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Jordan Triple Systems by the Grid Approach

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INTRODUCTION

Some background and definitions

In these notes we study Jordan triple systems. A prominent class of examples of Jordan triple systems are associative algebras. To consider an associative algebra A with a bilinear product $(x,y) \to xy$ as a Jordan triple system means to forget the bilinear product of A and its unit element and instead work just with the triple product $(x,y) \to P(x)y = xyx$ and its linearization $(x,y,z) \to \{xyz\} = xyz + zyx$. There are good reasons for doing this, some of which are indicated below. Of course, the associative law has to be rephrased in terms of the triple product, leading to the general definition due to K. Meyberg: A Jordan triple system consists of a module V over a commutative associative ring k and a quadratic map $P: V \to \mathrm{End}_k V$ (the triple product) satisfying the following identities in all scalar extensions:

(1) $L(x,y)P(x) = P(x)L(y,x)$

(2) $L(P(x)y,y) = L(x,P(y)x)$

(3) $P(P(x)y) = P(x)P(y)P(x)$,

where the linear map $L(x,y) \in \mathrm{End}_k V$ is defined by $L(x,y)z = \{xyz\} = P(x+z)y - P(x)y - P(z)y$.

Jordan triple systems received their name because they are generalizations of Jordan algebras (which in turn are generalizations of associative algebras): although this is not the usual definition, a Jordan algebra can be defined as a Jordan triple system which contains a unit element, i.e. an element e with $P(e) = \mathrm{Id}$. Over the past 20 years the theory of Jordan algebras has seen major advances mainly due to the work of N. Jacobson, K. McCrimmon and the Russian School, notably E.I. Zel'manov. An exposition of this has been given by N. Jacobson in his book [19] and his lecture notes [20] [21] and by K. McCrimmon in [41].

Besides Jordan algebras another class of Jordan triple systems, which has been thoroughly investigated in recent years, is the class of Jordan pairs. A Jordan pair is a Jordan triple system which has a polarization: V is the direct sum of two submodules, $V = V^+ \oplus V^-$, satisfying

$$P(V^\varepsilon)V^\varepsilon = 0 = P(V^+,V^-)V, \quad P(V^\varepsilon)V^{-\varepsilon} \subset V^\varepsilon.$$

The theory of Jordan pairs was proposed by K. Meyberg and pursued and described by O. Loos in his lecture notes [31].

One of the driving forces of the development of the Jordan pair and

Jordan triple theory was the increasing number of applications found in other areas of mathematics showing that Jordan triple systems are more than just generalizations of associative and Jordan algebras.

The most important of these applications was found by M. Koecher, the category of circled bounded symmetric domains in $\mathbb{C}^n$ is equivalent to a certain category of Jordan triple systems (namely finite-dimensional positive hermitian Hilbert triples in the terminology of these notes, see IV §2). An exposition of this fundamental theorem, from different points of view, was given by M. Koecher in his notes [28], O. Loos in [32] and I. Satake in [52]. This theorem has been generalized by W. Kaup and his collaborators to infinite dimensions, using certain infinite-dimensional Banach Jordan triple systems (see IV §3). An exposition of this part of Jordan theory can be found in H. Upmeier's notes [55]. Banach Jordan algebras are studied in [11] and [17].

There are many more applications of Jordan structures known - so many that we can only mention some areas of applications without going into details: Lie algebras, algebraic and Lie groups, symmetric spaces, in particular symmetric R-spaces, Siegel domains, cones, various types of geometries and mathematical physics. Some of these applications have been described in the surveys [6, 37, 16].

The grid approach motivated by examples

In these notes we present a new method in Jordan theory: the grid approach. For a first understanding of grids it might be helpful to keep in mind the theory of split semisimple Lie algebras in characteristic 0. Roughly speaking, covering grids in Jordan triple systems play the same rôle as Chevalley bases for split semisimple Lie algebras, a spanning system of elements closed under the product. We will make this more precise in the course of this introduction starting with two instructive examples. But first we need to introduce some general notions.

