THE THEORY OF PARALLELS

LOBACHEVSKI

GEOMETRICAL RESEARCHES

ON

THE THEORY OF PARALLELS

BY

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BERLIN, 1840

Translated from the Original

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NEW EDITION

CHICAGO—LONDON
OPEN COURT PUBLISHING COMPANY
1914

TRANSLATOR'S PREFACE.

Lobachevski was the first man ever to publish a non-Euclidean geometry.

Of the immortal essay now first appearing in English Gauss said, "The author has treated the matter with a master-hand and in the true geometer's spirit. I think I ought to call your attention to this book, whose perusal can not fail to give you the most vivid pleasure."

Clifford says, "It is quite simple, merely Euclid without the vicious assumption, but the way things come out of one another is quite lovely." * * * "What Vesalius was to Galen, what Copernicus was to Ptolemy, that was Lobachevski to Euclid."

Says Sylvester, "In Quaternions the example has been given of Algebra released from the yoke of the commutative principle of multiplication—an emancipation somewhat akin to Lobachevski's of Geometry from Euclid's noted empirical axiom."

Cayley says, "It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobachevski constructed a perfectly consistent theory, wherein this axiom was assumed not to hold good, or say a system of non-Euclidean plane geometry. There is a like system of non-Euclidean solid geometry."

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May 1, 1891.

TRANSLATOR'S INTRODUCTION.

"Prove all things, hold fast that which is good," does not mean demonstrate everything. From nothing assumed, nothing can be proved. "Geometry without axioms," was a book which went through several editions, and still has historical value. But now a volume with such a title would, without opening it, be set down as simply the work of a paradoxer.

The set of axioms far the most influential in the intellectual history of the world was put together in Egypt; but really it owed nothing to the Egyptian race, drew nothing from the boasted lore of Egypt's priests.

The Papyrus of the Rhind, belonging to the British Museum, but given to the world by the erudition of a German Egyptologist, Eisenlohr, and a German historian of mathematics, Cantor, gives us more knowledge of the state of mathematics in ancient Egypt than all else previously accessible to the modern world. Its whole testimony confirms with overwhelming force the position that Geometry as a science, strict and self-conscious deductive reasoning, was created by the subtle intellect of the same race whose bloom in art still overawes us in the Venus of Milo, the Apollo Belvidere, the Laocoon.

In a geometry occur the most noted set of axioms, the geometry of Euclid, a pure Greek, professor at the University of Alexandria.

Not only at its very birth did this typical product of the Greek genius assume sway as ruler in the pure sciences, not only does its first efflorescence carry us through the splendid days of Theon and Hypatia, but unlike the latter, fanatics can not murder it; that dismal flood, the dark ages, can not drown it. Like the phœnix of its native Egypt, it rises with the new birth of culture. An Anglo-Saxon, Adelard of Bath, finds it clothed in Arabic vestments in the land of the Alhambra. Then clothed in Latin, it and the new-born printing press confer honor on each other. Finally back again in its original Greek, it is published first in queenly Basel, then in stately Oxford. The latest edition in Greek is from Leipsic's learned presses.

How the first translation into our cut-and-thrust, survival-of-the-fittest English was made from the Greek and Latin by Henricus Billingsly, Lord Mayor of London, and published with a preface by John Dee the Magician, may be studied in the Library of our own Princeton, where they have, by some strange chance, Billingsly's own copy of the Arabic-Latin version of Campanus bound with the Editio Princeps in Greek and enriched with his autograph emendations. Even to-day in the vast system of examinations set by Cambridge, Oxford, and the British government, no proof will be accepted which infringes Euclid's order, a sequence founded upon his set of axioms.

The American ideal is success. In twenty years the American maker expects to be improved upon, superseded. The Greek ideal was perfection. The Greek Epic and Lyric poets, the Greek sculptors, remain unmatched. The axioms of the Greek geometer remained unquestioned for twenty centuries.

How and where doubt came to look toward them is of no ordinary interest, for this doubt was epoch-making in the history of mind.

Among Euclid's axioms was one differing from the others in prolixity, whose place fluctuates in the manuscripts, and which is not used in Euclid's first twenty-seven propositions. Moreover it is only then brought in to prove the inverse of one of these already demonstrated.

