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INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

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ICTP, Trieste, Italy   8-19 February 1988

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FUNCTIONAL ANALYTIC METHODS IN COMPLEX ANALYSIS AND APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

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INTRODUCTION

The volume contains the proceedings of the Workshop on Functional Analytic Methods in Complex Analysis and Applications to Partial Differential Equations held in Trieste, Italy, from 8 to 19 February 1988, at the International Centre for Theoretical Physics of the International Atomic Energy Agency (Vienna, Austria) and United Nations Educational, Scientific and Cultural Organization (Paris, France) on the invitation of its Director, Professor Abdus Salam, and its Director of Mathematics, Professor James Eells. The Workshop was directed by Professor Dr. Wolfgang Tutschke (Martin Luther University, Halle, German Democratic Republic) and Dr. Ali Seif Mshimba (University of Dar es Salaam, Tanzania). It had been included in the ICTP 1988 programme on the initiative of Professor Eells. The organization of the Workshop rested with the ICTP. The preparation for the contents of the programme was graced through the support of Professor Dr. G.F. Mandzhavidze (I.N. Vekua Institute for Applied Mathematics, Tbilissi State University, U.S.S.R.).

Those who learned complex function theory from a usual lecture for mathematicians or physicists could get the impression that this theory is closed in essence. Such an impression could arise because many problems of complex function theory are very natural, and, moreover, most of them are solved by standard methods and formulated in natural theorems too. Thus one could get the impression that there are no problems in complex function theory, so that could lead to a new furious development of mathematics.

A deeper look at the continuous growth of problems and knowledge in mathematics shows, however, that complex methods still play a significant role and contribute to new ideas and results today. One important reason for a new revival of complex analysis lies in the fact that the relations between complex analysis and the theory of partial differential equations have been acquiring a new quality. Whereas the applications of complex analysis to partial differential equations were side results in the past, one of the main aims of complex analysis today is its systematic application to general classes of partial differential equations. Such new tendencies were started by S. Bergman's theory of integral operators, I.N. Vekau's theory of generalized analytic functions and L. Bers' theory of pseudo-analytic functions. The latter two theories are applicable to uniformly elliptic linear systems of first order for two unknown real-valued functions in the plane. Enlarging these aims, today the complex analysis is applicable to systems for more than two unknown real-valued functions, to systems in $\mathbb{R}^n$ and to nonlinear differential equations as well.

There are three aspects of this enlargement of the aims of complex analysis: First it is possible to interpret the peculiarities of holomorphic functions as properties of solutions of special systems of partial differential equations. Secondly, complex analysis becomes applicable to general classes of differential equations, not only to special ones. And thirdly, the new general complex analysis is able to construct solutions and to describe the properties of given solutions with the help of solutions of corresponding problems for holomorphic functions. The third aspect is significant, since in the case of nonlinear equations it essentially means their reduction to linear problems. In addition this third aspect shows that the general complex analysis is able to make use of results of classical function theory and of such results which originally have not been connected with partial differential equations.

In the light of the above said, the workshop set to introduce mathematicians and physicists to functional-analytic methods in complex analysis and to show how these can be applied to partial differential equations, to survey current related knowledge and to draw attention to research problems. The core topics, as can be seen from the table of contents, were:

— weak solutions of partial differential equations
— basic integral operators in complex analysis
— solution of boundary value problems for elliptic partial differential equations in the plane by complex methods
— generalized analytic functions
— generalizations to higher dimensions

The success of any project depends on dedication of many. With regards to our workshop we would like to thank all the lecturers for their efforts before and during the course. Our thanks go as well to participants of the workshop as well as to those applicants who were not selected to attend for the interest they have shown. We know that in turn all will join us in expressing our appreciation of the truly exceptional administration and secretarial staff of the Centre.

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CONTENTS

INTRODUCTION : THE NEW METHODS OF COMPLEX ANALYSIS

(Wolfgang Tutschke - Martin Luther University)

0.1. Features of the classical complex function theory

The complex function theory has been developed as a theory of differentiable functions of a complex variable. A function $w = f(z)$ depending of a complex variable z is said to be differentiable in the complex sense at z_o if the limit of the complex difference quotient

$$\frac{f(z) - f(z_o)}{z - z_o}$$

exists and does not depend on the direction in which z tends to z_o. Complex differentiability is more restrictive than the differentiability with respect to a real variable. Therefore holomorphic functions, i.e. functions differentiable in the complex sense, possess many special properties such as the following:

0.1.1. Inside a closed curve γ the values of a holomorphic function $w = w(z)$ can be represented by the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad (1)$$

i.e. the values of f inside γ are uniquely determined by its values on γ.