Grids will be special families of tripotents. An element e in a Jordan triple system V over k is called a <u>tripotent</u> if $P(e)e = e$, in which case V has a <u>Peirce decomposition</u> with respect to e:

$$V = V_2(e) \oplus V_1(e) \oplus V_0(e),$$

with $V_i(e) = \{v \in V;\ L(e,e)v = iv\}$ if $1/2 \in k$, for a general k see I §1.3. We can define <u>generalized Peirce spaces</u> with respect to an arbitrary family E of tripotents:

$$V_I(E) = \bigcap_{e \in E} V_{i(e)}(e),\quad I = (i(e))_{e \in E} \in \{0,1,2\}^E$$

In general, V will not be the sum of the Peirce spaces $V_I(E)$ nor will it be determined by E. Some information can only be expected for the <u>cover of</u> E which is

$$C_V(E) = \oplus \{V_I(E);\ E \cap V_I(E) \neq \emptyset\}.$$

We say E <u>covers</u> V if $C_V(E) = V$.

<u>Example 1</u> (rectangular matrix system): For an associative k-algebra D with involution $d \to \bar{d}$ the $(p\times q)$-matrices over D form a Jordan triple system $\mathrm{Mat}(p,q;D,^-)$ over k with quadratic representation $P(x)y = x\bar{y}^t x$. A Chevalley-type D-basis of $V = \mathrm{Mat}(p,q;D,^-)$ is given by the matrix units E_{ij}:

$$\mathcal{R}(p,q) := \{E_{ij};\ 1\le i\le p, 1\le j\le q\},$$

is a concrete realization of what has been called a rectangular grid. It is straightforward to check that $\mathcal{R}(p,q)$ has the following two decisive properties:

(G) $\mathcal{R}(p,q)$ is a family of tripotents which is closed under triple products (the "grid property"), i.e. for $i \neq k$, $l \neq j$ one has

$$\{E_{ij}\ E_{ij}\ E_{kj}\} = E_{kj} = \{E_{kl}\ E_{kl}\ E_{kj}\},\ i\neq k,\ l\neq j,$$
$$\{E_{ij}\ E_{kj}\ E_{kl}\} = E_{il},$$

whereas all other types of products vanish,

(C) $\mathcal{R}(p,q)$ covers $V = \mathrm{Mat}(p,q;D,^-)$:

$$V = \oplus DE_{ij}$$

where DE_{ij} is a Peirce space of V relative to $\mathcal{R}(p,q)$:

$$DE_{ij} = V_2(E_{ij}) \subset V_1(E_{ik}) \cap V_1(E_{lj}) \cap V_0(E_{lk})$$

It is easily seen that in this example the whole Jordan triple structure, i.e. the module and the triple product, is determined by

(i) the covering rectangular grid $\mathcal{R}(p,q)$ and

(ii) the coordinate system $(D,^-)$.

<u>Example 2</u> (hermitian matrix system): Let $(D,^-)$ be as in Example 1 and π be a second involution of D commuting with $^-$. Then the π-hermitian $(p\times p)$-matrices $x = x^{\pi t}$ form a Jordan triple system $H_p(D,\pi,^-)$ with triple product $P(x)y = x\bar{y}^t x$ (which is a Jordan algebra iff $\pi = {}^-$). In this case a Chevalley-type (D,π)-basis (use $\mathrm{Fix}\,\pi$ on the diagonal) is formed by the hermitian matrix units $H_{ii} = E_{ii}$, $H_{ij} = H_{ji} = E_{ij} + E_{ji}$, $i \neq j$:

$$\mathcal{H}(p) = \{H_{ij};\ 1\le i\le j\le p\}$$

is a concrete realization of a hermitian grid. It has similar properties to $\mathcal{R}(p,q)$:

(G) $\mathcal{H}(p)$ is a family of tripotents closed under triple products in the following sense: For $i,j,k \neq$ one has

$\{H_{ij}\ H_{ij}\ H_{ii}\} = 2H_{ii}$, $\{H_{ii}\ H_{ii}\ H_{ij}\} = H_{ij}$

$\{H_{ij}\ H_{ij}\ H_{ik}\} = H_{ik}$,

$P(H_{ij})H_{ii} = H_{jj}$

$\{H_{im}\ H_{mn}\ H_{nj}\} = H_{ij}$ (m arbitrary) ,

$\{H_{im}\ H_{mn}\ H_{ni}\} = 2H_{ii}$, (m,n arbitrary)

whereas all other types of products vanish.

