George Grätzer

Universal Algebra

Second Edition

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INTRODUCTION TO THE SECOND EDITION

About a year ago I was approached by the publisher of the present volume about a new edition of the book, Universal Algebra. This was a good opportunity to review the book I wrote in 1964–1965, about thirteen years ago, to find out whether I can still subscribe to the presentation of the book or a complete revision is necessary.

It is my opinion that the definitions and results presented in this book form a foundation of universal algebra as much today as they did a decade ago. Some concepts became more important and some new ones appeared, but the foundation has not changed much.

On the other hand, my point of view changed rather substantially in a number of areas. Compare the elementary approach to the congruence lattice characterization theorem of a decade ago with the axiomatic approach of today (see Appendix 7).

The obstacles in the way of a complete revision appeared just as formidable. An initial appraisal put the number of papers written since the bibliography of Universal Algebra was closed (around 1967) near to 1000, making it very difficult for someone to pretend to be an expert on all the major developments in universal algebra. At twenty-seven I thought nothing of establishing as my goal "to give a systematic treatment of the most important results"; at forty (with a thousand more papers to contend with and in the middle of proofreading my General Lattice Theory) I was not so sure of being able to undertake the same.

So I decided to obtain the help of a number of experts to review various aspects of recent developments. B. Jónsson agreed to survey congruence varieties, a fast evolving chapter of universal algebra, based on his 1974 lecture at the Vancouver meeting of the International Mathematical Union (Appendix 3). Walter Taylor consented to have an abbreviated version of his survey on equational theories included (Appendix 4). R. W. Quackenbush undertook to present primal algebras and their generalizations, a vast field containing many important results (Appendix 5). Finally, G. H. Wenzel agreed to survey equational compactness (Appendix 6).

In addition, Appendix 1 surveys the developments of the last decade: in §55, the survey follows the sections of the book; §56 surveys related structures and §57 outlines some important new topics. Appendix 2 reviews the problems given in the first edition. Finally, Appendix 7 contains a proof of the independence of congruence lattices, automorphism groups, and subalgebra lattices of infinitary algebras and the characterization of type-2 congruence lattices by modularity; these have not previously appeared in print.

Referencing to the new bibliography is by year of publication, e.g., [1975], [1975 a]; items not in print at the time of the original compilation are listed as [a], [b], etc.

All the appendices and the new bibliography have been widely circulated. I would like to thank all those who sent in corrections and additions, especially H. Andréka, J. Berman, C. C. Chen, A. P. Huhn, L. Márki, I. Németi, B. M. Schein, W. Taylor, and A. Waterman.

In the compilation of the new bibliography I was greatly assisted by M. E. Adams. The typing and clerical work was done by L. Gushulak, M. McTavish, and S. Padmanabhan. In the proofreading I was helped by M. E. Adams, W. J. Blok, and P. Köhler.

INTRODUCTION TO THE FIRST EDITION

In A. N. Whitehead's book on Universal Algebra,† published in 1898, the term universal algebra had very much the same meaning that it has today.

Universal algebra started to evolve when mathematics departed from the study of operations on real numbers only. Hamilton's quaternions, Boole's symbolic logic, and so forth, brought to light operations on objects other than real numbers and operations which are very different from the traditional ones.

"Such algebras have an intrinsic value for separate detailed study; also they are worthy of a comparative study, for the sake of the light thereby thrown on the general theory of symbolic reasoning, and on algebraic symbolism in particular. The comparative study necessarily presupposes some previous separate study, comparison being impossible without knowledge"; so wrote Whitehead in 1898 and his point of view is still shared by many.

Thus universal algebra is the study of finitary operations on a set, and the purpose of research is to find and develop the properties which such diverse algebras as rings, fields, Boolean algebras, lattices, and groups may have in common.

Although Whitehead recognized the need for universal algebra, he had no results. The first results were published by G. Birkhoff in the thirties. Some thirty years elapsed between Whitehead's book and Birkhoff's first paper, despite the fact that the goal of research was so beautifully stated in Whitehead's book. However, to generalize, one needs experience, and before the thirties most of the branches of modern algebra were not developed sufficiently to give impetus to the development of universal algebras.

