Lecture Notes in Mathematics

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Joseph Lipman
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Foundations of Grothendieck Duality for Diagrams of Schemes

$$\begin{array}{ccc}
\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F, f^!G) & \xrightarrow{\ \sim\ } & \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, G) \\
\downarrow & & \uparrow \\
\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*\mathbf{R}f_*F, f^!G) & \xrightarrow{\ \sim\ } & \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, \mathbf{R}f_*f^!G)
\end{array}$$

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Preface

This volume contains two related, though independently written, monographs.

In *Notes on Derived Functors and Grothendieck Duality* the first three chapters treat the basics of derived categories and functors, and of the rich formalism, over ringed spaces, of the derived functors, for unbounded complexes, of the sheaf functors $\otimes$, $\mathcal{H}om$, f_* and f^* where f is a ringed-space map. Included are some enhancements, for concentrated (i.e., quasi-compact and quasi-separated) schemes, of classical results such as the projection and Künneth isomorphisms. The fourth chapter presents the abstract foundations of Grothendieck Duality—existence and tor-independent base change for the right adjoint of the derived functor $\mathbf{R}f_*$ when f is a quasi-proper map of concentrated schemes, the twisted inverse image pseudofunctor for separated finite-type maps of noetherian schemes, refinements for maps of finite tor-dimension, and a brief discussion of dualizing complexes.

In *Equivariant Twisted Inverses* the theory is extended to the context of diagrams of schemes, and in particular, to schemes with a group-scheme action. An equivariant version of the twisted inverse-image pseudofunctor is defined, and equivariant versions of some of its important properties are proved, including Grothendieck duality for proper morphisms, and flat base change. Also, equivariant dualizing complexes are dealt with. As an application, a generalized version of Watanabe's theorem on the Gorenstein property of rings of invariants is proved.

More detailed overviews are given in the respective Introductions.

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Part I
Joseph Lipman: Notes on Derived Functors and Grothendieck Duality

Abstract

This is a polished version of notes begun in the late 1980s, largely available from my home page since then, meant to be accessible to mid-level graduate students. The first three chapters treat the basics of derived categories and functors, and of the rich formalism, over ringed spaces, of the derived functors, for unbounded complexes, of the sheaf functors $\otimes$, $\mathcal{H}om$, f_* and f^* (where f is a ringed-space map). Included are some enhancements, for concentrated (= quasi-compact and quasi-separated) schemes, of classical results such as the projection and Künneth isomorphisms. The fourth chapter presents the abstract foundations of Grothendieck Duality—existence and tor-independent base change for the right adjoint of the derived functor $\mathbf{R}f_*$ when f is a quasi-proper map of concentrated schemes, the twisted inverse image pseudofunctor for separated finite-type maps of noetherian schemes, some refinements for maps of finite tor-dimension, and a brief discussion of dualizing complexes.

Introduction

(0.1) The first three chapters of these notes[1] treat the basics of derived categories and functors, and of the formalism of four of Grothendieck's "six operations" ([**Ay**], [**Mb**]), over, say, the category of ringed spaces (topological spaces equipped with a sheaf of rings)—namely the derived functors, for complexes which need not be bounded, of the sheaf functors $\otimes$, $\mathcal{H}om$, and of the direct and inverse image functors f_* and f^* relative to a map f. The axioms of this formalism are summarized in §3.6 under the rubric *adjoint monoidal Δ-pseudofunctors,* with values in closed categories (§3.5).

Chapter 4 develops the abstract theory of the *twisted inverse image* functor $f^!$ associated to a finite-type separated map of schemes $f\colon X \to Y$. (Suppose for now that Y is noetherian and separated, though for much of what we do, weaker hypotheses will suffice.) This functor maps the derived category of cohomologically bounded-below $\mathcal{O}_Y$-complexes with quasi-coherent homology to the analogous category over X. Its characterizing properties are:

- *Duality.* If f is proper then $f^!$ is right-adjoint to the derived direct image functor $\mathbf{R}f_*$.
- *Localization.* If f is an open immersion (or even étale), then $f^!$ is the usual inverse image functor f^*.
- *Pseudofunctoriality* (or *2-functoriality*). To each composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ we can assign a natural functorial isomorphism $(gf)^! \xrightarrow{\sim} f^!g^!$, in such a way that a kind of associativity holds with respect to any composition of three maps, see §(3.6.5).

