ORDINARY DIFFERENTIAL EQUATIONS

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WITH DIAGRAMS

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PREFACE

In accordance with the tradition which allows an author to make his preface serve rather as an epilogue, I submit that my aim has been to introduce the student into the field of Ordinary Differential Equations, and thereafter to guide him to this or that standpoint from which he may see the outlines of unexplored territory. Naturally, I have not covered the whole domain of the subject, but have chosen a path which I myself have followed and found interesting. If the reader would pause at any point where I have hurried on, or if he would branch off into other tracks, he may seek guidance in the footnotes. In the earlier stages I ask for little outside knowledge, but for later developments I do assume a growing familiarity with other branches of Analysis.

For some time I have felt the need for a treatise on Differential Equations whose scope would embrace not merely that body of theory which may now be regarded as classical, but which would cover, in some aspects at least, the main developments which have taken place in the last quarter of a century. During this period, no comprehensive treatise on the subject has been published in England, and very little work in this particular field has been carried out; while, on the other hand, both on the Continent and in America investigations of deep interest and fundamental importance have been recorded. The reason for this neglect of an important branch of Analysis is that England has but one school of Pure Mathematics, which implies a high development in certain fields and a comparative neglect of others. To spread the energies of this school over the whole domain of Pure Mathematics would be to scatter and weaken its forces; consequently its interests, which were at no time particularly devoted to the subject of Differential Equations, have now turned more definitely into other channels, and that subject is denied the cultivation which its importance deserves. The resources of those more fortunate countries, in which several schools of the first rank flourish, are adequate to deal with all branches of Mathematics. For this reason, and because of more favourable traditions, the subject of Differential Equations has not elsewhere met with the neglect which it has suffered in England.

In a branch of Mathematics with a long history behind it, the prospective investigator must undergo a severer apprenticeship than in a field more recently opened. This applies in particular to the branch of Analysis which lies before us, a branch in which the average worker cannot be certain of winning an early prize. Nevertheless, the beginner who has taken the pains to acquire a sound knowledge of the broad outlines of the subject will find manifold opportunities for original work in a special branch. For instance, I may draw attention to the need for an intensive study of the groups of functions defined by classes of linear equations which have a number of salient features in common.

Were I to acknowledge the whole extent of my indebtedness to others, I should transfer to this point the bibliography which appears as an appendix. But passing over those to whom I am indebted through their published work, I feel it my duty, as it is my privilege, to mention two names in particular. To the late Professor George Chrystal I owe my introduction to the subject; to Professor E. T. Whittaker my initiation into research and many acts of kind encouragement. And also I owe to a short period of study spent in Paris, a renewal of my interest in the subject and a clarifying of the ideas which had been dulled by war-time stagnation.

In compiling this treatise, I was favoured with the constant assistance of Mr. B. M. Wilson, who read the greater part of the manuscript and criticised it with helpful candour. The task of proof-correction had hardly begun when I was appointed to my Chair in the Egyptian University at Cairo, and had at once to prepare for the uprooting from my native country and transplanting to a new land. Unassisted I could have done no more than merely glance through the proof-sheets, but Mr. S. F. Grace kindly took the load from my shoulders and read and re-read the proofs. These two former colleagues of mine have rendered me services for which I now declare myself deeply grateful. My acknowledgments are also due to those examining authorities who have kindly allowed me to make use of their published questions; it was my intention to add largely to the examples when the proof stage was reached, but the circumstances already mentioned made this impossible. And lastly, I venture to record my appreciation of the consideration which the publishers, Messrs. Longmans, Green and Co., never failed to show, a courtesy in harmony with the traditions of two hundred years.

If this book is in no other respect worthy of remark, I can claim for it the honour of being the first to be launched into the world by a member of the Staff of the newly-founded Egyptian University. In all humility I trust that it will be a not unworthy forerunner of an increasing stream of published work bearing the name of the Institution which a small band of enthusiasts hopes soon to make a vigorous outpost of scientific enquiry.

E. L. INCE.

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PART I

DIFFERENTIAL EQUATIONS IN THE REAL DOMAIN

CHAPTER I

INTRODUCTORY

1·1. Definitions.—The term *æquatio differentialis* or differential equation was first used by Leibniz in 1676 to denote a relationship between the differentials dx and dy of two variables x and y.* Such a relationship, in general, explicitly involves the variables x and y together with other symbols $a, b, c, \ldots$ which represent constants.