In the period from 1935 to 1950 most papers were written along the lines suggested by Birkhoff's papers: free algebras, the homomorphism theorem and the isomorphism theorems, congruence lattices, and subalgebra lattices were discussed. Many of the results of this period can be found in Birkhoff's book [6].

† According to A. N. Whitehead, the subject matter originated with W. R. Hamilton and A. DeMorgan, and the name for it was coined by J. J. Sylvester.

In the meantime, mathematical logic developed to the point where it could be applied to algebra. K. Gödel's completeness theorem (Gödel [1]), A. Tarski's definition of satisfiability, and so on, made mathematicians realize the possibility of applications. Such applications came about slowly. A. I. Mal'cev's 1941 paper [2] was the first one, but it went unnoticed because of the war. After the war, A. Tarski, L. A. Henkin, and A. Robinson began working in this field and they started publishing their results about 1950.

A. Tarski's lecture at the International Congress of Mathematicians (Cambridge, Massachusetts, 1950) may be considered as the beginning of the new period.

The model-theoretic aspect of universal algebras was mostly developed by Tarski himself and by C. C. Chang, L. A. Henkin, B. Jónsson, H. J. Keisler, R. C. Lyndon, M. Morley, D. Scott, R. L. Vaught, and others, and to a certain extent by A. I. Mal'cev.

In the late fifties E. Marczewski [2] emphasized the importance of bases of free algebras; he called them independent sets. As a result Marczewski, J. Mycielski, W. Narkiewicz, W. Nitka, J. Płonka, S. Świerczkowski, K. Urbanik, and others were responsible for more than 50 papers on the algebraic theory of free algebras.

There are a number of individuals who have not been mentioned yet and who have made significant contributions to universal algebra. It is hoped that the references in the text will give everyone his due credit.

Because of the way in which universal algebras developed, many elementary results have never been published but have been used without any reference in the papers, sometimes only in the form of a "therefore". It is hoped that this book will give an adequate background for the explanation of the "therefore's".

The purpose of this book is to give a systematic treatment of the most important results in the field of universal algebras. We will consider generalizations of universal algebras only to the extent that they are necessary for the development of the theory of universal algebras themselves. Therefore, the particular problems of partial algebras and structures are not discussed. Infinitary algebras will be touched upon only in the exercises. Multi-algebras are scarcely mentioned at all. This limitation is quite natural. First of all, to keep the length of a book within reasonable bounds, some limitations are necessary. Secondly, it so happens that most of the results on universal algebras can be extended in each of the directions mentioned, at the expense of more involved notations. Since the purpose of a book should be, in the author's opinion, to present ideas and methods, the framework of universal algebras is sufficiently wide enough to accomplish this. However, each of these directions has problems of its own. For instance, infinitary partial algebras contain topological spaces as special cases, and the nature of these investigations does not have much to do with the topic of this book. Topological universal algebras and partially ordered universal algebras have not been included because of lack of material.

Category theory is excluded from this book because a superficial treatment seems to present pedagogical difficulties and it would be mathematically not too effective; moreover, those topics that can be treated in depth in a categorical framework (in particular, parts of Chapters 4 and 6) are to be discussed by S. Eilenberg in a book (universal algebra and automata theory) and by F. W. Lawvere in a book (on elementary theories), in lecture notes (on algebraic theories), and in an expository article (on the category of sets). However, there are a number of exercises originating in category theory.

Since a short description of the content is given at the beginning of each chapter, we will include here only a brief outline of the book.

In Chapter 0 the set-theoretic notations together with some basic facts are given, of course, without proof. The last section is on a special type of lattices that are useful in algebraic applications. One can hardly expect everyone to agree with the presentation of Chapter 0. Some will find it too short, some too long. However, it is hoped that the reader without set-theoretic knowledge will find sufficient background material there for an understanding of the remainder of the book, and, if he wants to delve deeper into set theory, at least he will know what to look for.

