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A. L. Onishchik E. B. Vinberg (Eds.)

Lie Groups and Lie Algebras III

Structure of Lie Groups and Lie Algebras

李群与李代数 III

李群与李代数的结构

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

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Structure of Lie Groups and Lie Algebras

V. V. Gorbatsevich, A. L. Onishchik, E. B. Vinberg

Translated from the Russian
by V. Minachin

Contents

Introduction

This article builds on Vinberg and Onishchik [1988] and is devoted to an exposition of the main results on the structure of Lie groups and finite-dimensional Lie algebras. The greater part of the article is concerned with theorems on the structure and classification of semisimple Lie groups (algebras) and their subgroups (subalgebras). The tables given at the end of the article can be used as reference material in any work on Lie groups.

We consider only the results of the classical theory of Lie groups. Some classes of infinite-dimensional Lie groups and Lie algebras, as well as Lie supergroups and superalgebras, will be dealt with in special articles of one of the following volumes of this series. The same applies to the theory of Lie algebras over fields of finite characteristic. However, the results on Lie algebras given in the present article can be extended to more general fields of characteristic 0 (e.g., the field $\mathbb{C}$ of complex numbers can be replaced by any algebraically closed field of characteristic 0).

For the theory of linear representations of Lie groups and algebras, the reader is referred to the volumes especially devoted to this theory, although we had to include in this article some classical theorems on finite-dimensional representations, which form an inseparable part of the structural theory. We also use some results from the theory of algebraic groups. Almost all of them can be found in Springer [1989], and some in Chap. 1, Sect. 6. On the other hand, the results on complex and real algebraic groups contained in Springer [1989] can be treated as results on Lie groups. Some of them (e.g. the Bruhat decomposition) are not dealt with in this volume.

The authors have tried, whenever possible, to give the reader the ideas of the proofs.

The terminology and notation of the article follow that of Vinberg and Onishchik [1988]. In particular, Lie groups are denoted by upper-case Roman letters, and their tangent algebras by lower-case Gothic.

Chapter 1
General Theorems

All vector spaces and Lie algebras considered in this chapter are assumed to be finite-dimensional. The ground field is denoted by K, which is either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers.

§ 1. Lie's and Engel's Theorems

1.1. Lie's Theorem. Denote by $T_n(K)$ the subgroup of $\mathrm{GL}_n(K)$ consisting of all nondegenerate upper triangular matrices, and by $\mathfrak{t}_n(K)$ the subalgebra of the Lie algebra $\mathfrak{gl}_n(K)$ consisting of all triangular matrices. The group $T_n(K)$ (respectively, Lie algebra $\mathfrak{t}_n(K)$) can be interpreted as a subgroup of the full linear group $\mathrm{GL}(V)$ (respectively, subalgebra of the full linear algebra $\mathfrak{gl}(V)$), where V is an n-dimensional vector space over K consisting of operators preserving some full flag, i.e. a set of subspaces $V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V$, where $\dim V_i = i$. The group $T_n(K)$ and the Lie algebra $\mathfrak{t}_n(K)$ are solvable (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.5). The following theorem, first proved by Sophus Lie, shows that the subgroup $T_n(\mathbb{C})$ (subalgebra $\mathfrak{t}_n(\mathbb{C})$) is, up to conjugation, the only maximal connected solvable Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ (respectively, maximal solvable subalgebra of $\mathfrak{gl}_n(\mathbb{C})$).

Theorem 1.1 (see Bourbaki [1975], Jacobson [1962]). (1) *Let $R: G \to \mathrm{GL}(V)$ be a complex linear representation of a connected solvable Lie group G. Then there is a full flag in V invariant under $R(G)$.*

(2) *Let $\mathfrak{g}$ be a solvable Lie algebra, and $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ a complex linear representation of it. Then there is a full flag in V invariant under $\rho(\mathfrak{g})$.*

Because of the correspondence between solvable Lie groups and Lie algebras (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.5), statements (1) and (2) of the theorem are equivalent. We now give an outline of the proof of statement (1).

We start with some definitions and simple auxiliary statements.

