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Polynomial Identities and Combinatorial Methods

edited by

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Preface

This volume contains the proceedings of the conference on Polynomial Identities and Combinatorial Methods, held on the island of Pantelleria, Italy. It was the fourth in a series of meetings in the last decade concerning the theory of associative and nonassociative algebras satisfying polynomial identities (PI-algebras). The first of these meetings was a small workshop in Palermo, Italy, in 1992. The second was the conference entitled Methods in Ring Theory, held in 1997 in Trento, Italy. The proceedings of that conference were published in the Marcel Dekker series Lecture Notes in Pure and Applied Mathematics, Volume 198, and it is now a standard reference for specialists working in the area of polynomial identities.

Considerable progress could be observed in the theory of algebras with polynomial identities during the years following the Trento conference. Some of the most important achievements in this area were due to a combination of algebraic techniques with analytical and combinatorial methods in the study of various numerical characteristics of PI-algebras and their identities. Leading specialists in this area met in Rehovot, Israel, in May 2000, in the Workshop on Growth Phenomena in Associative and Lie PI-algebras. Unfortunately, the results presented in Rehovot were not collected in one volume. To make up for this deficiency we hereby offer the proceedings of the Pantelleria conference, presenting the up-to-date status and tendencies in this area.

The conference featured the latest results in the theory of polynomial identities and a presentation of different methods and techniques pertaining to different areas, such as algebraic combinatorics, invariant theory, and representation theory, both of the symmetric and the classical groups, and of Lie algebras and superalgebras.

During the conference one-hour invited lectures were given by Y. Bahturin, A. Belov, O. Di Vincenzo, M. Domokos, V. Drensky, E. Formanek, A. Giambruno, A. Guterman, P. Koshlukov, S. Mishchenko, V. Petrogradsky, C. Procesi, A. Regev, L. H. Rowen, I. Shestakov, and M. Zaicev. In addition, several other invited talks of shorter lengths were presented. This volume includes the papers of most of the principal speakers and some other invited contributions related to the conference.

Even though the contents of this volume cover a broad range of themes, from ring theory to combinatorics to invariant theory, they still have a common thread in the theory of polynomial identities. The book will be useful to all researchers working with polynomial identities, varieties of associative algebras, Lie and Liebnitz algebras, and their generalizations. It will also be of interest to specialists in free algebras, growth functions of algebras, and, more generally, to mathematicians who apply numerical and analytical methods in algebra.

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Contents

Linearization method of computing $\mathbf{Z}_2$ - codimensions of identities of the Grassmann algebra

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1 INTRODUCTION

Let A be an associative algebra over a field F of characteristic 0, and let G be a finite group of automorphisms and anti-automorphisms of A. Also let $X = \{x_1, x_2, \ldots\}$ be a countable set of indeterminates. The free algebra with the set of free generators $< X|G >= \{x_i^g | i \in \mathbf{N}, g \in G\}$ is called the algebra of G-polynomials. The space

$$V_n(x, G) = \text{span}_F\{x_{\sigma(1)}^{g_{\sigma(1)}} \ldots x_{\sigma(n)}^{g_{\sigma(n)}} | \sigma \in S_n, g \in G\}$$

is called the space of multilinear G-polynomials in the variables $x_1, \ldots, x_n$.

The polynomial $f(x_1, \ldots, x_n, G) \in V_n(x, G)$ is called an n-linear G-identity if for arbitrary elements $a_1, \ldots, a_n$ of A, $f(a_1, \ldots, a_n, G) = 0$; here we use the notation $a^g = g(a)$ for all $a \in A$, $g \in G$. All n-linear G-identities form the ideal $\text{Id}_n(A, G)$ of n-linear G-identities. The sequence

$$c_n(A, G) = \dim_F \frac{V_n(x|G)}{\text{Id}_n(x, G)}$$

is called the sequence of G-codimensions of the ideal of the G-identities of the algebra A. If the group G is generated by an element φ one uses the notation of "φ-identities" instead of "G-identities". We remind that the basic notions of PI theory were introduced by A. Regev in [1] and then generalized by A. Giambruno and A. Regev in [2].