(C) $H(p)$ covers $V = H_p(D,\pi,^-)$:

$V = \oplus_i (\mathrm{Fix}\ \pi)\ H_{ii} \oplus (\oplus_{i<j} DH_{ij})$

where for $i,j,k,m \neq$:

$$(\mathrm{Fix}\ \pi)H_{ii} = V_2(H_{ii}) = V_2(H_{ij}) \cap V_0(H_{jj}) \cap V_0(H_{jk})$$

$$DH_{ij} = V_2(H_{ij}) \cap V_1(H_{ii}) \cap V_1(H_{jj}) \subset V_1(H_{jk}) \cap V_0(H_{km}).$$

Analogously to the rectangular matrix case the Jordan triple $H_p(D,\pi,^-)$ is determined by

(i) the covering hermitian grid $H(p)$ and

(ii) the coordinate system $(D,\pi,^-)$.

These two examples suggest the following program: Develop a theory for grids in general such that $R(p,q)$ and $H(p)$ become examples and such that interesting classes of Jordan triple systems allow the

GRID APPROACH (finite-dimensional version):

Look at a Jordan triple system as an object determined by:

(i) a finite covering grid (carrying all the combinatorial information), and

(ii) a coordinate system.

We want to point out that this philosophy appears in different areas in mathematics, for example in the theory of split semisimple Lie algebras or algebraic groups or in the theory of W* - algebras.

Although we will find that important classes of Jordan triple systems are accessible to the finite-dimensional version of the grid approach, the Jordan triple systems appearing in some applications of Jordan theory require an extended version which we want to motivate by the following.

Example 3 (Hilbert-Schmidt operators). The Hilbert-Schmidt operators from E to F, where E and F are Hilbert spaces over $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, form a real Jordan triple system $L_2(E,F;\mathbb{K})$ with triple product $P(x)y = xy^*x$ (y^* = adjoint of y). With respect to Hilbert space bases $(e_j)_{j\in J}$ of E and $(e_i)_{i\in I}$ of F we can identify $L_2(E,F;\mathbb{K})$ with matrices

$x = (x_{ij})_{i\in I, j\in J}$ over $\mathbb{K}$ such that $\sum_{i,j}|x_{ij}|^2 < \infty$. Then the product becomes $P(x)y = x\bar{y}^t x$ ($^-$ = canonical involution of $\mathbb{K}$), and the whole triple system behaves like an infinite rectangular matrix system. Indeed, we again have a rectangular grid

$$\mathcal{R}(I,J) = \{e_i \otimes e_j^* \; ; \; i\in I, j\in J\},$$

satisfying the multiplication rules (G) of Example 1. However, a new phenomenon appears: $\mathcal{R}(I,J)$ does not cover $V = L_2(E,F;\mathbb{K})$, we only have that

($\bar{C}$) the cover of $\mathcal{R}(I,J)$ is dense in V with respect to the Hilbert-Schmidt operator norm: $V = \overline{C_V(\mathcal{R}(I,J))}$.

Nevertheless, one can recover the structure of $L_2(E,F;\mathbb{K})$ from the cover of $\mathcal{R}(I,J)$. Thus, here we need a topology as a third ingredient (besides a grid and a coordinate system) in order to completely determine the structure of $L_2(E,F;\mathbb{K})$. This example suggests the

GRID APPROACH (topological version):

Look at a Jordan triple system as an object determined by

(i) a grid of arbitrary size,

(ii) a coordinate system, and

(iii) a topology with respect to which the cover of the grid is dense.

We see that both versions of the grid approach lead to the following four fundamental questions which will be answered in these notes:

- What is the general definition of a grid so that $\mathcal{R}(p,q)$, $\mathcal{H}(p)$ and $\mathcal{R}(I,J)$ are examples of grids? - Chapter I
- Can grids be classified? - Chapter II
- What does the cover of a grid look like? - Chapter III
- Which classes of Jordan triple systems allow the grid approach? - Chapter I § 5,6 and Chapter IV.

A short description of the contents can be found below ("the new results") or at the beginning of each chapter and section.

The origins of grids

One special grid has long been known to Jordan algebraists: In our terminology Jacobson's Coordinatization Theorem states that a Jordan algebra covered by a hermitian grid $\mathcal{H}(p)$, $3 \geq p$, is a hermitian matrix algebra, a fundamental result in the classification theory of Jordan algebras. Since semi-primitive Jordan algebras with capacity are always covered by a hermitian grid, it was however not necessary to develop a general theory for grids in Jordan algebras.