Let $R: G \to \mathrm{GL}(V)$ be a linear representation of a group G over an arbitrary field K. For any character χ of the group G, i.e. a homomorphism $\chi: G \to K^\times$, where $K^\times$ is the multiplicative group of the field K, we set

$$V_\chi = V_\chi(G) = \{v \in V \mid R(g)v = \chi(g)v \quad \text{for all} \quad g \in G\}.$$

If $V_\chi \neq 0$, then the character χ is said to be a *weight* of the representation R, the subspace V_χ is called the *weight subspace*, and its nonzero vectors the *weight vectors* corresponding to the weight χ. Similarly, for any linear representation ρ of the Lie algebra $\mathfrak{g}$ over the field K and any linear form

$\lambda \in \mathfrak{g}^*$ let

$$V_\lambda(\mathfrak{g}) = \{v \in V | \rho(x)v = \lambda(x)v \quad \text{for all} \quad x \in \mathfrak{g}\}.$$

If $V_\lambda(\mathfrak{g}) \neq 0$, then the form λ is said to be a *weight* of the representation ρ, the subspace $V_\lambda(\mathfrak{g})$ is called the *weight subspace*, and its nonzero vectors the *weight vectors* corresponding to the weight λ.

Weight subspaces corresponding to different weights are linearly independent. Thus a finite-dimensional linear representation may have only finitely many weights.

The proof of Lie's theorem is based on the following property of weight subspaces.

Lemma 1.1. *Let H be a normal subgroup of the group G, χ the character of H, and $R: G \to \mathrm{GL}(V)$ a linear representation. Then for any $g \in G$ we have*

$$R(g)V_\chi(H) = V_{\chi^g}(H),$$

where $\chi^g(h) = \chi(g^{-1}hg)$ $(h \in H)$.

Outline of the proof of Theorem 1.1. First, one shows by induction on $\dim G$ that R has at least one weight in V. For $\dim G = 1$ the statement is evident. In the general case, the definition of a solvable Lie group implies that there is a virtual normal Lie subgroup H of G of codimension 1. Clearly, $G = CH$, where C is a connected virtual one-dimensional Lie subgroup. By the inductive hypothesis, $V_\chi(H) \neq 0$ for some character χ of the group H. In view of Lemma 1.1, the operators $R(g)$, $g \in G$, permute the weight subspaces of the group H. Since G is connected, $V_\chi(H)$ is invariant under $R(G)$.

The one-dimensional subgroup C has a one-dimensional invariant subspace in $V_\chi(H)$, which is evidently invariant under the action of the entire group G.

Thus, there is a one-dimensional subspace in V invariant under G. The existence of a full flag in V invariant under G is then proved by induction on $\dim V$. □

Corollary 1. *Any irreducible complex linear representation of a connected solvable Lie group or a solvable Lie algebra is one-dimensional.*

Corollary 2. *Let $G \subset \mathrm{GL}(V)$ be a connected irreducible complex linear Lie group. Then either G is semisimple, or* $\mathrm{Rad}\, G = \{cE | c \in \mathbb{C}^\times\}$.

Proof. Suppose that G is not semisimple. Consider the vector subspace $W = V_\chi(\mathrm{rad}\, G) \neq 0$. Lemma 1.1 implies that it is invariant under G. Hence $W = V$, i.e. $\mathrm{Rad}\, G$ contains scalar operators only. □

Corollary 3. *A Lie algebra $\mathfrak{g}$ over $K = \mathbb{C}$ or $\mathbb{R}$ is solvable if and only if the Lie algebra $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}_n(K)$ is nilpotent.*

Proof. If $\mathfrak{g} = \mathfrak{t}_n(K)$, then $[\mathfrak{g}, \mathfrak{g}]$ is the nilpotent Lie algebra of all upper diagonal matrices with zeros on the diagonal. In the general case one can assume, using the complexification procedure if necessary, that $K = \mathbb{C}$. We now see, by Lie's theorem, that if $\mathfrak{g}$ is solvable, then the Lie algebra $\operatorname{ad}[\mathfrak{g},\mathfrak{g}] = [\operatorname{ad}\mathfrak{g}, \operatorname{ad}\mathfrak{g}]$ is nilpotent and therefore $\mathfrak{g}$ is also nilpotent. □

1.2. Generalizations of Lie's Theorem. First we consider the possibilities of generalizing Lie's theorem to Lie algebras over an arbitrary field K. If a representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{g}$ over K has an invariant full flag, then the characteristic numbers of all operators $\rho(x)$, $x \in \mathfrak{g}$, must belong to the field K, which is far from being always true if K is not algebraically closed. If $\operatorname{char} K = 0$, then the above mentioned property of the operators $\rho(x)$, $x \in \mathfrak{g}$, turns out to be also sufficient for the existence of an invariant flag.

Theorem 1.2. *Let $\mathfrak{g}$ be a solvable Lie algebra over a field K of characteristic 0 and $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ a linear representation of it over K. If all characteristic numbers of all operators $\rho(x)$, $x \in \mathfrak{g}$, belong to K, then there is a full flag in V invariant under $\rho(\mathfrak{g})$.*

The proof is similar to that of Theorem 1.1, and makes use of the following analogue of Lemma 1.1.

