ELEMENTARY MATRICES

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AND SOME APPLICATIONS TO DYNAMICS AND DIFFERENTIAL EQUATIONS

by

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PREFACE

The number of published treatises on matrices is not large, and so far as we are aware this is the first which develops the subject with special reference to its applications to differential equations and classical mechanics. The book is written primarily for students of applied mathematics who have no previous knowledge of matrices, and we hope that it will help to bring about a wider appreciation of the conciseness and power of matrices and of their convenience in computation. The general scope of the book is elementary, but occasional discussions of advanced questions are not avoided. The sections containing these discussions, which may with advantage be omitted at the first reading, are distinguished by an asterisk.

The first four chapters give an account of those properties of matrices which are required later for the applications. Chapters I to III introduce the general theory of matrices, while Chapter IV is devoted to various numerical processes, such as the reciprocation of matrices, the solution of algebraic equations, and the calculation of latent roots of matrices by iterative methods.

The remainder of the book is concerned with applications. Chapters V and VI deal in some detail with systems of linear ordinary differential equations with constant coefficients, and Chapter VII contains examples of numerical solutions of systems of linear differential equations with variable coefficients. The last six chapters take up the subject of mechanics. They include an account of the kinematics and dynamics of systems, a separate discussion of motions governed by linear differential equations, illustrations of iterative methods of numerical solution, and a treatment of simple dynamical systems involving solid friction. The part played by friction in the motions of dynamical systems is as yet very incompletely understood, and we have considered it useful to include a very brief description of some experimental tests of the theory.

A considerable number of worked numerical examples has been included. It is our experience that the practical mathematician, whose requirements we have mainly considered, is often able to grasp the significance of a general algebraic theorem more thoroughly when it is illustrated in terms of actual numbers. For examples of applications of dynamical theory we have usually chosen problems relating to the oscillations of aeroplanes or aeroplane structures. Such problems conveniently illustrate the properties of dissipative dynamical systems, and they have a considerable practical importance.

A word of explanation is necessary in regard to the scheme of numbering adopted for paragraphs, equations, tables, and diagrams. The fourth paragraph of Chapter I, for example, is denoted by § 1·4. The two equations introduced in § 1·4 are numbered (1) and (2), but when it is necessary in later paragraphs to refer back to these equations they are described, respectively, as equations (1·4·1) and (1·4·2). Tables and diagrams are numbered in each paragraph in serial order: thus, the two consecutive tables which appear in § 7·13 are called Tables 7·13·1 and 7·13·2, while the single diagram introduced is Fig. 7·13·1.

The list of references makes no pretence to be complete, and in the case of theorems which are now so well established as to be almost classical, historical notices are not attempted. We believe that much of the subject-matter—particularly that relating to the applications—presents new features and has not appeared before in text-books. However, in a field so extensive and so widely explored as the theory of matrices, it would be rash to claim complete novelty for any particular theorem or method.

The parts of the book dealing with applications are based very largely on various mathematical investigations carried out by us during the last seven years for the Aeronautical Research Committee. We wish to express our great indebtedness to that Committee and to the Executive Committee of the National Physical Laboratory for permission to refer to, and expand, a number of unpublished reports, and for granting many other facilities in the preparation of the book. We wish also to record our appreciation of the care which the Staff of the Cambridge University Press has devoted to the printing.

Our thanks are also due to Miss Sylvia W. Skan of the Aerodynamics Department of the National Physical Laboratory for considerable assistance in computation and in the reading of proofs.

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ADDENDA ET CORRIGENDA

p. 120, 2nd table, 2nd row, for "$r_6 - 2r_9$" read "$r_6 - 3r_9$".

p. 144, para. beginning at line 8 should read

"If there are p distinct dominant roots $\lambda_1 \lambda_2 \dots \lambda_p$ and if $\kappa_1 \kappa_2 \dots \kappa_p$ are the corresponding modal rows, the procedure is as follows. Partition the $(\bar{p}, n)$ matrix $\{\kappa_1, \kappa_2, \dots, \kappa_p\}$ in the form $[\alpha, \beta]$, where α is a $(\bar{p}, p)$ submatrix, assumed to be non-singular (rearrangement of the rows of u and columns of $[\alpha, \beta]$ may be necessary to satisfy this condition). In this case the required matrix w is constructed in the partitioned form

$$w = \begin{bmatrix} I, & \alpha^{-1}\beta \\ 0, & 0 \end{bmatrix}$$

and then
$$v = u(I - w) = u\begin{bmatrix} 0, & -\alpha^{-1}\beta \\ 0, & I \end{bmatrix}.$$

Evidently v has p zero columns and hence p zero latent roots. If rearrangement has been required u must be in the corresponding rearranged form.

The choice of a non-singular submatrix α is a generalization of the choice of a non-zero element κ_{r1} in the elimination of a single dominant root.

This process is in effect that which is applied in the numerical example on p. 330."

p. 176, equation (4), denominator of third fraction, for

$$\text{“}\Delta^{(1)}(\lambda_r)(\lambda-\lambda_r)\text{” read “}\lambda_r\Delta^{(1)}(\lambda_r)(\lambda-\lambda_r)\text{”.}$$

p. 252, equation at bottom, interchange first and third matrices on the right-hand side.

p. 277. It is to be noted that in the definition of x at line 8, α is used to denote a set of parameters. Thereafter α denotes the components of acceleration.

p. 291, § 9·9. The following is a simple alternative proof of the reality of the roots of the determinantal equation $\Delta_m(z) = 0$ when A and E are real and symmetrical.