Properties of such sequences were widely investigated only in the traditional case $G = \{\mathrm{id}\}$; the general case is very incomplete. Moreover, there are only a few algebras whose sequence of codimensions of identities have been computed exactly. One of these algebras is the infinite-dimensional Grassmann algebra Λ, $c_n(\Lambda) = 2^{n-1}$ (see [3]). Note that the Grassmann algebra is of fundamental importance in PI theory, for example, it generates a minimal variety of exponential growth. Questions that arise naturally are: to compute the involutive and the $\mathbf{Z}_2$-codimensions of the identities of the Grassmann algebra, and to describe the ideals of $\mathbf{Z}_2$-identities of Λ. These questions were partially answered in [4]. For example, the sequence of involutive codimensions of the Grassmann algebra was computed for an arbitrary involution. But the general case of $\mathbf{Z}_2$-action on Λ was investigated in [4] only for automorphisms of order two with linear action on generators. In this article we point out the deep connection between arbitrary $\mathbf{Z}_2$-codimensions of identities and $\mathbf{Z}_2$-codimensions of identities for linear automorphisms which were computed in [4]. The statements of the theorems of Section 2 contain an additional assumption on the automorphism. By computing the structure of some automorphisms of the Grassmann algebra we show in Section 3 that the additional assumption is natural for such automorphisms.

We now list some definitions and properties used later. Let Λ be the infinite-dimensional Grassmann algebra with generators $e_1, e_2, \dots$ and defining relations $e_ie_j + e_je_i = 0$. Then the set of all ordered monomials $\{e_{i_1} \dots e_{i_k} | k \geq 1, 1 \leq i_1 < \dots < i_k\}$ form a basis D_Λ of Λ. There is a natural $\mathbf{Z}_2$-grading on Λ, $\Lambda = \Lambda_0 \oplus \Lambda_1$, where Λ_0 and Λ_1 are spanned by the basic monomials of even and of odd length correspondingly. The $\mathbf{Z}_2$-graded elements of Λ commute by the following rule: if $g \in \Lambda_i$ and $h \in \Lambda_j$, then $gh = (-1)^{ij}hg$. The length $|a|$ of the monomial $a \in D_\Lambda$ is the number of generators in a: $|e_{i_1} \dots e_{i_k}| = k$.

Let φ be an automorphism of order two on the algebra Λ. If it's action on the generators is defined by the formula

$$\varphi(e_i) = \sum_j \alpha_{ji} e_j + \sum_{\substack{a_m \in D_\Lambda \\ |a_m| \geq 2}} \beta_{mi} a_m,$$

then the mapping defined on the generators by the formula

$$\varphi_l(e_i) = \sum_j \alpha_{ji} e_j$$

and homomorphically continued on Λ is also an automorphism of order two on the Grassmann algebra. The automorphism φ_l is a linear operator in the space $L = \text{span}_F\{e_1, e_2, \ldots\}$. In further considerations we will assume without loss of generality that the generators of the Grassmann algebra form the eigenbasis for the operator φ_l in the space L. The subspaces corresponding to the eigenvalues 1 and -1 are denoted by L_1 and L_{-1}.

2 Z_2-IDENTITIES OF THE GRASSMANN ALGEBRA

First we note that for an arbitrary automorphism φ of the Grassmann algebra Λ with $\dim L_1 = \dim L_{-1} = \infty$ the φ-codimensions of the identities of Λ were computed in [4]. But $c_n(\Lambda, \varphi_l)$ are known precisely only if one of the eigenspaces L_1 or L_{-1} is finite-dimensional. In this section the coincidence of $\text{Id}_n(\Lambda, \varphi)$ and $\text{Id}_n(\Lambda, \varphi_l)$ (and of corresponding codimensions) is proved, provided there are some additional assumption on φ.