[1] That are a polished version of notes written largely in the late 1980s, available in part since then from < `www.math.purdue.edu/~lipman` >. I am grateful to Bradley Lucier for his patient instruction in some of the finer points of TEX, and for setting up the appearance macros in those days when canned style files were not common—and when compilation was several thousand times slower than nowadays.

J. Lipman, M. Hashimoto, *Foundations of Grothendieck Duality for Diagrams of Schemes*, Lecture Notes in Mathematics 1960.

Additional basic properties of $f^!$ are its compatibility with *flat base change* (Theorems (4.4.3), (4.8.3)), and the existence of canonical functorial maps, for $\mathcal{O}_Y$-complexes E and F having quasi-coherent homology:

$$\mathbf{R}\mathcal{H}om(\mathbf{L}f^*E, f^!F) \to f^!\mathbf{R}\mathcal{H}om(E, F)$$

$$\mathbf{L}f^*E \underline{\underline{\otimes}} f^!F \to f^!(E \underline{\underline{\otimes}} F)$$

(where $\underline{\underline{\otimes}}$ denotes the left-derived tensor product), of which the first is an isomorphism when E is cohomologically bounded above, with coherent homology, and F is cohomologically bounded below, (Exercise (4.9.3)(b)), and the second is an isomorphism whenever f has finite tor-dimension (Theorem (4.9.4)) or E is a bounded flat complex (Exercise (4.9.6)(a)).

The existence and uniqueness, up to isomorphism, of the twisted inverse image pseudofunctor is given by Theorem (4.8.1), and compatibility with flat base change by Theorem (4.8.3). These are culminating results in the notes. Various approximations to these theorems have been known for decades, see, e.g., [**H**, p. 383, 3.4]. At present, however, the proofs of the theorems, as stated here, seem to need, among other things, a compactification theorem of Nagata, that any finite-type separable map of noetherian schemes factors as an open immersion followed by a proper map, a fact whose proof was barely accessible before the appearance of [**Lt**] and [**C′**] (see also [**Vj**]); and even with that compactification theorem, I am not aware of any complete, detailed exposition of the proofs in print prior to the recent one by Nayak [**Nk**].[2] There must be a more illuminating treatment of this awesome pseudofunctor in the Plato-Erdös Book!

(0.2) The theory of $f^!$ was conceived by Grothendieck [**Gr′**, pp. 112–115], as a generalization of Serre's duality theorems for smooth projective varieties over fields. Grothendieck also applied his ideas in the context of étale cohomology. The fundamental technique of derived categories was developed by Verdier, who used it in establishing a duality theorem for locally compact spaces that generalizes classical duality theorems for topological manifolds. Deligne further developed the methods of Grothendieck and Verdier (cf. [**De′**] and its references).

Hartshorne gave an account of the theory in [**H**]. The method there is to treat separately several distinctive special situations, such as smooth maps, finite maps, and regular immersions (local complete intersections), where $f^!$ has a nice explicit description; and then to do the general case by pasting together special ones (locally, a general f can be factored as smooth ∘ finite). The fact that this approach works is indicative of considerable depth in the underlying structure, in that the special cases, that don't *a priori* have to

[2] In fact Nayak's methods, which are less dependent on compactifications, apply to other contexts as well, for example flat finitely-presentable separated maps of not-necessarily-noetherian schemes, or separated maps of noetherian formal schemes, see [**Nk**, §7]. See also the summary of Nayak's work in [**S′**, §§3.1–3.3].

be related at all, can in fact be melded; and in that the reduction from general to special involves several choices (for example, in the just-mentioned factorization) of which the final results turn out to be independent. Proving the existence of $f^!$ and its basic properties in this manner involves many compatibilities among those properties in their various epiphanies, a notable example being the "Residue Isomorphism" [**H**, p. 185]. The proof in [**H**] also makes essential use of a pseudofunctorial theory of dualizing complexes,[3] so that it does not apply, e.g., to arbitrary separated noetherian schemes.

On first acquaintance, [**De′**] appears to offer a neat way to cut through the complexity—a direct abstract proof of the existence of $f^!$, with indications about how to derive the concrete special situations (which, after all, motivate and enliven the abstract formalism). Such an impression is bolstered by Verdier's paper [**V′**]. Verdier gives a reasonably short proof of the flat base change theorem, sketches some corollaries (for example, the finite tor-dimension case is treated in half a page [*ibid.*, p. 396], as is the smooth case [*ibid.*, pp. 397–398]), and states in conclusion that "all the results of [**H**], except the theory of dualizing and residual complexes, are easy consequences of the existence theorem." In short, Verdier's concise summary of the main features, together with some background from [**H**] and a little patience, should suffice for most users of the duality machine.