The first hint about grids in Jordan triple systems can be found in K. McCrimmon's review [38] of O. Loos' notes [31]. To explain this, let us recall the procedure of [31] for classifying Jordan pairs: Choose a maximal tripotent e so that $V = V_2(e) \oplus V_1(e)$. If $V_1(e) = 0$ then V is a mutation of a Jordan algebra hence is known modulo Jordan algebra theory. Otherwise V is the standard imbedding of an alternative polarized triple system (called alternative pair in [31]) which is defined on $V_1(e)$. In view of the "complicated and asymmetric identities of alternative pairs" McCrimmon remarked that it may be possible to "analyse the Jordan pair directly using 'collinear' rather than merely 'orthogonal' family of tripotents", [38] p. 689.

Subsequently ([40]) he considered "compatible" families of tripotents and then defined special classes of grids. However, he did not develop a theory for grids in general nor did he give a classification-free proof that covering grids exist for certain classes of semisimple Jordan triple systems. Nevertheless the merits of grids became clear in the paper [42] by K. McCrimmon and K. Meyberg, where the covers of finite rectangular, symplectic and hermitian grids were coordinatized.

Independently of [40] and [42] (actually before their publication) and unaware of McCrimmon's remark in [38] the author was led to grids by studying weight space decompositions of compact Jordan triple systems with respect to the structure algebra: The weight spaces turn out to be exactly the generalized Peirce spaces of a suitable grid. Details will appear elsewhere.

The new results

Since grids are a new concept in Jordan theory, most of the results in these notes are new. In case a result has been previously proven we have indicated this to the best of our knowledge. We apologize in advance for any omissions.

The results can be summarized according to the various themes of the chapters:

Chapter I ("What is the general definition of a grid so that $\mathcal{R}(p,q)$, $\mathcal{H}(p)$ and $\mathcal{R}(I,J)$ are examples of grids ?") - We develop a theory for families of Peirce compatible tripotents (called cogs), leading to the fundamental notion of a grid. Roughly speaking a grid is a cog which is multiplicatively closed up to association. For example, $\mathcal{R}(p,q) \subset \mathrm{Mat}(p',q';D,^-)$, $p \leq p'$, $q \leq q'$, is a grid, but also

$$\mathcal{R}(p,q,(\zeta_{ij})) = (\zeta_{ij} E_{ij};\ 1 \leq i \leq p,\ 1 \leq j \leq q)$$

for any choice of scalars $\zeta_{ij} \in D$ satisfying $\zeta_{ij}\bar{\zeta}_{ij} = 1$.

Chapter II ("Can grids be classified?") - Indeed they can be classified modulo "association". We show that every grid is a unique disjoint union of connected grids and that every connected grid is associated to one of seven standard grids. For example, $\mathcal{R}(p,q,(\zeta_{ij}))$ is associated to $\mathcal{R}(p,q)$. The seven standard grids fall into 5 types of grids of arbitrary size (for example $\mathcal{R}(I,J)$ is one type) and 2 exceptional grids consisting of 16 resp. 27 tripotents. This last grid, called Albert grid, naturally carries a geometry, which is the same as the geometry of the 27 lines on a cubic surface.

Chapter III ("What does the cover of a grid look like?") - We generalize the coordinatization theorems of McCrimmon and Meyberg to grids of arbitrary size and prove new coordinatization theorems for the remaining four types of grids.

Chapter IV ("Which classes of Jordan triple systems allow the grid approach?") - The results of chapters I - III give, among others, a classification of a new class of Jordan triple systems, namely the ones covered by a grid. This class is distinct from the class of Jordan triple systems recently classified by E. Zel'manov ([56]). In the intersection of both classes lie the simple Jordan triple systems covered by a grid. These are identified in IV §1. In the last two sections we give examples of how the topological version of the grid approach works. We classify Hilbert triples over $\mathbb{R}$ and $\mathbb{C}$, generalizing results of Kaup, and we present the theory of atomic JBW*-triples in a new way which is independent of the elaborate theory of JB-algebras.

The advantages of grids

Grids provide a more direct and natural approach to Jordan triple systems than presently existing ones, thus justifying McCrimmon's remark as quoted above. Grids make it superfluous to leave the category of Jordan triple systems (or Jordan pairs) and study alternative triple systems (or alternative pairs) first in order to gain information about Jordan triple systems.