This restricted use of the term was soon abandoned; differential equations are now understood to include any algebraical or transcendental equalities which involve either differentials or differential coefficients. It is to be understood, however, that the differential equation is not an identity.†

Differential equations are classified, in the first place, according to the number of variables which they involve. An *ordinary* differential equation expresses a relation between an independent variable, a dependent variable and one or more differential coefficients of the dependent with respect to the independent variable. A *partial* differential equation involves one dependent and two or more independent variables, together with partial differential coefficients of the dependent with respect to the independent variables. A *total* differential equation contains two or more dependent variables together with their differentials or differential coefficients with respect to a single independent variable which may, or may not, enter explicitly into the equation.

The *order* of a differential equation is the order of the highest differential coefficient which is involved. When an equation is polynomial in all the differential coefficients involved, the power to which the highest differential coefficient is raised is known as the *degree* of the equation. When, in an ordinary or partial differential equation, the dependent variable and its derivatives occur to the first degree only, and not as higher powers or products, the equation is said to be *linear*. The coefficients of a linear equation are therefore either constants or functions of the independent variable or variables.

Thus, for example,

$$\frac{d^2y}{dx^2}+y=x^3$$

is an ordinary linear equation of the second order;

$$(x+y)^2\frac{dy}{dx}=1$$

is an ordinary non-linear equation of the first order and the first degree;

* A historical account of the early developments of this branch of mathematics will be found in Appendix A.

† An example of a differential identity is:

$$\left(\frac{dx}{dy}\right)^2\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\cdot\frac{d^2x}{dy^2}+3\frac{d^2y}{dx^2}\cdot\frac{d^2x}{dy^2}=0;$$

this is, in fact, equivalent to:

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=1.$$

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}=3\frac{d^2y}{dx^2}$$

is an ordinary equation of the second order which when rationalised by squaring both members is of the second degree ;

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z=0$$

is a linear partial differential equation of the first order in two independent variables ;

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

is a linear partial differential equation of the second order in three independent variables ;

$$\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=0$$

is a non-linear partial differential equation of the second order and the second degree in two independent variables ;

$$udx+vdy+wdz=0,$$

where u, v, and w are functions of x, y and z, is a total differential equation of the first order and the first degree, and

$$x^2dx^2+2xydxdy+y^2dy^2-z^2dz^2=0$$

is a total differential equation of the first order and the second degree.

In the case of a total differential equation any one of the variables may be regarded as independent and the remainder as dependent, thus, taking x as independent variable, the equation

$$udx+vdy+wdz=0$$

may be written

$$u+v\frac{dy}{dx}+w\frac{dz}{dx}=0,$$

or an auxiliary variable t may be introduced and the original variables regarded as functions of t, thus

$$u\frac{dx}{dt}+v\frac{dy}{dt}+w\frac{dz}{dt}=0.$$

1·2. Genesis of an Ordinary Differential Equation.—Consider an equation

(A) $$f(x, y, c_1, c_2, \ldots, c_n)=0,$$

in which x and y are variables and $c_1, c_2, \ldots, c_n$ are arbitrary and independent constants. This equation serves to determine y as a function of x ; strictly speaking, an n-fold infinity of functions is so determined, each function corresponding to a particular set of values attributed to $c_1, c_2, \ldots, c_n$. Now an ordinary differential equation can be formed which is satisfied by every one of these functions, as follows.

Let the given equation be differentiated n times in succession, with respect to x, then n new equations are obtained, namely,

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'=0,$$

$$\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}y'+\frac{\partial^2 f}{\partial y^2}y'^2+\frac{\partial f}{\partial y}y''=0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\partial^n f}{\partial x^n}+\ \cdots\ +\frac{\partial f}{\partial y}y^{(n)}=0,$$

where

$$y'=\frac{dy}{dx},\ y''=\frac{d^2y}{dx^2},\ \ldots,\ y^{(n)}=\frac{d^ny}{dx^n}.$$

Each equation is manifestly distinct from those which precede it;* from the aggregate of $n+1$ equations the n arbitrary constants $c_1, c_2, \ldots, c_n$ can be eliminated by algebraical processes, and the eliminant is the differential equation of order n:

$$F(x, y, y', y'', \ldots, y^{(n)})=0.$$

It is clear from the very manner in which this differential equation was formed that it is satisfied by every function $y=\phi(x)$ defined by the relation (A). This relation is termed the *primitive* of the differential equation, and every function $y=\phi(x)$ which satisfies the differential equation is known as a *solution*.† A solution which involves a number of essentially distinct arbitrary constants equal to the order of the equation is known as the *general solution*.‡ That this terminology is justified, will be seen when in Chapter III. it is proved that one solution of an equation of order n and one only can always be found to satisfy, for a specified value of x, n distinct conditions of a particular type. The possibility of satisfying these n conditions depends upon the existence of a solution containing n arbitrary constants. The general solution is thus essentially the same as the primitive of the differential equation.