Chapters 1–3 develop the basic results. In Chapter 1, polynomials, polynomial symbols, homomorphisms, congruence relations, and subalgebras are discussed and the standard results, the isomorphism theorems, and the like are given. The same results for partial algebras are presented in Chapter 2, but only from the point of view of applications to algebras. To show the usefulness of partial algebras, the last two sections of Chapter 2 give the characterization theorem of congruence lattices of algebras, due to E. T. Schmidt and the author. Constructions of new algebras from given ones play a central role in universal algebras. Direct products, subdirect products, direct and inverse limits, and many related constructions are given in Chapter 3.

In Chapters 4 and 5 one of the most important concepts of universal algebras, namely that of free algebras, is discussed. The constructions, basic properties, and several applications of free algebras are given in Chapter 4, and in Chapter 5 we consider the bases of free algebras, a concept identical with E. Marczewski's notion of independence.

A short introduction to model theory is given in Chapter 6. The basic tool is J. Łoś' concept of prime product.

In Chapter 7 these results are applied to determine the properties that are preserved under certain algebraic constructions using generalized atomic formulas of H. J. Keisler; for direct products, the method of S. Feferman and R. L. Vaught is used.

It is hoped that most experts will agree that in these chapters the author has selected the most important topics not biased by his own research, but this obviously does not apply to Chapter 8, which is the author's theory of free structures over first order axiom systems. However, this topic seems to be as good as any to yield further applications of the methods developed in Chapter 6 to the purely algebraic problems of free algebras.

Each chapter is followed by exercises and problems. There are more than 650 exercises and over 100 research problems in the book. Many of the exercises are simple illustrations of new concepts, some ask for (or give) counterexamples, and some review additional results in the field. The problems list some open questions which the author thought interesting.

The numbering system of theorems, lemmas, corollaries, definitions, exercises, and problems is self-explanatory. Within each section, theorems and lemmas are numbered consecutively. A single corollary to a theorem or lemma is not numbered; however, if more than one corollary follows a lemma or theorem, they are numbered from one in each case. Theorem 2 refers to Theorem 2 of the section in which it occurs; Theorem 38.2 refers to Theorem 2 of §38; Exercise 3.92 refers to Exercise 92 of Chapter 3.

The present book is intended for the mathematician who wants to use the methods and results of universal algebra in his own field and also for those who want to specialize in universal algebra. For applications of universal algebra to groups, rings, Lie algebras, and so on, the reader should consult P. M. Cohn [1] and §6 of the author's report [14].

The first version of the Bibliography was sent out to about 50 experts. Numerous suggestions were received, for which the author wants to thank each contributor. In the compilation of the original bibliography, and also of the revised form, the author was helped by Catherine M. Gratzer.

This book is based on the notes of the lectures delivered at the Pennsylvania State University between October 1, 1964, and November 1, 1965. Professor Leo F. Boron took notes of the lectures, and after his notes were reviewed (many times, rewritten), he typed them up and had them duplicated. He worked endless hours on this. The author finds it hard to find the words which would express his gratitude for Professor Boron's unselfish help. These lectures notes were completely rewritten by the author and mimeographed. Thanks are due to the Mathematics Department of the Pennsylvania State University for providing partial funds for this project and to Mrs. L. Moyer who did all the typing of this second version.

The author cannot be too grateful to the large number of mathematicians who took the time and trouble to read the second mimeographed version and to send him detailed lists of suggestions and corrections, major and minor. The author wants to thank all of them for their help, especially P. M. Cohn, K. H. Diener, B. Jónsson, H. F. J. Lowig, D. Monk, M. Novotny, H. Ribeiro, B. M. Schein, J. Schmidt, and A. G. Waterman; their generous interest was invaluable in writing the third, final version.