It is also no longer necessary to reduce triple theory to algebra theory as has been customary until now. In fact the theory of grids as presented in these notes is independent of the theory of Jordan algebras. Indeed, parts of Jordan algebra theory (e.g. classification) are special cases of our results.

Since grids can have arbitrary size, systems of infinite rank are treated at the same time as the finite rank case. This is not only a technical improvement from the algebraic point of view, but also allows a more direct application of the algebraic theory in a functional analytic setting (see for example IV§§ 2,3).

Besides these fundamental advantages we would like to point out two more innovations: The coordinatization theorems proven by McCrimmon and Meyberg in [42] and supplemented here in Chapter III immediately give information about unital bimodules of Jordan triple systems covered by a grid. Details have not been included here, however they are easily deduced along the lines of [42] § 6.

And, at last we give a confirmation (via the Albert grid) of the conjecture that there is a connection between the 27 lines on a cubic surface and the 27 dimensional exceptional Jordan structures.

Concluding remarks

The notes are self-contained up to some elementary facts concerning Peirce decompositions (to be found in [31] § 5 or [44]) and the results of the papers [40] and [42]. However all that is needed is put together in I §1, including the necessary definitions.

These notes contain the proofs of the results of the survey [47]. These results were also announced at the Oberwolfach meeting on Jordan algebras in 1982 and in an improved form again in 1985.

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TABLE OF CONTENTS

CHAPTER I. SPECIAL FAMILIES OF COMPATIBLE TRIPOTENTS

This first chapter is the foundation for the whole notes. After a review of the basic definitions, results and examples for Jordan triple systems in §1 we consider in the following sections special families of compatible tripotents, i.e. tripotents whose Peirce projections commute. We derive the fundamental properties of these families (§2-§5) and prove existence theorems (§§ 5,6).

§1 Basic definitions, known results, examples

In this section we recall some of the basic facts about Jordan triple systems which are needed in the following. We also give the examples which will play a fundamental role throughout these notes.

1.1. The results of this section are mostly contained in [30], [31] or [44] which are the standard texts on Jordan triple systems. To fix our notation and assumptions we begin with the definition:

A Jordan triple system V is a module V over an arbitrary ring k of scalars together with a quadratic map $P\colon V \to \mathrm{End}\, V$ such that the following identities hold in all scalar extensions:

(1.1) $\quad L(x,y)P(x) = P(x)L(y,x)$

(1.2) $\quad L(P(x)y,y) = L(x,P(y)x)$

(1.3) $\quad P(P(x)y) = P(x)P(y)P(x)$

where $L(x,y)z = P(x,z)y = P(x+z)y - P(x)y - P(z)y$. We use the notation

$$\{xyz\} = L(x,y)z = P(x,z)y = \{zyx\}$$

We refer to P resp. $\{\ldots\}$ as the triple product.

A homomorphism $\varphi\colon V \to V'$ of Jordan triple systems over k is a k-linear map such that $\varphi P(x)y = P(\varphi x)\varphi y$ for all $x,y \in V$. Isomorphisms and automorphisms are defined in the usual way.

A subsystem (resp. an ideal) of a Jordan triple system V is a submodule W of V such that $P(W)W \subset W$ (resp. $P(V)W + P(W)V + \{VVW\} \subset W$).

The defining identities for Jordan triple systems imply of course more identities (eg. by linearization, see [31]§2), of which we list in particular the following

$$(1.4)\qquad L(x,y)P(x) = P(x,P(x)y) = P(x)L(y,x)$$

$$(1.5)\qquad P(x,z)L(y,x) + P(x)L(y,z) = P(x,\{xyz\}) + P(z,P(x)y) = L(x,y)P(x,z) + L(z,y)P(x)$$

$$(1.6)\qquad L(x,\{yxz\}) + P(x)P(y,z) = L(x,z)L(x,y) + L(P(x)y,z) = L(x,y)L(x,z) + L(P(x)z,y)$$

$$(1.7)\qquad L(\{xyz\},y) = L(z,P(y)x) + L(x,P(y)z)$$

$$(1.8)\qquad L(x,\{yxz\}) = L(P(x)y,z) + L(P(x)z,y)$$

$$(1.9)\qquad L(x,y)L(z,y) = P(x,z)P(y) + L(x,P(y)z)$$

$$(1.10)\qquad P(x,z)L(y,x) = P(P(x)y,z) + L(z,y)P(x)$$

$$(1.11)\qquad L(x,y)P(x,z) = P(P(x)y,z) + P(x)L(y,z)$$

$$(1.12)\qquad L(x,y)P(z) + P(z)L(y,x) = P(z,\{xyz\})$$

$$(1.13)\qquad L(x,y)L(x,z) = L(P(x)y,z) + P(x)P(y,z)$$

$$(1.14)\qquad [L(x,y),L(u,v)] = L(\{xyu\},v) - L(u,\{yxv\}),$$

where as usual $[A,B] = AB - BA$.