All this suggested, at Europe's renaissance, not a doubt of the axiom, but the possibility of getting along without it, of deducing it from the other axioms and the twenty-seven propositions already proved. Euclid demonstrates things more axiomatic by far. He proves what every dog knows, that any two sides of a triangle are together greater than the third. Yet when he has perfectly proved that lines making with a transversal equal alternate angles are parallel, in order to prove the inverse, that parallels cut by a transversal make equal alternate angles, he brings in the unwieldly postulate or axiom:

"If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles."

Do you wonder that succeeding geometers wished by demonstration to push this unwieldly thing from the set of fundamental axioms.

Numerous and desperate were the attempts to deduce it from reasonings about the nature of the straight line and plane angle. In the "Encyclopœdie der Wissenschaften und Kunste; Von Ersch und Gruber;" Leipzig, 1838; under "Parallel," Sohncke says that in mathematics there is nothing over which so much has been spoken, written, and striven, as over the theory of parallels, and all, so far (up to his time), without reaching a definite result and decision.

Some acknowledged defeat by taking a new definition of parallels, as for example the stupid one, "Parallel lines are everywhere equally distant," still given on page 33 of Schuyler's Geometry, which that author, like many of his unfortunate prototypes, then attempts to identify with Euclid's definition by pseudo-reasoning which tacitly assumes Euclid's postulate, e. g. he says p. 35: "For, if not parallel, they are not everywhere equally distant; and since they lie in the same plane; must approach when produced one way or the other; and since straight lines continue in the same direction, must continue to approach if produced farther, and if sufficiently produced, must meet." This is nothing but Euclid's assumption, diseased and contaminated by the introduction of the indefinite term "direction."

How much better to have followed the third class of his predecessors who honestly assume a new axiom differing from Euclid's in form if not in essence. Of these the best is that called Playfair's; "Two lines which intersect can not both be parallel to the same line."

The German article mentioned is followed by a carefully prepared list of ninety-two authors on the subject. In English an account of like attempts was given by Perronet Thompson, Cambridge, 1833, and is brought up to date in the charming volume, "Euclid and his Modern Rivals," by C. L. Dodgson, late Mathematical Lecturer of Christ Church, Oxford, the Lewis Carroll, author of Alice in Wonderland.

All this shows how ready the world was for the extraordinary flaming-forth of genius from different parts of the world which was at once to overturn, explain, and remake not only all this subject but as consequence all philosophy, all ken-lore. As was the case with the discovery of the Conservation of Energy, the independent irruptions of genius, whether in Russia, Hungary, Germany, or even in Canada gave everywhere the same results.

At first these results were not fully understood even by the brightest intellects. Thirty years after the publication of the book he mentions, we see the brilliant Clifford writing from Trinity College, Cambridge, April 2, 1870, "Several new ideas have come to me lately: First I have procured Lobachevski, 'Études Géométriques sur la Théorie des Parallels' - - - a small tract of which Gauss, therein quoted, says: L'auteur a traité la matière en main de maître et avec le véritable esprit géométrique. Je crois devoir appeler votre attention sur ce livre, dont la lecture ne peut manquer de vous causer le plus vif plaisir.'" Then says Clifford: "It is quite simple, merely Euclid without the vicious assumption, but the way the things come out of one another is quite lovely."

The first axiom doubted is called a "vicious assumption," soon no man sees more clearly than Clifford that all are assumptions and none vicious. He had been reading the French translation by Hoüel, published in 1866, of a little book of 61 pages published in 1840 in Berlin under the title Geometrische Untersuchungen zur Theorie der Parallellinien by Nicolas Lobachevski (1793-1856), the first public expression of whose discoveries, however, dates back to a discourse at Kasan on February 12, 1826.

Under this commonplace title who would have suspected the discovery of a new space in which to hold our universe and ourselves.

A new kind of universal space; the idea is a hard one. To name it, all the space in which we think the world and stars live and move and have their being was ceded to Euclid as his by right of pre-emption, description, and occupancy; then the new space and its quick-following fellows could be called Non-Euclidean.

Gauss in a letter to Schumacher, dated Nov. 28, 1846, mentions that as far back as 1792 he had started on this path to a new universe. Again he says: "La géométrie non-euclidienne ne renferme en elle rien de contradictoire, quoique, à prèmiere vue, beaucoup de ses résultats aien l'air de paradoxes. Ces contradictions apparents doivent être regardées comme l'effet d'une illusion, due à l'habitude que nous avons prise de bonne heure de considérer la géométrie euclidienne comme rigoureuse."