The author's students, especially M. I. Gould, G. H. Wenzel, and also R. M. Vancko and C. R. Platt, contributed many suggestions, simplifications of proofs, and corrections at all stages of the work. They also helped in checking the Bibliography and in presenting papers in the seminar. The task of the final revision of the manuscript, including a final check of the Bibliography, was undertaken by C. C. Chen. E. C. Johnston, W. A. Lampe, H. Pesotan, C. R. Platt, R. M. Vancko, and G. H. Wenzel aided the author in the proofreading.

Thanks are also due to the Mathematics Department of McMaster University, Hamilton, Ontario, Canada, and especially to Professors B. Banaschewski and G. Bruns, for making it possible for the author to give three series of lectures (December 1964, June 1965, and December 1965) on parts of this book, and for their several suggestions.

TABLE OF NOTATION

Following a symbol the page number of the first occurrence is given in parentheses.

Algebra, structure	$\mathfrak{A}$ $\mathfrak{B}$ $\mathfrak{C}$ $\mathfrak{D}$ $\mathfrak{E}$ $\mathfrak{F}$ $\mathfrak{G}$ $\mathfrak{H}$ $\mathfrak{I}$ $\mathfrak{K}$ $\mathfrak{L}$ $\mathfrak{M}$ $\mathfrak{N}$ $\mathfrak{O}$ $\mathfrak{P}$ $\mathfrak{Q}$ $\mathfrak{R}$ $\mathfrak{S}$ $\mathfrak{T}$ $\mathfrak{U}$ $\mathfrak{V}$ $\mathfrak{W}$ $\mathfrak{X}$ $\mathfrak{Y}$ $\mathfrak{Z}$
Base set	A B C D E F G H I K L M N O P Q R S T U V W X Y Z

Algebras, structures. $\langle A; F\rangle$ (8, 33, 224), $\langle A; F, R\rangle$ (223), $\langle A, R\rangle$ (8, 224), $\mathfrak{A}/\Theta$ (36, 82), $\mathfrak{P}^{(n)}(\mathfrak{A})$ (38), $\mathfrak{P}^{(n)}(\tau)$ (40, 84) $\mathfrak{P}^{(\alpha)}(\tau)$ (41), $\mathfrak{P}^{(\alpha)}(K)$ (43), $\mathfrak{C}(\mathfrak{A})$ (51), $\mathfrak{E}(\mathfrak{A})$ (67, 98), $\mathfrak{G}(\mathfrak{A})$ (68), $\mathfrak{E}_F(\mathfrak{A})$, $\mathfrak{E}_S(\mathfrak{A})$ (98), $\mathfrak{L}(\mathfrak{A})$ (72), $\mathfrak{L}(\tau)$ (172), $\mathfrak{F}_K(\mathfrak{m})$, $\mathfrak{F}_K(\alpha)$ (163), $\mathfrak{F}_K(\mathfrak{A})$ (180), $\mathfrak{F}_K(\alpha, \Omega)$ (183), $\mathfrak{F}_\Sigma(\alpha)$ (307), $\langle\mathfrak{A}, \mathrm{a}\rangle$ (239).

Classes of algebras, structures. $K(\tau)$ (34, 223), $K(\mathfrak{D}, X)$ (165), $\hat{K}$ (278), $Sp(K)$ (159), $Id(K)$ (170), $v(K)$, $f(K)$ (191), $A(K)$, $\hat{A}(K)$ (327).

Algebraic constructions. $\mathfrak{A}/\Theta$ (36, 82), $\prod(\mathfrak{A}_i \mid i \in I)$ (118), $\mathfrak{A}^I$ (119), $\prod_{\mathscr{D}}(\mathfrak{A}_i \mid i \in I)$ (144, 145, 240), $\mathfrak{A}_{\mathscr{D}}{}^I$ (144, 145, 246), $\underrightarrow{\lim}\, \mathscr{A}$ (129, 130), $\underleftarrow{\lim}\, \mathscr{A}$ (131), $\mathscr{A}/P$ (135, 136), $\mathfrak{A}[\mathfrak{B}]$ (147), $\mathbf{I}$, $\mathbf{S}$, $\mathbf{H}$, $\mathbf{P}$, $\mathbf{P}^*$, $\mathbf{P}_S$, $\mathbf{P}_S{}^*$, $\underrightarrow{\mathbf{L}}$, $\underleftarrow{\mathbf{L}}$, (152), $\mathbf{C}$ (158), $\mathbf{P}_P$ (244).