The map

$$B(x,y) = \mathrm{Id} - L(x,y) + P(x)P(y) \in \mathrm{End}\, V$$

is called the <u>Bergman operator</u> (because of reasons explained in [28], [32]). It is used to define the <u>Jacobson radical</u>

$$(1.15)\qquad \mathrm{Rad}\, V = \{x \in V;\ B(x,y) \text{ is invertible for all } y \in V\},$$

see [31]§4 or [44]XIII. If $\mathrm{Rad}\, V = 0$ one calls V <u>semisimple</u>. The Bergman operator satisfies the identity

$$(1.16)\qquad P(B(x,y)z) = B(x,y)P(z)B(y,x),$$

so $B(x,y)$ is an automorphism of V if it is invertible with inverse $B(y,x)$. A special example of this fact is Theorem 1.13.

<u>1.2.</u> In this subsection we present the basic examples of Jordan triple systems.

<u>Example 1.1.</u> Let J be a quadratic Jordan algebra over k with quadratic representation U (see [20], [21], [31] or [44] for a definition). By forgetting the squaring (or the identity element if J has one) and putting $P = U$ one obtains a Jordan triple system, denoted by $V(J)$ and called <u>the Jordan triple system associated to</u> J. For an abstract characterization of such Jordan triple systems see [48].

<u>Example 1.2.</u> Let $V = (V^+,V^-)$ be a Jordan pair over k and put $V = V^+ \oplus V^-$, $P(x^+ \oplus x^-)(y^+ \oplus y^-) = P(x^+)y^- \oplus P(x^-)y^+$. Then V is a Jordan triple system called <u>the Jordan triple system of a Jordan pair</u>. The

Jordan triple systems arising in this way are exactly the polarized Jordan triple systems, i.e. Jordan triple systems $V = V^+ \oplus V^-$ where the P-operator satisfies ($\varepsilon = \pm$)

$$P(V^\varepsilon)V^\varepsilon = 0, \quad P(V^\varepsilon)V^{-\varepsilon} \subset V^\varepsilon ,$$
$$L(V^\varepsilon,V^\varepsilon)V^{-\varepsilon} = P(V^\varepsilon,V^{-\varepsilon})V^\varepsilon = 0.$$

In this way we will identify Jordan pairs = polarized Jordan triple systems.

Example 1.3. (rectangular matrix system) This class of examples consists of two subclasses:

a) Let V be the 1×2 matrices with entries in a unital alternative algebra D over k, which has a k-linear involution $d \to \bar{d}$, and define the quadratic representation by $P(x)y = x({}^t\bar{y}x)$. We denote this Jordan triple system by Mat(1,2;D) or Mat(1,2;D,$^-$).

b)(i) Before we define the second subclass we make the following remark: For any unital algebra D over k with an involution $d \to \bar{d}$ we may look at the $p \times q$ matrices Mat(p,q;D), $p + q \geq 3$, together with the product $P(x)y = x({}^t\bar{y}x)$. It was shown in [34] that this defines a Jordan triple system iff D is alternative and even associative if $p + q \geq 4$. We will generalize this example to matrices of arbitrary size.

(ii) Let I and J be arbitrary index sets and D an associative algebra over k. We define a matrix of type $I \times J$ over D as a family $x = (x_{ij})_{(i,j) \in I \times J}$ of elements of D such that for every $i \in I$ the i-th row $(x_{ij})_{j \in J}$ and for every $j \in J$ the j-th column $(x_{ij})_{i \in I}$ contain only a finite number of non-zero elements. Assuming that $d \to \bar{d}$ is a k-linear involution of D we can define a Jordan triple system on the space of all matrices of type $I \times J$ over D by putting as above $P(x)y = x{}^t\bar{y}x$. (Note that $x{}^t\bar{y}x$ makes sense). This triple system is denoted by Mat(I,J;D) or Mat(I,J;D,$^-$) if we want to exhibit the involution. In case $\#I = p < \infty$ and $\#J = q < \infty$ we also write Mat(p,q;D) for Mat(I,J;D).