But here we see in the last word the same imperfection of view as in Clifford's letter. The perception has not yet come that though the non-Euclidean geometry is rigorous, Euclid is not one whit less so.

A former friend of Gauss at Gœttingen was the Hungarian Wolfgang Bolyai. His principal work, published by subscription, has the following title:

Tentamen Juventutem studiosam in elementa Matheseos purae, elementaris ac sublimioris, methodo intuitiva, evidentiaque huic propria, introducendi. Tomus Primus, 1832; Secundus, 1833. 8vo. Maros-Vásárhelyini.

In the first volume with special numbering, appeared the celebrated Appendix of his son John Bolyai with the following title:

APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens: *a veritate aut falsitate Axiomatis XI Euclidei (a priori haud unquam decidenda) independentem.* Auctore JOHANNE BOLYAI de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo. (26 pages of text).

This marvellous Appendix has been translated into French, Italian, English and German.

In the title of Wolfgang Bolyai's last work, the only one he composed in German (88 pages of text, 1851), occurs the following:

"und da die Frage, *ob zwey von der dritten geschnittene Geraden, wenn die summe der inneren Winkel nicht=2R, sich schneiden oder nicht?* niemand auf der Erde ohne ein Axiom (wie *Euclid* das XI) aufzustellen, beantworten wird; die davon unabhængige Geometrie abzusondern; und eine auf die *Ja*-Antwort, andere auf das *Nein* so zu bauen, dass die Formeln der letzen, auf ein Wink auch in der ersten gültig seyen."

The author mentions Lobachevski's Geometrische Untersuchungen, Berlin, 1840, and compares it with the work of his son John Bolyai, "au sujet duquel il dit: 'Quelques exemplaires de l'ouvrage publié ici ont été envoyés à cette époque à Vienne, à Berlin, à Gœttingue. . . De Goettingue le géant mathématique, [Gauss] qui du sommet des hauteurs embrasse du même regard les astres et la profondeur des abîmes, a écrit qu'il était ravi de voir exécuté le travail qu'il avait commencé pour le laisser après lui dans ses papiers.' "

In fact this first of the Non-Euclidean geometries accepts all of Euclid's axioms but the last, which it flatly denies and replaces by its contradictory, that the sum of the interior angles made on the same side of a transversal by two straight lines may be less than a straight angle without the lines meeting. A perfectly consistent and elegant geometry then follows, in which the sum of the angles of a triangle is always less than a straight angle, and not every triangle has its vertices concyclic.

THEORY OF PARALLELS.

In geometry I find certain imperfections which I hold to be the reason why this science, apart from transition into analytics, can as yet make no advance from that state in which it has come to us from Euclid.

As belonging to these imperfections, I consider the obscurity in the fundamental concepts of the geometrical magnitudes and in the manner and method of representing the measuring of these magnitudes, and finally the momentous gap in the theory of parallels, to fill which all efforts of mathematicians have been so far in vain.

For this theory Legendre's endeavors have done nothing, since he was forced to leave the only rigid way to turn into a side path and take refuge in auxiliary theorems which he illogically strove to exhibit as necessary axioms. My first essay on the foundations of geometry I published in the Kasan *Messenger* for the year 1829. In the hope of having satisfied all requirements, I undertook hereupon a treatment of the whole of this science, and published my work in separate parts in the "*Gelehrten Schriften der Universitæt Kasan*" for the years 1836, 1837, 1838, under the title "New Elements of Geometry, with a complete Theory of Parallels." The extent of this work perhaps hindered my countrymen from following such a subject, which since Legendre had lost its interest. Yet I am of the opinion that the Theory of Parallels should not lose its claim to the attention of geometers, and therefore I aim to give here the substance of my investigations, remarking beforehand that contrary to the opinion of Legendre, all other imperfections—for example, the definition of a straight line—show themselves foreign here and without any real influence on the theory of parallels.

In order not to fatigue my reader with the multitude of those theorems whose proofs present no difficulties, I prefix here only those of which a knowledge is necessary for what follows.

1. A straight line fits upon itself in all its positions. By this I mean that during the revolution of the surface containing it the straight line does not change its place if it goes through two unmoving points in the surface: (*i. e.*, if we turn the surface containing it about two points of the line, the line does not move.)

2. Two straight lines can not intersect in two points.

3. A straight line sufficiently produced both ways must go out beyond all bounds, and in such way cuts a bounded plain into two parts.