Let z, k respectively denote any root and its associated modal column, and let $\bar{z}$, $\bar{k}$ be the corresponding conjugates (see § 1·17). Then

$$zAk = Ek. \qquad \text{......(1)}$$

Premultiplication by $\bar{k}'$ yields

$$z\bar{k}'Ak = \bar{k}'Ek, \qquad \text{......(2)}$$

and by transposition

$$zk'A\bar{k} = k'E\bar{k}.$$

The conjugate relation is

$$\bar{z}\bar{k}'Ak = \bar{k}'Ek. \qquad \text{......(3)}$$

Comparison of (2) and (3) gives $z = \bar{z}$, which shows that z is real. Thus by (1) k is real, and by (2) z is positive when the potential energy function is positive and definite.

p. 310, § 10·2 (e), second sentence should read "The principle shows that first order errors in the mode yield only second order errors in the frequency as calculated by the equation of energy".

Also line 10 should read "used, and when U happens to be symmetrical, a convenient ...".

p. 315, line 9 from bottom, for "Rayleigh's principle will next be applied" read "Since U is symmetrical, the extension of Rayleigh's principle given in § 10·2 (e) can be applied ...".

p. 363, § 12·3, line 4, for "given" read "are given".

CHAPTER I

FUNDAMENTAL DEFINITIONS AND ELEMENTARY PROPERTIES

1·1. Preliminary Remarks. Matrices are sets of numbers or other elements which are arranged in rows and columns as in a double entry table and which obey certain rules of addition and multiplication. These rules will be explained in §§ 1·3, 1·4.

Rectangular arrays of numbers are of course very familiar in geometry and physics. For example, an ordinary three-dimensional vector is represented by three numbers called its components arranged in one row, while the state of stress at a point in a medium can be represented by nine numbers arranged in three rows and three columns. However, two points must be emphasised in relation to matrices. Firstly, the idea of a matrix implies the treatment of its elements taken as a whole and in their proper arrangement. Secondly, matrices are something more than the mere arrays of their elements, in view of the rules for their addition and multiplication.

1·2. Notation and Principal Types of Matrix. (*a*) *Rectangular Matrices.* The usual method of representing a matrix is to enclose the array of its elements within brackets, and in general square brackets are used for this purpose.* For instance, the matrix formed from the array

$$\begin{matrix} 1 & 12 & 0 \\ 5 & 6 & 1 \end{matrix}$$

is represented by

$$\begin{bmatrix} 1 & 12 & 0 \\ 5 & 6 & 1 \end{bmatrix}.$$

The meaning of other special brackets will be explained later. If a matrix contains lengthy numbers or complicated algebraic expressions, the elements in the rows can be shown separated by commas to avoid confusion.

The typical element of a matrix such as

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

* Some writers employ thick round brackets or double lines.

can be denoted by A_{ij}, where the suffices i and j are understood to range from 1 to m and from 1 to n, respectively. A convenient abbreviated notation for the complete matrix is then $[A_{ij}]$, but in cases where no confusion can arise it is preferable to omit the matrix brackets and the suffices altogether and to write the matrix simply as A.

The letters i, j are generally used in the sense just explained as suffices for a typical element of a matrix. Specific elements will generally have other suffices, such as m, n, r, s.

(*b*) *Order*. A matrix having m rows and n columns is said to be of order m by n. For greater brevity, such a matrix will usually be referred to as an $(\overline{m}, n)$ matrix; the bar shows which of the two numbers m, n relates to the rows.*

(*c*) *Column Matrices and Row Matrices*. A matrix having m elements arranged in a single column—namely, an $(\overline{m}, 1)$ matrix—will be called a *column matrix*. A column of numbers occupies much vertical space, and it is often preferable to adopt the convention that a single row of elements enclosed within braces represents a column matrix. For instance,

$$\{x_1, x_2, x_3\} \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A literal matrix such as the above can be written in the abbreviated form $\{x_i\}$.

In the same way a matrix with only a single row of elements will be spoken of as a *row matrix*.† When it is necessary to write a row matrix at length, the usual square brackets will be employed; but the special brackets $\lfloor\ \rfloor$ will be used to denote a literal row matrix in the abbreviated form. For example,

$$\lfloor y_j \rfloor \equiv [y_1, y_2, y_3].$$

In accordance with the foregoing conventions, the matrix formed from the rth column of an $(\overline{m}, n)$ matrix $[A_{ij}]$ is

$$\{A_{1r}, A_{2r}, \dots, A_{mr}\},$$

and this can be represented as $\{A_{ir}\}$, provided that i is always taken to be the typical suffix and r the specific suffix. In the same way the matrix formed from the sth row of $[A_{ij}]$ is

$$[A_{s1}, A_{s2}, \dots, A_{sn}],$$

and this can be expressed as $\lfloor A_{sj} \rfloor$.

* An alternative notation, which is in current use, is $[A]_m^n$.

† A row matrix is often called a *line matrix*, a *vector of the first kind*, or a *prime*; while a column matrix is referred to as a *vector of the second kind*, or a *point*.