(iii) A D-basis of Mat(I,J;D) is given by all rectangular matrix units $E_{ij} = (x_{kl})$ with $x_{kl} = \delta_{ki}\delta_{lj}$:

$$R := R(I,J) := \{E_{ij};\ i \in I,\ j \in J\}$$

We have a decomposition

$$\mathrm{Mat}(I,J;D) = \bigoplus_{i \in I} \bigoplus_{j \in J} DE_{ij}$$

and the product is given by the following rules which are stated in such a way that they also hold for the example a):

$$P(aE_{ij})bE_{ij} = a(\bar{b}a)E_{ij},$$
$$\{aE_{ij}\ bE_{ij}\ cE_{kj}\} = (c\bar{b})aE_{kj}, \quad \{aE_{ij}\ bE_{ij}\ cE_{il}\} = a(\bar{b}c)E_{il},$$
$$\{aE_{ij}\ bE_{kj}\ cE_{kl}\} = a(\bar{b}c)E_{il},$$

all other products (modulo symmetry in the outer variables) vanish.

(iv) That Mat(I,J;D) actually is a Jordan triple system may be best seen in the following more general context: Let A be an associative algebra over k with an involution $a \to a^*$ and define $P(a)b = ab^*a$. Then the identities (1.1) - (1.3) are easily checked in $K \otimes A$ for any extension $K \supset k$. Hence we have a Jordan triple system; call it $V(A,*)$. Now every subsystem W of $V(A,*)$ is again a Jordan triple system. For $A = \mathrm{Mat}(I\dot{\cup}J, I\dot{\cup}J; D)$, $a^* = \bar{a}^t$, we can imbed Mat(I,J;D) into A in a canonical way so that it becomes a subsystem of A.

Example 1.4. (symplectic matrix system) Let I be an arbitrary index set and define the alternating matrices of type $I \times I$ as those skew-symmetric matrices $x = -x^t$ of type $I \times I$ with diagonal elements $x_{ii} = 0$. We assume that the entries of x belong to an extension $K \supset k$, i.e. a unital commutative and associative algebra over k. Moreover, let $a \to \bar{a}$ be a k-linear automorphism of K with period 2. Then the alternating matrices form a subsystem of $\mathrm{Mat}(I,I;K,^-)$ with product $P(x)y = -x\bar{y}x$, which we denote by $A(I;K,^-)$ or $A(I;K)$ for short or $A(p;K)$ in case $\#I = p$ and which we call the symplectic (or alternating in case $^- = \mathrm{Id}$) matrix system. We point out that the involution $^-$ is only needed to define the product, but not the underlying module of $A(I;K,^-)$.

The symplectic matrix units are the matrices $F_{ij} = E_{ij} - E_{ji} = -F_{ji}$ for $i,j \in I$, $i \neq j$. To write down a K-basis for $A(I;K)$ we have to use a total order $\leq$ on I:

$$S = S(I) = \{F_{ij};\ i<j\}$$

The triple product of $A(I;K)$ is completely determined by the following rules (for $i,j,k,l \neq$):

$P(aF_{ij})bF_{ij} = a\bar{b}aF_{ij}$, $\{aF_{ij}\ bF_{ij}\ cF_{ik}\} = a\bar{b}cF_{ik}$,

$\{aF_{ij}\ bF_{kj}\ cF_{kl}\} = a\bar{b}cF_{il}$,

all other products (modulo symmetry) vanish.

Finally, since $A(1;K) = 0$, $A(2;K) \cong K$ and $A(3;K) \cong \mathrm{Mat}(1,3;K)$ (see [40](0.11)) we always assume $\#I \geq 4$ for this example.