4. Two straight lines perpendicular to a third never intersect, how far soever they be produced.

5. A straight line always cuts another in going from one side of it over to the other side: (*i. e.*, one straight line must cut another if it has points on both sides of it.)

6. Vertical angles, where the sides of one are productions of the sides of the other, are equal. This holds of plane rectilineal angles among themselves, as also of plane surface angles: (*i. e.*, dihedral angles.)

7. Two straight lines can not intersect, if a third cuts them at the same angle.

8. In a rectilineal triangle equal sides lie opposite equal angles, and inversely.

9. In a rectilineal triangle, a greater side lies opposite a greater angle. In a right-angled triangle the hypothenuse is greater than either of the other sides, and the two angles adjacent to it are acute.

10. Rectilineal triangles are congruent if they have a side and two angles equal, or two sides and the included angle equal, or two sides and the angle opposite the greater equal, or three sides equal.

11. A straight line which stands at right angles upon two other straight lines not in one plane with it is perpendicular to all straight lines drawn through the common intersection point in the plane of those two.

12. The intersection of a sphere with a plane is a circle.

13. A straight line at right angles to the intersection of two perpendicular planes, and in one, is perpendicular to the other.

14. In a spherical triangle equal sides lie opposite equal angles, and inversely.

15. Spherical triangles are congruent (or symmetrical) if they have two sides and the included angle equal, or a side and the adjacent angles equal.

From here follow the other theorems with their explanations and proofs.

16. All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes — into *cutting* and *not-cutting*.

The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line.*

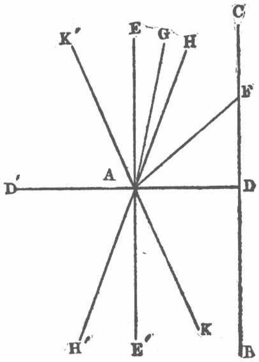

FIG. 1.

From the point A (Fig. 1) let fall upon the line BC the perpendicular AD, to which again draw the perpendicular AE.

In the right angle EAD either will all straight lines which go out from the point A meet the line DC, as for example AF, or some of them, like the perpendicular AE, will not meet the line DC. In the uncertainty whether the perpendicular AE is the only line which does not meet DC, we will assume it may be possible that there are still other lines, for example AG, which do not cut DC, how far soever they may be prolonged. In passing over from the cutting lines, as AF, to the not-cutting lines, as AG, we must come upon a line AH, parallel to DC, a boundary line, upon one side of which all lines AG are such as do not meet the line DC, while upon the other side every straight line AF cuts the line DC.

The angle HAD between the parallel HA and the perpendicular AD is called the parallel angle (angle of parallelism), which we will here designate by Π (p) for AD $=$ p.

If Π (p) is a right angle, so will the prolongation AE′ of the perpendicular AE likewise be parallel to the prolongation DB of the line DC, in addition to which we remark that in regard to the four right angles, which are made at the point A by the perpendiculars AE and AD, and their prolongations AE′ and AD′, every straight line which goes out from the point A, either itself or at least its prolongation, lies in one of the two right angles which are turned toward BC, so that except the parallel EE′ all others, if they are sufficiently produced both ways, must intersect the line BC.

If Π (p) $< \frac{1}{2}\pi$, then upon the other side of AD, making the same angle DAK $= \Pi$ (p) will lie also a line AK, parallel to the prolongation DB of the line DC, so that under this assumption we must also make a distinction of *sides in parallelism.*

All remaining lines or their prolongations within the two right angles turned toward BC pertain to those that intersect, if they lie within the angle HAK $= 2\ \Pi$ (p) between the parallels; they pertain on the other hand to the non-intersecting AG, if they lie upon the other sides of the parallels AH and AK, in the opening of the two angles EAH $= \frac{1}{2}\pi - \Pi$ (p), E'AK $= \frac{1}{2}\pi - \Pi$ (p), between the parallels and EE' the perpendicular to AD. Upon the other side of the perpendicular EE' will in like manner the prolongations AH' and AK' of the parallels AH and AK likewise be parallel to BC; the remaining lines pertain, if in the angle K'AH', to the intersecting, but if in the angles K'AE, H'AE' to the non-intersecting.

In accordance with this, for the assumption Π (p) $= \frac{1}{2}\pi$. the lines can be only intersecting or parallel; but if we assume that Π (p) $< \frac{1}{2}\pi$, then we must allow two parallels, one on the one and one on the other side; in addition we must distinguish the remaining lines into non-intersecting and intersecting.