Sets. $\in$, $\notin$, $\subseteq$, $\subset$, $\cup$, $\cap$, $'$, $-$, $\bigcup$, $\bigcap$, $\varnothing$, (1, 4), $P(A)$ (1), $(a_i \mid i \in I)$, $\{a_i \mid i \in I\}$ (4) Part(A) (1), ι, ι_A, ω, ω_A (2), $A \times B$, $\prod(A_i \mid i \in I)$, A^n (2, 4, 5), $E(A)$ (3), $|A|$ (13), A/ε (6), $[x]\varepsilon$ (6), ε_φ (7).

Mappings. $\varphi\colon A \to B$, $a\varphi$, $\varphi(a)$, $A\varphi$, $D(\varphi)$ (3), $D(f, \mathfrak{A})$ (80), φ_A (4), A^B (4), A^α (16), $e_i{}^I$, $e_i{}^n$ (5), $\varphi\psi$ (6), ε (17), $M(A)$, $M_O(A)$, $M_I(A)$ (17), $\mathbf{p}_{\mathfrak{A}}$, $\mathbf{p}^{\mathfrak{A}}$ (40, 304), $p(\bar{a})$ (42) a, a_k, $\mathrm{a}(k/b)$ (227).

Partially ordered sets, lattices, Boolean algebras. $\leqq$, $<$ (8), l.u.b.(H), g.l.b. (H) (10), $\vee$, $\wedge$, $\bigvee$, $\bigwedge$, 0, 1, $'$ (8, 9, 10, 11), $\mathfrak{P}(A)$ (9), $(H]$, $(a]$ (20), $[H)$, $[a)$ (26), $I(\mathfrak{L})$, $\mathfrak{I}(\mathfrak{L})$ (20).

Cardinals, ordinals. $|A|$ (13) $\aleph_0$, $\mathfrak{c}$ (13), $\aleph_\alpha$ (16), $\mathfrak{m} \leqq \mathfrak{n}$, $\mathfrak{m}\cdot\mathfrak{n}$, $\mathfrak{m}+\mathfrak{n}$, $\mathfrak{m}^{\mathfrak{n}}$, $\Sigma\,(\mathfrak{m}_i \mid i \in I)$, $\prod\,(\mathfrak{m}_i \mid i \in I)$ (13), $\bar{\alpha}$ (15), ω (14), $\omega_{\mathfrak{m}}$ (15), ω_α (16), $\alpha+\beta$, $\alpha\cdot\beta$, $\Sigma\,(\alpha_i \mid i \in I)$, $\lim\,(\alpha_i \mid i \in I)$ (15).

Closures. $[X]_{\mathscr{E}}$ (6), $[H]_{\mathscr{A}}$ (24), $[H]$ (24, 35), $[H]_\Sigma$ (303), $\mathscr{S}(\mathfrak{A})$ (45, 96), $\mathscr{S}^+(\mathfrak{A})$ (49), $\mathscr{S}^0(\mathfrak{A})$ (72), $[a_0, \cdots, a_{n-1}]$ (202).

Congruence relations. $\Theta_{\mathfrak{a}}$ (42, 84), Θ_K, $\Theta_{\mathfrak{A}}$ (42), $\Theta(H)$, $\Theta(a, b)$ (52), ι, ω (2), Φ/Θ (59), Θ^σ (105), Θ_L (144), $\mathcal{O}_K(a)$ (214), $\Sigma\Theta_i$ (215).