Example 1.5. (hermitian matrix system) Here we start with a unital alternative algebra D over k with a nuclear k-linear involution $a \to a^\pi$ (i.e. all norms aa^π are in the nucleus $N(D) = \{n \in D;\ (nd_1)d_2 = n(d_1d_2)$ for all $d_1, d_2 \in D\}$) and an ample subspace D_0 (i.e. a subspace of π-symmetric elements in the nucleus of D containing the unit 1 of D and closed under $aD_0a^\pi \subset D_0$ for all $a \in D$). Also, let I be an index set with $\#I \geq 2$. The underlying module of this example is formed by the $I \times I$ hermitian matrices $(x^\pi = x^t \in \mathrm{Mat}(I,I;D))$ over D whose diagonal elements lie in D_0. To define the triple product we moreover assume that we have another involution $a \to \bar{a}$ of D commuting with the given involution π and leaving D_0 invariant ($a^\pi = \bar{a}$ allowed).

For $\#I \leq 3$ it is well-known that the hermitian matrices over D form a Jordan algebra J (see e.g. [20]). Let U be its quadratic representation and define a new quadratic representation by $P(x)y = U(x)\bar{y}^t$. That this so-called mutation actually defines a Jordan triple system follows from [44]10 Theorem 2. For $\#I \geq 4$ we assume in addition that D is associative and define the quadratic representation for the hermitian matrices by $P(x)y = x\bar{y}^t x$, i.e. we consider them as a subsystem of $\mathrm{Mat}(I,I;D,^-)$.

In both cases the resulting triple system is called a <u>hermitian matrix system</u> and denoted by $H_I(D,D_0,\pi,^-)$ or $H_p(D,D_0,\pi,^-)$ if $\#I = p$. It is denoted by $H_I(D,D_0,^-)$ in case $\bar{a} = a^\pi$. We remark that $H_I(D,D_0,^-)$ is actually the Jordan triple system associated to the obvious Jordan algebra. Often the choice of D_0 is clear (for example, if $\frac{1}{2} \in k$ then $D_0 = \mathrm{Fix}\,\pi$ is the only possible choice for D_0), then $H_I(D,D_0,\pi,^-) = H_I(D,\pi,^-)$ and $H_I(D,^-,^-) = H_I(D,^-)$. In particular for $D = k$ the only choices are $D_0 = k$, $\pi = {}^- = \mathrm{Id}$ and we put $H_I(k,k,\mathrm{Id},\mathrm{Id}) = H_I(k)$ or $H_p(k)$ for $\#I = p$. In general, $H_I(D,D_0,\pi,^-)$ is spanned by elements of type

$$a[ij] = aE_{ij} + a^\pi E_{ji} \quad (a \in D,\ i \neq j),\quad a_0[ii] = a_0 E_{ii} \quad (a_0 \in D_0)$$

with products for distinct i,j,k,l

$$P(a_0[ii])b_0[ii] = a_0(\bar{b}_0 a_0)[ii]$$
$$P(a[ij])b[ij] = a(\bar{b}a)[ij]$$
$$P(a[ij])b_0[jj] = a(\bar{b}_0 a^\pi)[ii]$$
$$\{a[ij]\ b[ij]\ c[ik]\} = a(\bar{b}c)[ik]$$
$$\{a[ij]\ b[kj]\ c[ki]\} = (a(\bar{b}c) + (c^\pi \bar{b}^\pi)a^\pi)[ii] \quad (k = i \text{ allowed})$$
$$\{a[ij]\ b[kj]\ c[kl]\} = a(\bar{b}c)[il]$$
$$(k = j \text{ or } i = j \text{ or } k = i = j \text{ or } i = j,\ k = l \text{ allowed})$$

whereas all other products between these generators vanish. Finally, in analogy to the symplectic matrix units we define the <u>hermitian matrix units</u> H_{ij} as the matrices $H_{ij} = 1[ij] = H_{ji}$ ($i = j$ allowed)
Note that

$$H(I) = \{H_{ij}, i,j \in I\}$$

is a (D,D_0)-basis of $H_I(D,D_0,\pi,^-)$.

<u>Example 1.6.</u> (<u>quadratic form triple</u>) Here the ingredients are the following: K is an extension over k, i.e. a unital commutative associative algebra over k, with a k-linear automorphism $\kappa: K \to K: a \to \bar{a}$ of period 2, V is a module over K, $q: V \to K$ is a quadratic form with bilinearization $q(x,y) = q(x+y)-q(x)-q(y)$ and finally $S: V \to V$ is a k-linear map with $S^2 = \mathrm{Id}$, $S(cv) = \bar{c}S(v)$ for $c \in K$, $v \in V$ and $q(Sv) =$