LEMMA 2.1 *Let $\varphi : \Lambda \to \Lambda$ be a graded automorphism of the Grassmann algebra and let k be a natural number. If for any k generators $e_{i_1}, \ldots, e_{i_k}$*

$$\prod_{j=1}^{k} (\varphi(e_{i_j}) - e_{i_j}) = 0,$$

then for any basic monomials $a_{i_1}, \ldots, a_{i_k} \in D_\Lambda$

$$\prod_{j=1}^{k} (\varphi(a_{i_j}) - a_{i_j}) = 0.$$

Proof. Consider arbitrary basic monomials $a_{i_1}, \ldots, a_{i_k} \in D_\Lambda$. We argue by induction on the total length $\sum_{j=1}^{k} |a_{i_j}|$ of the chosen monomials. The base of induction follows from the condition of the lemma for k arbitrary generators. Next assume $\sum_{j=1}^{k} |a_{i_j}| = m > k$ and that the statement of the lemma is true for all sets of basic monomials with total length less than m. Since $m > k$, at least one of the chosen monomials $a_{i_1}, \ldots, a_{i_k}$ is of a non-unit length. Since φ is a graded automorphism we may assume, without loss of generality, that $a_{i_1} = e_{i_1} a'_{i_1}$ is of non-unit length. Then

$$\prod_{j=1}^{k} (\varphi(a_{i_j}) - a_{i_j}) = (\varphi(e_{i_1} a'_{i_1}) - e_{i_1} a'_{i_1}) \prod_{j=2}^{k} (\varphi(a_{i_j} - a_{i_j}) =$$

$$= ((\varphi(e_{i_1}) - e_{i_1} + e_{i_1})(\varphi(a'_{i_1}) - a'_{i_1} + a'_{i_1}) - e_{i_1} a'_{i_1}) \prod_{j=2}^{k} (\varphi(a_{i_j}) - a_{i_j}) =$$

$$= \big(\varphi(e_{i_1})(\varphi(a'_{i_1}) - a'_{i_1}) + (\varphi(e_{i_1}) - e_{i_1})a'_{i_1}\big) \prod_{j=2}^{k} (\varphi(a_{i_j}) - a_{i_j}).$$

Since $|a'_{i_1}| + \sum_{j=2}^{k} |a_{i_j}| = m - 1 < m$, therefore by induction

$$(\varphi(a'_{i_1}) - a'_{i_1}) \prod_{j=2}^{k} (\varphi(a_{i_j}) - a_{i_j}) = 0.$$

Since φ is a graded automorphism, the element $\varphi(e_{i_1}) - e_{i_1}$ is also graded and hence commutes (or anticommutes) with the monomial a'_{i_1}. So finally we obtain

$$\prod_{j=1}^{k} (\varphi(a_{i_j}) - a_{i_j}) = (\varphi(e_{i_1}) - e_{i_1})a'_{i_1} \prod_{j=2}^{k} (\varphi(a_{i_j}) - a_{i_j}) =$$

$$= \pm a'_{i_1} (\varphi(e_{i_1}) - e_{i_1}) \prod_{j=2}^{k} (\varphi(a_{i_j}) - a_{i_j}) = 0$$

since $1 + \sum_{j=2}^{k} |a_{i_j}| < m$ and so we may apply the inductive assumption to the set of monomials $e_{i_1}, a_{i_2}, \ldots, a_{i_k}$.