0.1.2. A holomorphic function is locally representable by a power series in z, i.e.

$$f(z) = \sum_{k=o}^{\infty} a_k (z - z_o)^k \qquad (2)$$

if f is holomorphic in a neighbourhood of z_o.

0.1.3. Generalizing the power series representation (2), a holomorphic function can be represented by a Laurent series

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_o)^k$$

in a neighbourhood of an isolated singular point z_o . This representation leads to an easy classification of singular points.

0.2. Connections of the complex function theory with other fields in mathematics

Special features of holomorphic functions such as those mentioned in 0.1. enable to apply the complex function theory in many branches of mathematics such as the following:

0.2.1. Connections with geometry

A holomorphic function realizes a conformal mapping, i. e. the angle between curves remains unchanged by a mapping defined by a holomorphic function. This fact was the starting point of the geometrical function theory whose first main result was the Riemann mapping theorem (which states that a simply connected domain, with two boundary points at least, is conformally equivalent to the unit disk).

0.2.2. Connections with the global analysis

The global behaviour of a holomorphic function is uniquely determined by its local behaviour. This fact is based on the principle of analytic continuation: Carrying out all possible analytic continuations along curves (by rearrangement of the power series), a given power series (2) leads to a function defined, possibly multi-valued, in the z-plane.

The domain of definition of such a multi-valued function is a Riemann surface. To each Riemann surface one can construct a so-called universal covering surface which turns out to be simply connected. Applying an extended version of the Riemann mapping theorem (cf. 0.2.1), it follows that there exists an one-to-one mapping of the universal covering surface onto the unit disk, the z-plane and the extended z-plane resp. Restri cting this mapping to the Riemann surface originally given, one gets a uniformization of that Riemann surface (cf. R. Nevanlinna [26]).

0.2.3. Connections with algebra

Such a connection is given, for instance, by the fact that a Riemann surface is compact if and only if it is the corresponding Riemann surface to an algebraic function, i.e. to a function $w = f(z)$ defined by

$$\sum_{k,l} a_{kl} z^k w^l = 0$$

where the a_{kl} are (complex) constants and the summation is finite.

0.2.4. Connections with partial differential equations

Real part u and imaginary part v of a holomorphic function $w = u + iv = f(z)$ satisfy the well-known Cauchy-Riemann system

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} = 0 \tag{3}$$

and, consequently, the real part u and the imaginary part v as well are solutions of the Laplace equation; i.e.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4}$$

and

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Suppose that $w = f(z)$ is holomorphic in the (bounded) domain D and continuous in the closure $\overline{D}$. Taking into account that a solution of (4) in D is uniquely determined by its boundary values on ∂D, it follows immediately that the real part u of a holomorphic function is uniquely determined in D by its boundary values on ∂D. On the other hand, in view of (3) the real part u determines the imaginary part v up to a constant.

Summarizing these considerations, we see that a function $w = f(z)$ holomorphic in D and continuous in $\overline{D}$ is uniquely determined by the boundary values of the real part and the value of the imaginary part in one point of D (Dirichlet's boundary value problem for holomorphic functions). Instead of prescribing the real part on the boundary it is possible to prescribe a linear combination of real and imaginary part on the boundary (Riemann-Hilbert's boundary value problem).

Boundary value problems such as Dirichlet's and Riemann-Hilbert's ones for systems of partial differential equations more general than the system (3) can be solved by reducing these problems to analogous ones for holomorphic functions. For that purpose the solutions of general equations must be represented by holomorphic functions. The basic idea for deducing such representations is to solve the inhomogeneous Cauchy-Riemann system

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = h_1, \qquad \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} = h_2 \tag{5}$$

where h_1, h_2 are given functions in D. The system (5) can be solved by the so-called T_D-operator (cf. 0.3.3.)

0.3. Partial complex derivatives and corresponding integral operators

0.3.1. Partial complex derivatives (Wirtinger operators)

The ordinary complex differentiation $\frac{d}{dz}$ can be carried out only for holomorphic functions. In order to apply complex methods to partial differential equations more general than the Cauchy-Riemann system (3) and the Laplace equation (4) resp., it is necessary, consequently, to replace $\frac{d}{dz}$ by partial complex differentiations which are defined not only for holomorphic functions.

