2 \leq v$ is called *restricted*, and a *weakly restricted* linear space is a linear space satisfying $(b-v)^2 \leq b$.

Since a line of a partial linear space $\mathbf{L} = (p, L, I)$ is uniquely determined by its points, we shall identify a line with the set of its points. We write $p \in L$ instead of $(p,L) \in I$, and $\mathbf{L} = (p, L)$ instead of $\mathbf{L} = (p, L, I)$ where L is now seen as a set of subsets of p. The point of intersection of two intersecting lines L and H is denoted by $L \cap H$.

Let p' be a subset of p containing three non-collinear points. Then we can define the linear space $\mathbf{L}' = (p', \{L \cap p' \mid L \in L, |L \cap p'| \geq 2\})$ which is *induced by* $\mathbf{L}$ *on* p'. If $C := p - p'$, then $\mathbf{L}'$ is called *the complement of* C *in* $\mathbf{L}$ and it is denoted by $\mathbf{L}-C$. Obviously, $\mathbf{L}-C$ is embedded in $\mathbf{L}$. If L is a line of $\mathbf{L}$ with $|L \cap p'| \geq 2$, then $L' := L \cap p'$ is a line of $\mathbf{L}'$. In general L and L' do not coincide as sets of points. However, if no confusion is expected, then we identify L and L' as lines. E.g. if $p \in L - L'$, then we call p a point of L' outside of $\mathbf{L}'$.

(Partial) linear spaces with constant point degree n+1 occur very often. It is sometimes more comfortable to consider a little more general structure, the (partial) (n+1,1)-designs. An incidence structure I with constant point degree n+1 is called a *partial (n+1,1)-design*, if any two distinct points are contained in at most one line, and an *(n+1,1)-design*, if any two distinct points p and q lie on a unique line pq. Notice that a line is not uniquely determined by the set of its points. There may be two or more lines of degree 1 which contain the same point p, and also lines of degree 0 are allowed. Even though we regard the set of lines as a family of subsets of the set of points and we write $p \in L$ instead of $(p,L) \in I$.

A *projective plane* is a linear space P for which there is an integer $n \geq 2$ such that every point and line has degree n+1. The integer n is called the *order* of P. It is easy to see that P has n^2+n+1 points and lines and that any two lines of P intersect. A *generalized projective plane* is a near-pencil or a projective plane. An *affine plane* of *order* n is the complement of a line L in a projective plane of order n. The n+1 points of L are called the *infinite points* of the affine plane. It is known that a projective plane of order n exists whenever n is the power of a prime.

Suppose that $n = m^2$ is a perfect square and that P is a projective plane of order n with a *Baer-subplane* B = (q,Π), i.e. B is a projective plane of order m which is embedded in P. Then L := P$-q$ is also called the *complement of the Baer-subplane* B *in the projective plane* P. The lines of Π, considered as lines of L, form a parallel class of L. L ∞ Π is called the *closed complement of* B *in* P. It has n^2-m+1 points and n^2+n+1 lines. Furthermore, the point ∞ has degree $|\Pi| = n+\sqrt{n}+1$.

Now suppose that $n = m^2$ is a perfect square and that A is an affine plane of order n with a *Baer-subplane* B' = (q',Π'), i.e. B' is an affine plane of order m which is embedded in A. Then L' := A$-q'$ is also called the *complement of the Baer-subplane* B' *in the affine plane* A. Suppose that the lines of Π', considered as lines of L', form a parallel class of L'. Then L' ∞ Π is called the *closed complement of* B' *in* A. It has n^2-n points and n^2+n lines. Furthermore, the point ∞ has degree $|\Pi'| = n+\sqrt{n}$.

If C is the set consisting of the 2n+1 points of two lines of a projective plane P, then P$-$C is called the *complement of two lines in* P.