Lemma 1.2. *Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a Lie algebra $\mathfrak{g}$ over a field K of characteristic 0, $\mathfrak{h}$ an ideal in $\mathfrak{g}$, and $V_\lambda(\mathfrak{h})$ a weight subspace of the representation $\rho|\mathfrak{h}$. Then the following two equivalent statements hold: (1) $V_\lambda(\mathfrak{h})$ is invariant under $\rho(\mathfrak{g})$; (2) $\lambda(z) = 0$ for any $z \in [\mathfrak{g},\mathfrak{h}]$.*

Corollary 3 to Theorem 1.1 is extended to the case of an arbitrary field of characteristic 0. If a field of characteristic 0 is algebraically closed, then the analogues of Corollaries 1 and 2 hold.

The condition imposed by Theorem 1.2 on the characteristic is essential, as the following example shows.

Example. If $\operatorname{char} K = 2$, then the Lie algebra $\mathfrak{gl}_2(K)$ is solvable, but its identity representation in K^2 has no weight vectors.

Without going into details, we note that Lie's theorem can be extended to connected solvable linear algebraic groups over an algebraically closed field of arbitrary characteristic. This follows from Borel's fixed point theorem (see Springer [1989], Chap. 1, Sect. 3.5). We also state the following simple theorem on representations of abstract solvable groups.

Theorem 1.3 (see Merzlyakov [1987]). *Let G be a solvable group, and $R: G \to \mathrm{GL}(V)$ a complex linear representation of it. Then there is a full flag in V invariant under a subgroup of finite index $G_1 \subset G$.*

Proof. Consider the algebraic closure $H = {}^aR(G)$ of the subgroup $R(G)$ of $\mathrm{GL}(V)$. The solvable linear algebraic group H has a finite number of connected components. According to Theorem 1.1, there is a full flag in V invariant under H^0. But then it is also invariant under the subgroup $G_1 = R^{-1}(H^0)$, which is of finite index in G. □

In addition to the main statement of Theorem 1.3 one can also show that the subgroup G_1 can be chosen in such a way that its index does not exceed a number depending on $\dim V$ only (see Merzlyakov [1987]).

1.3. Engel's Theorem and Corollaries to It. The cornerstone in the theory of nilpotent Lie algebras and Lie groups is the following theorem first proved by F. Engel.

Theorem 1.4 (see Bourbaki [1975], Jacobson [1955]). *Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a Lie algebra $\mathfrak{g}$ over an arbitrary field K. Suppose that for each $x \in \mathfrak{g}$ the linear operator $\rho(x)$ is nilpotent. Then there is a basis in V with respect to which the operators $\rho(x)$, $x \in \mathfrak{g}$, are represented by upper triangular matrices with zeros on the diagonal. In particular, the Lie algebra $\rho(\mathfrak{g})$ is nilpotent.*

Proof. As for Lie's theorem, induction on $\dim V$ reduces the theorem to the proof of the existence of a weight vector (with the weight 0). The latter is achieved by induction on $\dim \mathfrak{g}$. For $\dim \mathfrak{g} = 1$ the statement is evident. Suppose that the statement holds for all Lie algebras of dimension less than m, and let $\dim \mathfrak{g} = m$. It follows from the statement of the theorem and the inductive hypothesis that there is an ideal $\mathfrak{h}$ of codimension 1 in $\mathfrak{g}$ (one can take for $\mathfrak{h}$ any maximal subalgebra of $\mathfrak{g}$). Then $\mathfrak{g} = \mathfrak{h} + \langle y \rangle$, where $y \in \mathfrak{g}$. Consider the weight subspace $V_0(\mathfrak{h}) \neq 0$. Since $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, Lemma 1.2 implies that $V_0(\mathfrak{h})$ is invariant under $\mathfrak{g}$. The operator $\rho(y)$ is nilpotent, whence there is a vector $v_0 \in V_0(\mathfrak{h})$, $v_0 \neq 0$, such that $\rho(y)v_0 = 0$. Evidently, v_0 is the desired weight vector with respect to $\mathfrak{g}$. □

Corollary 1. *If under the conditions of Theorem 1.4 the representation ρ is irreducible, then it is trivial and one-dimensional.*

An application of Engel's theorem to the adjoint representation easily yields the following corollary.