It has been assumed that the primitive actually contains n distinct constants $c_1, c_2, \ldots, c_n$. If there are only apparently n constants, that is to say if two or more constants can be replaced by a single constant without essentially modifying the primitive, then the order of the resulting differential equation will be less than n. For instance, suppose that the primitive is given in the form

$$f\{x, y, \phi(a, b)\}=0;$$

then it *apparently* depends upon two constants a and b, but *in reality* upon one constant only, namely $c=\phi(a, b)$. In this case the resulting differential equation is of the first and not of the second order.

Again, if the primitive is reducible, that is to say if $f(x, y, c_1, \ldots, c_n)$ breaks up into two factors, each of which contains y, the order of the resulting differential equation may be less than n. For if neither factor contains all the n constants, then each factor will give rise to a differential equation of order less than n, and it may occur that these two differential equations are identical, or that one of them admits of all the solutions of the other, and therefore is satisfied by the primitive itself. Thus let the primitive be

$$y^2-(a+b)xy+abx^2=0;$$

it is reducible and equivalent to the two equations

$$y-ax=0, \quad y-bx=0,$$

each of which, and therefore the primitive itself, satisfies the differential equation

$$y-xy'=0.$$

1·201. The Differential Equation of a Family of Confocal Conics.—Consider the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

where a and b are definite constants, and λ an arbitrary parameter which can assume all real values. This equation represents a family of confocal conics. The

* Needless to say, it is assumed that all the partial differential coefficients of f exist, and that $\dfrac{\partial f}{\partial y}$ is not identically zero.

† Originally the terms *integral* (James Bernoulli, 1689) and *particular integral* (Euler, *Inst. Calc. Int.* 1768) were used. The use of the word *solution* dates back to Lagrange (1774), and, mainly through the influence of Poincaré, it has become established. The term *particular integral* is now used only in a very restricted sense, cf. Chap. VI. *infra*.

‡ Formerly known as the *complete integral* or *complete integral equation* (*æquatio integralis completa*, Euler). The term *integral equation* has now an utterly different meaning (cf. § 3·2, *infra*), and its use in any other connection should be abandoned.

differential equation of which it is the primitive is obtained by eliminating λ between it and the derived equation

$$\frac{2x}{a^2+\lambda}+\frac{2yy'}{b^2+\lambda}=0.$$

From the primitive and the derived equation it is found that

$$a^2+\lambda=\frac{x^2y'-xy}{y'}, \quad b^2+\lambda=y^2-xyy',$$

and, eliminating λ,

$$a^2-b^2=\frac{x^2y'-xy}{y'}-y^2+xyy',$$

and therefore the required differential equation is

$$xyy'^2+(x^2-y^2-a^2+b^2)y'-xy=0;$$

it is of the first order and the second degree.

When an equation is of the first order it is customary to represent the derivative y' by the symbol p. Thus the differential equation of the family of confocal conics may be written:

$$xy(p^2-1)+(x^2-y^2-a^2+b^2)p=0.$$

1·21. Formation of Partial Differential Equations through the Elimination of Arbitrary Constants.—Let $x_1, x_2, \ldots, x_m$ be independent variables, and let z, the dependent variable, be defined by the equation

$$f(x_1, x_2, \ldots, x_m; z; c_1, c_2, \ldots, c_n)=0,$$

where $c_1, c_2, \ldots, c_n$ are n arbitrary constants. To this equation may be adjoined the m equations obtained by differentiating partially with respect to each of the variables $x_1, x_2, \ldots, x_m$ in succession, namely,

$$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x_1}=0, \ldots, \frac{\partial f}{\partial x_m}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x_m}=0.$$

If $m \geq n$, sufficient equations are now available to eliminate the constants $c_1, c_2, \ldots, c_n$. If $m<n$ the $\frac{1}{2}m(m+1)$ second derived equations are also adjoined; they are of the forms

$$\frac{\partial^2 f}{\partial x_r^2}+2\frac{\partial^2 f}{\partial x_r \partial z}\cdot\frac{\partial z}{\partial x_r}+\frac{\partial^2 f}{\partial z^2}\cdot\left(\frac{\partial z}{\partial x_r}\right)^2+\frac{\partial f}{\partial z}\cdot\frac{\partial^2 z}{\partial x_r^2}=0 \qquad (r=1, 2, \ldots, m),$$