Logic. $L(\tau)$ (225), $\mathbf{L}(\tau)$ (226), $L_{\mathfrak{A}}(\tau)$ (239), $L_{\mathfrak{A},\mathfrak{B}}(\tau)$ (271), $L(\tau \oplus \xi)$ (253), $L^{(\alpha)}(\tau)$ (249), $\wedge$, $\vee$, $\neg$, $\to$, $\leftrightarrow$, $\equiv$, $\bigwedge$, $\bigvee$, $\exists$, $\forall$ (225, 226), $\models$, $\Leftrightarrow$, $\mathfrak{A}\models$ (232), $\mathfrak{A} \equiv \mathfrak{B}$ (234), $\langle \mathfrak{A}, \mathsf{a} \rangle$ (239), K^* (255), Σ^* (170, 255), $S(\Phi, \mathfrak{f})$ (241), $\mathscr{S}(\Phi)$ (291), $e(\Phi)$ (301).

Properties and conditions. U_N (294), B_n, C_n (315), B, C (317), P (318), IP (322), P_n, P_ω (327), SP (35, 44), EIS (217).

Miscellaneous. $r_0 \cdot r_1$, r^{-1} (2), r_A (3), $\cong$ (34), $o(\tau)$ (33), $P^{(n)}(\mathfrak{A})$, $P^{(n,k)}(\mathfrak{A})$ (38), $\mathbf{P}^{(n)}(\tau)$ (40, 84), $T(\mu)$ (73), g^*, g, i, i_*, p (205, 206), τ_{a} (239), $f^{\vee}$ (240), $\mathbf{P}_n(\Sigma)$ (304), $E(\Sigma)$ (311).

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CHAPTER 0

BASIC CONCEPTS

In this chapter we will review briefly the basic concepts of set theory. The results that can be found in any standard book on set theory or algebra will be stated without proof. Those who are familiar with the basic concepts of set theory should only check the notations. Ideal theory of semilattices with complete proofs is presented in §6. This chapter, including the exercises, gives an adequate set theoretical background for the book.

§1. SETS AND RELATIONS

We accept the intuitive concept of a *set* as a collection of objects, called *elements* or *members* of the set. (See also the remark in §4 concerning classes.) The notation $a \in A$ means that a is an element of the set A. If a is not an element of A, we write $a \notin A$. If A and B are sets, $A \subseteq B$ denotes *inclusion*, that is, that A is a *subset* of B, or, all the elements of A are also in B. *Equality* of the sets A and B, in symbols $A = B$, holds if and only if $A \subseteq B$ and $B \subseteq A$. If $A = B$ does not hold, we write $A \neq B$. *Proper inclusion* is denoted by $A \subset B$; by definition $A \subset B$ means $A \subseteq B$ and $A \neq B$.

The *void set* is denoted by $\varnothing$; note that $\varnothing \subseteq A$ for every set A.

The *set theoretic operations* $\cup$, $\cap$, $-$ (they are called *union*, *intersection* and *difference*, respectively) have their usual meaning. If a set A is fixed, then for subsets B of A the *complement* B' of B is defined by $A - B$; by definition, $B \cup (A - B) = A$ and $B \cap (A - B) = \varnothing$. Note that $B \subseteq A$ is equivalent to $B = B \cap A$, which, in turn, is equivalent to $A = B \cup A$. If $A \cap B = \varnothing$, we say that A and B are *disjoint*.

If A is a set, then $P(A)$ (called the *power set* of A) denotes the set of all subsets of A.

A subset of $P(A)$ will be called a *system*, or more precisely, a system over A. A *partition* π of A is a system (over A) not containing $\varnothing$, satisfying the following property: every $a \in A$ is an element of exactly one $B \in \pi$. The members of π are called *blocks* of the partition π. We use Part(A) for the set of all partitions of A. Note that Part$(A) \neq \varnothing$; indeed, if $A = \varnothing$, then Part$(A) = P(A) = \{\varnothing\}$, and if $A \neq \varnothing$, then Part(A) contains the partition which has one block, namely A. If π_0 and π_1 are partitions of A, we will