Denote by $\mathcal{C}$ a class of linear spaces and call its elements $\mathcal{C}$-spaces. Suppose to every $\mathcal{C}$-space L is assigned an order n such that the parameters of L depend only on n. Then every linear space for which there is an integer n such that its parameters can be expressed in the same form in terms of n is called a *pseudo-*$\mathcal{C}$*-space* (*of order n*).

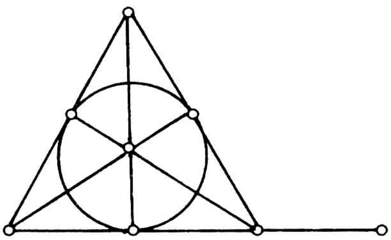
For example, a pseudo-complement of two lines in a projective plane of order n is a linear space with n^2-n points of degree n+1, n$-$1 lines of degree n, and n^2 lines of degree n$-$1. Notice that the definition does not imply that n is the order of a projective plane.

An *inflated affine plane* **D** consists of an affine plane **A** together with a linear space **L** imposed on some of its infinite points. That is, there is a projective plane **P** = (p,L), a line L of **P**, a partition $q_1 \cup q_2$ of the points of L, and a linear space **L** = (q_1, L') such that **D** = $(p-q_2,\ L' \cup \{X \cap (p-q_2) \mid X \in L-\{L\}\})$ and **A** = **P**−L. We also say that **D** is an affine plane of order n with **L** *at infinity*. If **L** is a near-pencil or a projective plane, then **D** is called a *projectively* inflated affine plane, and if **D** consists of all the infinite points of **A**, then it is called a *complete* inflated affine plane. The structure obtained by removing one of the finite points of an inflated affine plane is called an *inflated punctured affine plane*.

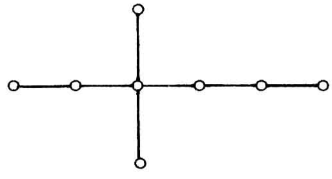
There is much more terminology for linear spaces. We do not want to give all definitions, because most explain themselves. For example, a *punctured projective plane* is a linear space which is obtained from a projective plane by removing one of its points, and an *affine plane with an infinite point* is a linear space **A** ∞ Π where **A** is an affine plane and Π is one of its parallel classes.

Now we know most of the linear spaces which occur in this work. There are a few 'exceptional' spaces, which will be denoted by $\mathbf{E_1}, \ldots, \mathbf{E_9}$. Instead of giving the set of points and lines, we define them with the help of a picture. Points are represented by little circles and a line is represented by a (not necessarily straight) line which joins its points. However, we only give a picture of the partial linear space $\mathbf{E'_j}$ obtained by removing the lines of degree 2 from $\mathbf{E_j}$, since $\mathbf{E_j}$ is uniquely determined by $\mathbf{E'_j}$.

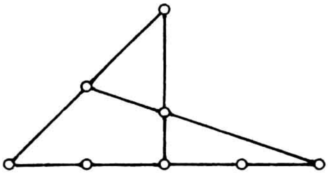
$\mathbf{E_1}$: v = 8, b = 11.



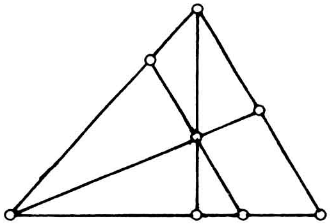
$\mathbf{E_2}$: v = 8, b = 12.



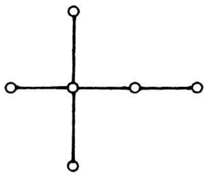
E3 : v = 8, b = 13.



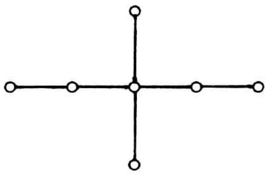
E4 : v = 8, b = 13.



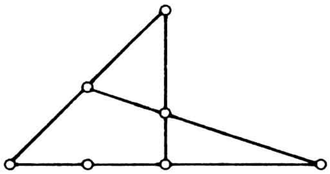
E5 : v = 6, b = 8.