□

THEOREM 2.1 *Let $\varphi : \Lambda \to \Lambda$ be a graded automorphism of order two of the Grassmann algebra Λ, let $\dim L_{-1} = l < \infty$ and assume that for any $l+1$ generators $e_{i_1}, \ldots, e_{i_{l+1}}$,*

$$\prod_{j=1}^{l+1} (\varphi(e_{i_j}) - e_{i_j}) = 0. \tag{1}$$

Then

$$\mathrm{Id}_n(\Lambda, \varphi) \cong \mathrm{Id}_n(\Lambda, \varphi_l),$$

$$c_n(\Lambda, \varphi) = c_n(\Lambda, \varphi_l) = \begin{cases} 4^{n-\frac{1}{2}}, & n \le l; \\ 2^{n-1} \sum_{j=0}^{l} C_n^j, & n > l. \end{cases}$$

Proof. For $n \le l$ the statement of the theorem follows from the chain of inequalities

$$4^{n-\frac{1}{2}} \overset{[3]}{=} 2^n c_n(\Lambda) \overset{[2]}{\ge} c_n(\Lambda, \varphi) \overset{[4]}{\ge} c_n(\Lambda, \varphi_l) \overset{[4]}{=} 4^{n-\frac{1}{2}}. \tag{2}$$

Now suppose $n > l$. In this case we prove the equality $c_n(\Lambda,\varphi) = c_n(\Lambda,\varphi_l)$ by proving the isomorphism of ideals $\mathrm{Id}_n(\Lambda,\varphi) \cong \mathrm{Id}_n(\Lambda,\varphi_l)$. Let $f \in V_n(x,\varphi)$, then:

$$f = \sum_{\substack{\sigma\in S_n \\ \vec{g}=(g_1,\dots,g_n)\in \mathbf{Z}_2^n}} \alpha_{\sigma,\vec{g}} x_{\sigma(1)}^{g_{\sigma(1)}} \dots x_{\sigma(n)}^{g_{\sigma(n)}}, \tag{3}$$

where $x_i^{g_i} = \begin{cases} x_i, & g_i = 0; \\ x_i^{\varphi}, & g_i = 1. \end{cases}$ We define the isomorphism $\tau : V_n(x,\varphi) \to V_n(x,\varphi_l)$ by the equality $\tau(x_i^{\varphi}) = x_i^{\varphi_l}$ and prove that if $(\tau f) \in \mathrm{Id}_n(\Lambda,\varphi_l)$ then $f \in \mathrm{Id}_n(\Lambda,\varphi)$. The inverse inclusion of these ideals of identities was proved in [4].

Choose a set $a_1,\dots,a_n$ of n arbitrary basic monomials of the Grassmann algebra Λ and recall some characteristics introduced in [4] for every such set of monomials:

- $I = I(a_1,\dots,a_n) = (i_1,\dots,i_n) \in \mathbf{Z}_2^n$ is called "the set of evens" for $a_1,\dots,a_n$ if $i_k = \begin{cases} 0, & |a_k| \text{ is even;} \\ 1, & |a_k| \text{ is odd;} \end{cases}$
- $J = J(a_1,\dots,a_n) = (j_1,\dots,j_n) \in \mathbf{Z}_2^n$ is called "the set of signs" for $a_1,\dots,a_n$ if j_k is defined by the formula $\varphi_l(a_k) = (-1)^{j_k} a_k$, $k = 1,\dots,n$;
- $f_I^{(n)}(\sigma)$ is defined by the equation

$$a_{\sigma(1)} \dots a_{\sigma(n)} = f_I^{(n)}(\sigma) a_1 \dots a_n; \tag{4}$$

- $g_J^{(n)}(g_1,\dots,g_n)$ is defined by the equation

  $$(\tau(x_{\sigma(1)}^{g_{\sigma(1)}} \dots x_{\sigma(n)}^{g_{\sigma(n)}}))(a_1,\dots,a_n) = g_J^{(n)}(g_1,\dots,g_n) a_{\sigma(1)} \dots a_{\sigma(n)}$$

  and may be computed by the formula

  $$g_{(j_1,\dots,j_n)}^{(n)}(g_1,\dots,g_n) = (-1)^{\sum_{k=1}^n j_k g_k}.$$

In [4] it was proved that $(\tau f) \in \mathrm{Id}_n(\Lambda,\varphi_l)$ if and only if

$$\sum_{\substack{\sigma\in S_n \\ \vec{g}=(g_1,\dots,g_n)\in \mathbf{Z}_2^n}} \Big(f_I^{(n)}(\sigma) g_J^{(n)}(g_1,\dots,g_n)\Big) \alpha_{\sigma,\vec{g}} = 0 \tag{5}$$

for all $(\sigma,\vec{g}) \in S_n \times \mathbf{Z}_2^n$, all $I \in \mathbf{Z}_2^n$ and all $J = (j_1,\dots,j_n)$ such that $\sum_{k=1}^n j_k \le l$.