$$\frac{\partial^2 f}{\partial x_r \partial x_s}+\frac{\partial^2 f}{\partial x_r \partial z}\cdot\frac{\partial z}{\partial x_s}+\frac{\partial^2 f}{\partial x_s \partial z}\cdot\frac{\partial z}{\partial x_r}+\frac{\partial^2 f}{\partial z^2}\cdot\frac{\partial z}{\partial x_r}\cdot\frac{\partial z}{\partial x_s}+\frac{\partial f}{\partial z}\cdot\frac{\partial^2 z}{\partial x_r \partial x_s}=0$$

$$(r, s=1, 2, \ldots, m;\ r \neq s).$$

This process is continued until enough equations have been obtained to enable the elimination to be carried out. In general, when this stage has been reached, there will be more equations available than there are constants to eliminate and therefore the primitive may lead not to one partial differential equation but to a system of simultaneous partial differential equations.

1·211. The Partial Differential Equations of all Planes and of all Spheres.—As a first example let the primitive be the equation

$$z=ax+by+c,$$

in which a, b, c are arbitrary constants. By a proper choice of these constants, the equation can be made to represent any plane in space except a plane parallel to the z-axis. The first derived equations are:

$$\frac{\partial z}{\partial x}=a, \quad \frac{\partial z}{\partial y}=b.$$

These are not sufficient to eliminate a, b, and c, and therefore the second derived equations are taken, namely,

$$\frac{\partial^2 z}{\partial x^2}=0, \quad \frac{\partial^2 z}{\partial x \partial y}=0, \quad \frac{\partial^2 z}{\partial y^2}=0.$$

They are free of arbitrary constants, and are therefore the differential equations required. It is customary to write

$$p=\frac{\partial z}{\partial x},\quad q=\frac{\partial z}{\partial y},\quad r=\frac{\partial^2 z}{\partial x^2},\quad s=\frac{\partial^2 z}{\partial x \partial y},\quad t=\frac{\partial^2 z}{\partial y^2}.$$

Thus any plane in space which is not parallel to the z-axis satisfies simultaneously the three equations

$$r=0,\quad s=0,\quad t=0.$$

In the second place, consider the equation satisfied by the most general sphere; it is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2,$$

where a, b, c and r are arbitrary constants. The first derived equations are

$$(x-a)+(z-c)p=0,\quad (y-b)+(z-c)q=0,$$

and the second derived equations are

$$1+p^2+(z-c)r=0,$$
$$pq+(z-c)s=0,$$
$$1+q^2+(z-c)t=0.$$

When $z-c$ is eliminated, the required equations are obtained, namely,

$$\frac{1+p^2}{r}=\frac{pq}{s}=\frac{1+q^2}{t}.$$

Thus there are two distinct equations. Let λ be the value of each of the members of the equations, then

$$\lambda^2(rt-s^2)=1+p^2+q^2>0.$$

Consequently, if the spheres considered are real, the additional condition

$$rt>s^2$$

must be satisfied.

1·22. A Property of Jacobians.—It will now be shown that the natural primitive of a single partial differential equation is a relation into which enter arbitrary functions of the variables. The investigation which leads up to this result depends upon a property of functional determinants or Jacobians.

Let $u_1, u_2, \ldots, u_m$ be functions of the independent variables $x_1, x_2, \ldots, x_n$, and consider the set of partial differential coefficients arranged in order thus :

$$\begin{matrix} \frac{\partial u_1}{\partial x_1}, & \frac{\partial u_1}{\partial x_2}, & \ldots, & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1}, & \frac{\partial u_2}{\partial x_2}, & \ldots, & \frac{\partial u_2}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial u_m}{\partial x_1}, & \frac{\partial u_m}{\partial x_2}, & \ldots, & \frac{\partial u_m}{\partial x_n} \end{matrix}$$

Then the determinant of order p whose elements are the elements common to p rows and p columns of the above scheme is known as a *Jacobian.** Let all the different possible Jacobians be constructed, then *if a Jacobian of order p, say*

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1}, & \ldots, & \frac{\partial u_1}{\partial x_p} \\ \cdot & \cdot & \cdot \\ \frac{\partial u_p}{\partial x_1}, & \ldots, & \frac{\partial u_p}{\partial x_p} \end{vmatrix}$$

is not zero for a chosen set of values $x_1=\xi_1, \ldots, x_n=\xi_n$, *but if every Jacobian of order* $p+1$ *is identically zero, then the functions* $u_1, u_2, \ldots, u_p$ *are*

* Scott and Mathews, *Theory of Determinants*, Chap. XIII.

independent, but the remaining functions $u_{p+1}, \ldots, u_m$ are expressible in terms of $u_1, \ldots, u_p$.