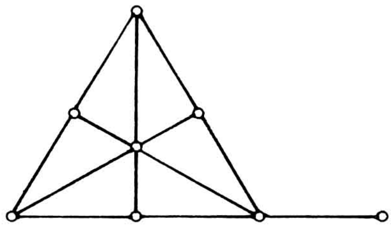
E6 : v = 7, b = 10.



E7 : v = 7, b = 10.



E8 : v = 8, b = 13.



The picture of our last exceptional space **E9** would already be too complicated. It can be defined as follows. Consider the projective plane $P = (p, L)$ of order 4, and let L_1, L_2, L_3 and L_4 be four lines of it which form a quadrilateral. Denote by q the set of the seven points $\neq L_1 \cap L_4, L_2 \cap L_4$ of $L_1 \cup L_2$, and by p_1, p_2, and p_3 the three points $\neq L_1 \cap L_3, L_2 \cap L_3$ of the line L_3. Then we denote the linear space $(p - q, (L - \{L_1, L_2, L_3\}) \cup \{\{p_1, p_2\}, \{p_1, p_3\}, \{p_2, p_3\}\})$ by **E9**. It has 14 points and 21 lines and a unique line of degree 5, which is the line L_4.

Some linear spaces have special names. The near-pencil on three points is called the *triangle*. *Tetrahedron* is another name for the affine plane of order 2, and the *Fano plane* is the projective plane of order 2. The *Fano quasi-plane* is the tetrahedron with the triangle at infinity, which can be obtained from the projective plane of order 2 by *breaking up* one of its lines into three lines of degree 2. The linear space $\mathbf{E}_5$ is called *Lin's Cross.*

Every incidence structure $\mathbf{I} = (\mathcal{p},\mathcal{L},\mathcal{I})$ satisfies the following *basic equation*

$$(B_1)\qquad \sum_{p\in \mathcal{p}} r_p = \sum_{L\in \mathcal{L}} k_L.$$

This is true, since both sides of the equation equal to $|\mathcal{I}|$. This equation has a lot of important consequences, which will be used throughout this paper.

For every linear space $\mathbf{L} = (\mathcal{p},\mathcal{L})$ we have

$$(B_2)\qquad v(v-1) = \sum_{L\in \mathcal{L}} k_L(k_L-1).$$

This follows from the basic equation for the incidence structure whose set of points is $P = \{(p,q) \mid p \text{ and } q \text{ are distinct points of } L\}$, whose lines are the sets $G_L = \{(p,q) \in P \mid p,q \in L\}$, $L \in \mathcal{L}$, and in which a point (p,q) lies on a line G_L if $p,q \in L$. Equation (B_2) reflects the fact that any two points of a linear space lie on a unique line.

If p is a point of a linear space $(\mathcal{p},\mathcal{L})$, then the equation ($B_1$) used for the partial linear space $(\mathcal{p}-\{p\},\{L \in \mathcal{L} \mid p \in L\})$ shows

$$(B_3)\qquad v-1 = \sum_{p\in L} (k_L-1).$$

This equation reflects the fact that the point p is joined to every other point by a unique line.

Let L be a line of a linear space $\mathbf{L} = (\mathcal{p},\mathcal{L})$ and denote by $\mathcal{M}$ the set of lines which are parallel to L. The basic equation (B_1) used for $(\mathcal{p}-L,\mathcal{M})$ gives

$$(B_4)\qquad \sum_{p\notin L} (r_p-k_L) = \sum_{L\in \mathcal{M}} k_L,$$

since every point p outside of L lies in r_p-k_L lines parallel to L.

Obviously, the equations (B_1), (B_2), (B_3), and (B_4) also hold for $(n+1,1)$-designs.

Now we prove two easy lemmas, which will be used frequently throughout this paper. We call them *Transfer-Lemma* and *Parallel-Lemma.*