Transform the value of the basic monomial $x_{\sigma(1)}^{g_{\sigma(1)}} \dots x_{\sigma(n)}^{g_{\sigma(n)}}$ of the space $V_n(x, \varphi)$, computed on the monomials $a_1, \dots, a_n$. Introduce the notation

$$a_i^{(h_i)} = \begin{cases} a_i, & h_i = 0; \\ \varphi(a_i) - a_i, & h_i = 1. \end{cases}$$

Let $g_{\sigma(i_1)} = \dots = g_{\sigma(i_k)} = 1$, and the other g_i are equal to 0. We also use a natural lexicographic partial order on binary sequences: $\vec{h} = (h_1, \dots, h_n) \leq (g_1, \dots, g_n) = \vec{g}$ if $h_i \leq g_i$ for all $i = 1, \dots, n$. Now we transform

$$a_{\sigma(1)}^{g_{\sigma(1)}} \dots a_{\sigma(n)}^{g_{\sigma(n)}} = a_{\sigma(1)} \dots (\varphi(a_{\sigma(i_1)}) - a_{\sigma(i_1)} + a_{\sigma(i_1)}) \dots$$

$$\dots (\varphi(a_{\sigma(i_k)}) - a_{\sigma(i_k)} + a_{\sigma(i_k)}) \dots a_{\sigma(n)} = a_{\sigma(1)} \dots a_{\sigma(n)} +$$

$$+ \sum_{\substack{\vec{h} \leq \vec{g} \\ \sum_{i=1}^n h_i = 1}} a_{\sigma(1)}^{(h_{\sigma(1)})} \dots a_{\sigma(n)}^{(h_{\sigma(n)})} + \dots + \sum_{\substack{\vec{h} \leq \vec{g} \\ \sum_{i=1}^n h_i = k}} a_{\sigma(1)}^{(h_{\sigma(1)})} \dots a_{\sigma(n)}^{(h_{\sigma(n)})}. \tag{6}$$

From condition (1) of the theorem and from Lemma 2.1 we obtain that for any natural number $m > l$,

$$\prod_{i=1}^{m} a_i^{(1)} = 0.$$

Since φ is a graded automorphism, one may continue Equation (6) as follows:

$$a_{\sigma(1)}^{g_{\sigma(1)}} \dots a_{\sigma(n)}^{g_{\sigma(n)}} = \sum_{k=0}^{\min\{l, \sum g_i\}} \sum_{\substack{\vec{h} \leq \vec{g} \\ \sum_{i=1}^n h_i = k}} a_{\sigma(1)}^{(h_{\sigma(1)})} \dots a_{\sigma(n)}^{(h_{\sigma(n)})}.$$

We substitute the above expression in the valuation of the polynomial f computed on the monomials $a_1, \dots, a_n$:

$$f(a_1, \dots, a_n) = \sum_{\sigma, \vec{g}} \alpha_{\sigma, \vec{g}} a_{\sigma(1)}^{g_{\sigma(1)}} \dots a_{\sigma(n)}^{g_{\sigma(n)}} =$$

$$= \sum_{\sigma, \vec{g}} \alpha_{\sigma, \vec{g}} \sum_{k=0}^{\min\{l, \sum g_i\}} \sum_{\substack{\vec{h} \leq \vec{g} \\ \sum_{i=1}^n h_i = k}} a_{\sigma(1)}^{(h_{\sigma(1)})} \dots a_{\sigma(n)}^{(h_{\sigma(n)})} =$$

$$= \sum_{k=0}^{l} \sum_{\substack{\sigma, \vec{g} \\ \sum_{i=1}^n g_i \geq k}} \alpha_{\sigma, \vec{g}} \sum_{\substack{\vec{h} \leq \vec{g} \\ \sum_{i=1}^n h_i = k}} a_{\sigma(1)}^{(h_{\sigma(1)})} \dots a_{\sigma(n)}^{(h_{\sigma(n)})}.$$