Take any continuously differentiable (complex-valued) function $w = f(z)$ defined in a neighbourhood of $z_o = x_o + iy_o$. Then the linearization $\tilde{f}$ of f at the point z_o is given by

$$w = \tilde{f}(z) = f(z_o) + c_1(x - x_o) + c_2(y - y_o) \qquad (6)$$

where $z = x + iy$, $z_o = x_o + iy_o$, and

$$c_1 = \frac{\partial f}{\partial x}(z_o) , \qquad c_2 = \frac{\partial f}{\partial y}(z_o) .$$

Since

$$z - z_o = (x - x_o) + i\,(y - y_o) ,$$

$$\overline{(z - z_o)} = (x - x_o) - i\,(y - y_o)$$

one gets

$$x - x_o = \frac{1}{2}\left((z - z_o) + \overline{(z - z_o)}\right),$$

$$y - y_o = \frac{1}{2}\left(\overline{(z - z_o)} - (z - z_o)\right)$$

and, therefore, the linearization (6) can be rewritten in the form

$$w = f(z) = f(z_o) + d_1\,(z - z_o) + d_2\,\overline{(z - z_o)}$$

where

$$d_1 = \frac{1}{2}(c_1 - ic_2) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_o) - i\frac{\partial f}{\partial y}(z_o)\right) ,$$

$$d_2 = \frac{1}{2}(c_1 + ic_2) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_o) + i\frac{\partial f}{\partial y}(z_o)\right).$$

The coefficients d_1, d_2 are called the partial complex derivatives of f at the point z_o and are denoted by

$$\frac{\partial f}{\partial z}(z_o) \text{ and } \frac{\partial f}{\partial \bar{z}}(z_o) \text{ resp. .}$$

Generally speaking we define

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

The differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are the so-called Wirtinger operators.

Using these operators, the Cauchy-Riemann system (3) can be written in the form

$$\frac{\partial w}{\partial \bar{z}} = 0,$$

whereas the inhomogeneous (5) can be rewritten as

$$\frac{\partial w}{\partial \bar{z}} = h$$

(where $h = h_1 + ih_2$). The Laplace equation (4) can be written in the form

$$4\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0 .$$

A first order system of type

$$H_j\left(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = 0, \quad j = 1, 2, \qquad (7)$$

can be rewritten as

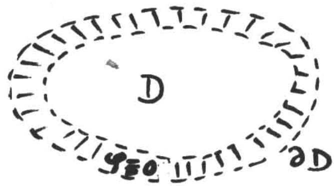
$$\frac{\partial w}{\partial \bar{z}} = F\left(z, w, \frac{\partial w}{\partial z}\right) \qquad (8)$$

provided the system (7) can be solved for suitably chosen variables.

0.3.2. Derivatives in Sobolev's sense

Suppose that f is continuously differentiable. Then the partial complex derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ exist and are continuous. Denote $\frac{\partial f}{\partial \bar{z}}$ by g.

Now take any test function φ in D, i.e. a continuously differentiable function which vanishes in a neighbourhood of the boundary ∂D of D.



Notice that the well-known Gauss-Ostrogradski integral formula can be written in the complex form

$$\iint_D \frac{\partial h}{\partial \bar{z}}\, dx\, dy = \frac{1}{2i} \int_{\partial D} h\, dz. \tag{9}$$

Consider $h = f\varphi$. Take into consideration that h vanishes on ∂D and, moreover, that

$$\frac{\partial h}{\partial \bar{z}} = f \frac{\partial \varphi}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{z}} \varphi = f \frac{\partial \varphi}{\partial \bar{z}} + g\varphi .$$

Then formula (9) implies that the relation

$$\iint_D \left(f \frac{\partial \varphi}{\partial \bar{z}} + g\varphi \right) dx\, dy = 0 \tag{10}$$

holds for any test function provided g is the derivative of f with respect to $\bar{z}$ (in the classical sense).

Formula (10) enables to generalize the concept of derivatives: Suppose that for a given function f (not necessarily differentiable in the classical sense) there exists a further (integrable) function g such that (10) is satisfied for any test function φ. Then g is said to be the derivative of f with respect to $\bar{z}$ in

Sobolev's sense and is also denoted by $\frac{\partial f}{\partial \bar{z}}$.

A variant of the above definition for the case of real variables can be found in S. L. Sobolev's book [4] . A still more general definition of derivatives (derivatives in distributional sense) is given in L. Schwartz' book [3] .