For both assumptions it serves as the mark of parallelism that the line becomes intersecting for the smallest deviation toward the side where lies the parallel, so that if AH is parallel to DC, every line AF cuts DC, how small soever the angle HAF may be.

17. *A straight line maintains the characteristic of parallelism at all its points.*

Given AB (Fig. 2) parallel to CD, to which latter AC is perpendicular. We will consider two points taken at random on the line AB and its production beyond the perpendicular.

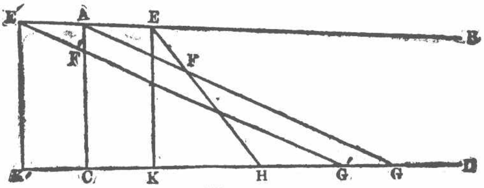

FIG. 2.

Let the point E lie on that side of the perpendicular on which AB is looked upon as parallel to CD.

Let fall from the point E a perpendicular EK on CD and so draw EF that it falls within the angle BEK.

Connect the points A and F by a straight line, whose production then (by Theorem 16) must cut CD somewhere in G. Thus we get a triangle ACG, into which the line EF goes; now since this latter, from the construction, can not cut AC, and can not cut AG or EK a second time (Theorem 2), therefore it must meet CD somewhere at H (Theorem 3).

Now let E′ be a point on the production of AB and E′K′ perpendicular to the production of the line CD; draw the line E′F′ making so small an angle AE′F′ that it cuts AC somewhere in F′; making the same angle with AB, draw also from A the line AF, whose production will cut CD in G (Theorem 16.)

Thus we get a triangle AGC, into which goes the production of the line E′F′; since now this line can not cut AC a second time, and also can not cut AG, since the angle BAG = BE′G′, (Theorem 7), therefore must it meet CD somewhere in G′.

Therefore from whatever points E and E′ the lines EF and E′F′ go out, and however little they may diverge from the line AB, yet will they always cut CD, to which AB is parallel.

18. *Two lines are always mutually parallel.*

Let AC be a perpendicular on CD, to which AB is parallel if we draw from C the line CE making any acute angle ECD with CD, and let fall from A the perpendicular AF upon CE, we obtain a right-angled triangle ACF, in which AC, being the hypothenuse, is greater than the side AF (Theorem 9.)

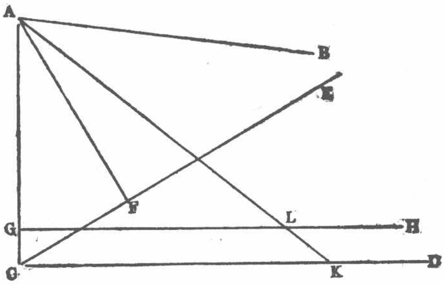

FIG. 3.

Make AG = AF, and slide the figure EFAB until AF coincides with AG, when AB and FE wil take the position AK and GH, such that the angle BAK = FAC, consequently AK must cut the line DC somewhere in K (Theorem 16), thus forming a triangle AKC, on one side of which the perpendicular GH intersects the line AK in L (Theorem 3), and thus determines the distance AL of the intersection point of the lines AB and CE on the line AB from the point A.

Hence it follows that CE will always intersect AB, how small soever may be the angle ECD, consequently CD is parallel to AB (Theorem 16.)

19. *In a rectilineal triangle the sum of the three angles can not be greater than two right angles.*

Suppose in the triangle ABC (Fig. 4) the sum of the three angles is equal to $\pi + a$; then choose in case of the inequality of the sides the smallest BC, halve it in D, draw from A through D the line AD and make the prolongation of it, DE, equal to AD, then join the point E to the point C by the straight line EC. In the congruent triangles ADB and CDE, the angle ABD = DCE, and BAD = DEC (Theorems 6 and 10); whence follows that also in the triangle ACE the sum of the three angles must be equal to $\pi + a$; but also the smallest angle BAC (Theorem 9) of the triangle ABC in passing over into the new triangle ACE has been cut up into the two parts EAC and AEC. Continuing this process, continually

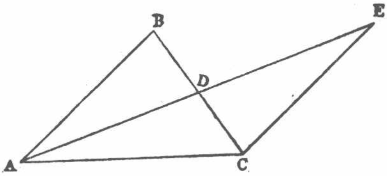

FIG. 4.