Personally speaking, it was in this spirit—not unlike that in which many scientists use mathematics—that I worked on algebraic and geometric applications in the late 1970s and early 1980s. But eventually I wanted to gain a better understanding of the foundations, and began digging beneath the surface. The present notes are part of the result. They show, I believe, that there is more to the abstract theory than first meets the eye.

(0.3) There are a number of treatments of Grothendieck duality for the Zariski topology (not to mention other contexts, see e.g., [**Gl′**], [**De**], [**LO**]), for example, Neeman's approach via Brown representability [**N**], Hashimoto's treatment of duality for *diagrams* of schemes (in particular, schemes with group actions) [**Hsh**], duality for *formal* schemes [**AJL′**], as well as various substantial enhancements of material in Hartshorne's classic [**H**], such as [**C**], [**S**], [**LNS**] and [**YZ**]. Still, some basic results in these notes, such as Theorem (3.10.3) and Theorem (4.4.1) are difficult, if not impossible, to find elsewhere, at least in the present generality and detail. And, as indicated below, there are in these notes some significant differences in emphasis.

It should be clarified that the word "Notes" in the title indicates that the present exposition is neither entirely self-contained nor completely polished. The goal is, basically, to guide the willing reader along one path to an understanding of all that needs to be done to prove the fundamental Theorems (4.8.1) and (4.8.3), and of how to go about doing it. The intent is to provide enough in the way of foundations, yoga, and references so that the reader can,

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[3] This enlightening theory—touched on in §4.10 below—is generalized to Cousin complexes over formal schemes in [**LNS**]. A novel approach, via "rigidity," is given in [**YZ**], at least for schemes of finite type over a fixed regular one.

more or less mechanically, fill in as much of what is missing as motivation and patience allow.

So what is meant by "foundations and yoga"?

There are innumerable interconnections among the various properties of the twisted inverse image, often expressible via commutativity of some diagram of natural maps. In this way one can encode, within a formal functorial language, relationships involving higher direct images of quasi-coherent sheaves, or, more generally, of complexes with quasi-coherent homology, relationships whose treatment might otherwise, on the whole, prove discouragingly complicated.

As a strategy for coping with duality theory, disengaging the underlying category-theoretic skeleton from the algebra and geometry which it supports has the usual advantages of simplification, clarification, and generality. Nevertheless, the resulting fertile formalism of adjoint monoidal pseudofunctors soon sprouts a thicket of rather complicated diagrams whose commutativity is an essential part of the development—as may be seen, for example, in the later parts of Chapters 3 and 4. Verifying such commutativities, fun to begin with, soon becomes a tedious, time-consuming, chore. Such chores must, eventually, be attended to.[4]

Thus, these notes emphasize purely formal considerations, and attention to detail. On the whole, statements are made, whenever possible, in precise category-theoretic terms, canonical isomorphisms are not usually treated as equalities, and commutativity of diagrams of natural maps—a matter of paramount importance—is not taken for granted unless explicitly proved or straightforward to verify. The desire is to lay down transparently secure foundations for the main results. A perusal of §2.6, which treats the basic relation "adjoint associativity" between the derived functors $\underline{\otimes}$ and $\mathbf{R}\mathcal{H}om$, and of §3.10, which treats various avatars of the tor-independence condition on squares of quasi-compact maps of quasi-separated schemes, will illustrate the point. (In both cases, total understanding requires a good deal of preceding material.)

Computer-aided proofs are often more convincing
than many standard proofs based on
diagrams which are claimed to commute,
arrows which are supposed to be the same,
and arguments which are left to the reader.

—J.-P. Serre [**R**, pp. 212–213].

In practice, the techniques used to decompose diagrams successively into simpler ones until one reaches those whose commutativity is axiomatic do not seem to be too varied or difficult, suggesting that sooner or later a computer might be trained to become a skilled assistant in this exhausting task. (For the general idea, see e.g., [**Sm**].) Or, there might be found a theorem in

[4] Cf. [**H**, pp. 117–119], which takes note of the problem, but entices readers to relax their guard so as to make feasible a hike over the seemingly solid crust of a glacier.