Since φ is a graded automorphism, for any i, $1 \le i \le n$, $a_i^{(1)}$ is a graded element and one may apply Equation (4) to $a_{\sigma(1)}^{(h_{\sigma(1)})} \dots a_{\sigma(n)}^{(h_{\sigma(n)})}$, which transforms the last equality as follows:

$$f(a_1, \dots, a_n) = \sum_{k=0}^{l} \sum_{\substack{\vec{h} \\ \sum_{i=1}^n h_i = k}} \Big(\sum_{\substack{\sigma \in S_n \\ \vec{g} \ge \vec{h}}} f_I^{(n)}(\sigma) \alpha_{\sigma,\vec{g}} \Big) a_1^{(h_1)} \dots a_n^{(h_n)}. \quad (7)$$

Prove that for any k, $0 \le k \le l$, and for any binary sequence $\vec{h} = (h_1, \dots, h_n)$ with $\sum_{i=1}^n h_i = k$,

$$\sum_{\substack{\sigma \in S_n \\ \vec{g} \ge \vec{h}}} f_I^{(n)}(\sigma) \alpha_{\sigma,\vec{g}} = 0, \quad (8)$$

using the system of Equations (5) on the coefficients $\alpha_{\sigma,\vec{g}}$ of the polynomial f. Consider the following linear combination of some equations of this system:

$$\sum_{\vec{f} \le \vec{h}} (-1)^{\sum_{i=1}^n f_i} \Big(\sum_{\sigma,\vec{g}} f_I^{(n)}(\sigma) g_{\vec{f}}^{(n)}(g_1, \dots, g_n) \alpha_{\sigma,\vec{g}} \Big), \quad (9)$$

where $I = I(a_1, \dots, a_n)$. Since $\sum_{i=1}^n h_i = k \le l$, therefore $\vec{f} \le \vec{h}$ holds $\sum_{i=1}^n f_i \le l$, and hence every term of the sum (9) (maybe multiplied by -1) is contained in the system of Equations (5). Hence the linear combination (9) is equal to 0. On the other hand this linear combination may be transformed as follows:

$$(9) = \sum_{\sigma,\vec{g}} \left(\sum_{\vec{f} \le \vec{h}} (-1)^{\sum f_i} g_{\vec{f}}^{(n)}(g_1, \dots, g_n) \right) f_I^{(n)}(\sigma) \alpha_{\sigma,\vec{g}}. \quad (10)$$

Fix $\vec{g}$ and σ and compute the coefficient of $f_I^{(n)}(\sigma)\alpha_{\sigma,\vec{g}}$. Without loss of generality we may assume that $h_1 = \dots = h_k = 1$, $h_{k+1} = \dots = h_n = 0$. Also we assume that $\vec{g}$ contains exactly r units on the first k entries and without loss of generality $g_1 = \dots = g_r = 1$. Also note that $g_{\vec{f}}^{(n)}(g_1, \dots, g_n) = (-1)^{\sum f_i g_i}$. Thus

$$\sum_{\vec{f} \le \vec{h}} (-1)^{\sum f_i} g_{\vec{f}}^{(n)}(g_1, \dots, g_n) = \sum_{\vec{f} \le \vec{h}} (-1)^{\sum_{i=1}^n f_i(g_i+1)} =$$

$$= \sum_{m=0}^{k} \sum_{\substack{\vec{f} \le \vec{h} \\ \sum f_i = m}} (-1)^{\sum_{i=1}^n f_i(g_i+1)}. \quad (11)$$