Corollary 2. *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if either of the following two conditions is satisfied:*

(1) *For any $x \in \mathfrak{g}$ the operator $\operatorname{ad} x$ is nilpotent.*
(2) *There is a basis $\{e_i\}$ in $\mathfrak{g}$ such that $[e_i, e_j]$ is a linear combination of the elements $e_k, e_{k+1}, \dots, e_m$, where $k = \max(i, j) + 1$.*

A Lie algebra $\mathfrak{g}$ is said to be *engelian* if all the operators $\operatorname{ad} x$, $x \in \mathfrak{g}$, are nilpotent. Corollary 2 implies that a finite-dimensional Lie algebra is engelian if and only if it is nilpotent. For an infinite-dimensional Lie algebra this statement does not hold, in general. If, however, $\mathfrak{g}$ is finitely generated and $(\operatorname{ad} x)^k = 0$ for some $k \in \mathbb{N}$ and all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

We also note that a stronger version of Engel's theorem is also valid, namely its conclusion holds for linear representations ρ of a Lie algebra $\mathfrak{g}$ such that $\rho(\mathfrak{g})$ is generated (as a Lie algebra) by a set of nilpotent operators closed under the commutator.

The next theorem lists other important properties of nilpotent Lie algebras proved with the use of Engel's theorem.

Theorem 1.5 (see Bourbaki [1975], Jacobson [1955], Serre [1987]). *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra. Then*

(i) $\operatorname{codim}[\mathfrak{g},\mathfrak{g}] \geq 2$.
(ii) *If* $\mathfrak{a}$ *is a subspace in* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{a} + [\mathfrak{g},\mathfrak{g}]$, *then* $\mathfrak{a}$ *generates* $\mathfrak{g}$ *as a Lie algebra.*
(iii) *If* $\mathfrak{h}$ *is an ideal in* $\mathfrak{g}$, *then* $\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}) \neq 0$.
(iv) *If* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{g}$, *then its normalizer* $\mathfrak{n}(\mathfrak{h})$ *strictly contains* $\mathfrak{h}$.

Finally, we note the following application of Engel's theorem to the theory of nilpotent Lie groups.

Theorem 1.6. *A connected Lie group* G *is nilpotent if and only if all operators* $\operatorname{Ad} g$ $(g \in G)$ *are unipotent. Any compact subgroup of a connected nilpotent Lie group* G *is contained in* $Z(G)$.

Proof. The first statement follows from Corollary 2 to Theorem 1.4 and the correspondence between nilpotent Lie groups and Lie algebras (see Vinberg and Onishchik [1988], Chap. 2, Theorem 5.13). To prove the second statement, consider the restriction R of the representation Ad to a compact subgroup $L \subset G$. Since R is completely reducible (see below Chap. 4, Corollary to Proposition 2.1), Corollary 1 to Theorem 1.4 implies that R is trivial. Hence $L \subset \operatorname{Ker} \operatorname{Ad} = Z(G)$.

1.4. An Analogue of Engel's Theorem in Group Theory. The following theorem can be considered as a group-theoretical analogue of Engel's theorem. It is not a formal consequence of Engel's theorem because it applies to groups that are not necessarily Lie groups.

Theorem 1.7 (Kolchin, see Merzlyakov [1987], Serre [1987]). *Let* G *be a group, and* $R: G \to \mathrm{GL}(V)$ *a linear representation of it over a field* K. *Suppose that* $V \neq 0$ *and all operators* $R(g)$, $g \in G$, *are unipotent. Then* $\chi \equiv 1$ *is a weight of the representation* R.

Proof. Consider the system of linear equations $(R(g) - E)v = 0$, where g runs over the entire group G. Since we are looking for nontrivial solutions of the system, the field K can be assumed to be algebraically closed. Replacing V by its minimal nonzero invariant subspace, one can also assume that R is irreducible. The Burnside theorem (see Kirillov [1987]) implies that the operators $R(g)$, $g \in G$, generate $\mathfrak{gl}(V)$ as a vector space.

On the other hand, let $Z = R(g) - E$. Then $\operatorname{tr} R(g) = \operatorname{tr} E + \operatorname{tr} Z = \dim \mathfrak{g}$ does not depend on $g \in G$. If $g,\ g' \in G$, then

$$\operatorname{tr}(ZR(g')) = \operatorname{tr}((R(g) - E)R(g')) = \operatorname{tr} R(gg') - \operatorname{tr} R(g') = 0.$$

Hence $\operatorname{tr}(ZX) = 0$ for any $X \in \mathfrak{gl}(V)$, whence $Z = 0$, i.e. $\rho(g) = E$. □