0.3.3. The T_D- and Π_D-operators

Many statements of real analysis are based on the fundamental theorem of the integral calculus in view of which the function defined by the integral

$$\int_a^x h(\xi)\, d\xi = H(x)$$

with variable upper limit is differentiable and its derivative at the point x is equal to

$$\frac{dH}{dx}(x) = h(x).$$

In other words, a special solution of the differential equation

$$\frac{dy}{dx} = h \tag{11}$$

is given by

$$H(x) = \int_a^x h(\xi)\, d\xi .$$

Notice that the general solution of (11) is given by

$$H(x) + \text{const}$$

because constants are the only functions whose derivatives vanish everywhere.

The functional-analytic methods of complex analysis to be explained here are based on the fact that for a (complex-valued) function h defined in the bounded domain D the partial complex differential equation

$$\frac{\partial w}{\partial \bar{z}} = h , \tag{12}$$

the so-called inhomogeneous Cauchy-Riemann equation, can also be solved by a function H defined explicitely by an integral, namely

$$H(z) = -\frac{1}{\pi} \iint_D \frac{h(\zeta)}{\zeta - z} \, d\xi \, d\eta \tag{13}$$

where $\zeta = \xi + i\eta$. For the sake of shortness the function H defined by (13) is denoted by $T_D h$, i.e. we have

$$\frac{\partial}{\partial \bar{z}} T_D h = h . \tag{14}$$

This differentiation must be interpreted in Sobolev's sense, in general.

We shall see that the T_D-operator defined by (13) can be used not only for solving the inhomogeneous Cauchy-Riemann equation (12) (see also (5)) but also for solving general (elliptic) systems of partial differential equations in real variables. For that purpose we must be in position to calculate the partial complex derivative of $T_D h$ with respect to z. This derivative (again in Sobolev's sense) can also be expressed explicitely by an integral operator, namely

$$\frac{\partial}{\partial z} T_D h = \Pi_D h \tag{15}$$

where $\Pi_D h$ is defined by

$$-\frac{1}{\pi} \iint_D \frac{h(\zeta)}{(\zeta - z)^2} \, d\xi \, d\eta .$$

For applying functional-analytic methods it is essential, further, that the T_D- and Π_D-operators turn out to be bounded in suitably chosen function spaces (cf. I. N. Vekua [11]). Both operators are bounded in the spaces $\mathcal{C}^\alpha(\bar{D})$, $0 < \alpha < 1$, and $L_p(D)$, $p > 2$. Remember that $\mathcal{C}^\alpha(\bar{D})$ contains all functions $w = w(z)$ satisfying the Hölder condition

$$|w(z_2) - w(z_1)| \leq K \cdot |z_2 - z_1|^\alpha$$

in $\overline{D}$ where K does not depend on z_1, z_2, whereas $L_p(D)$ is the space of all $w = w(z)$ for which $|w|^p$ is integrable (in Lebesque's sense). In addition, the T_D-operator is also a bounded operator in the space $\mathcal{C}(\overline{D})$ of all functions contin uous in $\overline{D}$.

With regard to the definition and properties of the varions function spaces we refer to L. A. Ljusternik and W. I. Sobolev [2] and to A. Adams [1] as well.

<u>0.3.4. Weyl's lemma</u>

Remember that any two solutions of the differential equations (11) differ from each other by a constant.

Analogously, let us consider the difference $w_o = w_1 - w_2$ of any two solutions w_1 and w_2 of the inhomogeneous Cauchy-Riemann equation (12). Then w_o satisfies the Cauchy-Riemann equation

$$\frac{\partial w_o}{\partial \overline{z}} = 0 . \qquad (16)$$

Conversely, let w_o be any solution of (16) and w_1 a special solution of the inhomegeneous Cauchy-Riemann equation (12) . Then $w_2 = w_1 + w_o$ is also a solution of (12) . Each holomorphic function is a solution of the differential equation (16). Hence it follows that $w_1 + \Phi$ is a solution of (12) if w_1 is a special solution of (12) and Φ is any holomorphic function.

Notice that the differential equation (16) must be understood in Sobolev's sense. Possibly there are solutions of (16) in Sobolev's sense not being classical holomorphic functions. This possibility can be excluded by the so-called Weyl lemma saying that (16) does not possess other solutions than classical holomorphic functions (cf. H. Weyl [5] and L. Schwartz [3]. An elementary proof of Weyl's lemma is given in [18]).

Summarizing these arguments, one sees that the general solution of the inhomogeneous Cauchy-Riemann equation (12) is given by