Suppose that, for values of $x_1, \ldots, x_n$ in the neighbourhood of $\xi_1, \ldots, \xi_n$, the functions $u_1, \ldots, u_p$ are not independent, but that there exists an identical relationship,

$$\phi(u_1, \ldots, u_p)=0.$$

Then the equations

$$\frac{\partial\phi}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1}+\ldots+\frac{\partial\phi}{\partial u_p}\cdot\frac{\partial u_p}{\partial x_1}=0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\partial\phi}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_p}+\ldots+\frac{\partial\phi}{\partial u_p}\cdot\frac{\partial u_p}{\partial x_p}=0,$$

are satisfied identically, and therefore

$$\frac{\partial(u, \ldots, u_p)}{\partial(x_1, \ldots, x_p)}\equiv\begin{vmatrix} \frac{\partial u_1}{\partial x_1}, & \ldots, & \frac{\partial u}{\partial x_p} \\ \cdot & \cdot & \cdot \\ \frac{\partial u_p}{\partial x_1}, & \ldots, & \frac{\partial u_p}{\partial x_p} \end{vmatrix}=0$$

identically in the neighbourhood of $\xi_1, \ldots, \xi_n$, which is contrary to the hypothesis. Consequently, the first part of the theorem, namely, that $u_1, \ldots, u_p$ are independent, is true.

In $u_{p+1}, \ldots, u_m$ let the variables $x_1, \ldots, x_p, x_{p+1}, \ldots, x_n$ be replaced by the new set of independent variables $u_1, \ldots, u_p, x_{p+1}, \ldots, x_n$. It will now be shown that if u_r is any of the functions $u_{p+1}, \ldots, u_m$, and x_s any one of the variables $x_{p+1}, \ldots, x_n$, then u_r is explicitly independent of x_s, that is

$$\frac{\partial u_r}{\partial x_s}=0.$$

Let

$$u_1=f_1(x_1, \ldots, x_n), \ldots, u_m=f_m(x_1, \ldots, x_n),$$

and let $x_1, \ldots, x_p$ be replaced by their expressions in terms of the new independent variables $u_1, \ldots, u_p, x_{p+1}, \ldots, x_n$, then differentiating both sides of each equation with respect to x_s,

$$0=\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial x_s}+\ldots+\frac{\partial f_1}{\partial x_p}\cdot\frac{\partial x_p}{\partial x_s}+\frac{\partial f_1}{\partial x_s},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$0=\frac{\partial f_p}{\partial x_1}\cdot\frac{\partial x_1}{\partial x_s}+\ldots+\frac{\partial f_p}{\partial x_p}\cdot\frac{\partial x_p}{\partial x_s}+\frac{\partial f_p}{\partial x_s};$$

$$\frac{\partial u_r}{\partial x_s}=\frac{\partial f_r}{\partial x_1}\cdot\frac{\partial x_1}{\partial x_s}+\ldots+\frac{\partial f_r}{\partial x_p}\cdot\frac{\partial x_p}{\partial x_s}+\frac{\partial f_r}{\partial x_s},$$

$(r=p+1, \ldots, m)$.

The eliminant of $\frac{\partial x_1}{\partial x_s}, \ldots, \frac{\partial x_p}{\partial x_s}$ is

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1}, & \ldots, & \frac{\partial f_1}{\partial x_p}, & \frac{\partial f_1}{\partial x_s} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_p}{\partial x_1}, & \ldots, & \frac{\partial f_p}{\partial x_p}, & \frac{\partial f_p}{\partial x_s} \\ \frac{\partial f_r}{\partial x_1}, & \ldots, & \frac{\partial f_r}{\partial x_p}, & \frac{\partial f_r}{\partial x_s}-\frac{\partial u_r}{\partial x_s} \end{vmatrix}=0,$$

or

$$\frac{\partial(f_1, \ldots, f_p, f_r)}{\partial(x_1, \ldots, x_p, x_s)} = \frac{\partial u_r}{\partial x_s} \cdot \frac{\partial(f_1, \ldots, f_p)}{\partial(x_1, \ldots, x_p)}.$$

But since, by hypothesis,

$$\frac{\partial(f_1, \ldots, f_p, f_r)}{\partial(x_1, \ldots, x_p, x_s)} = 0, \quad \frac{\partial(f_1, \ldots, f_p)}{\partial(x_1, \ldots, x_p)} \neq 0,$$

it follows that

$$\frac{\partial u_r}{\partial x_s} = 0 \qquad (r=p+1, \ldots, m;\ s=p+1, \ldots, n).$$

Consequently each of the functions $u_{p+1}, \ldots, u_m$ is expressible in terms of the functions $u_1, \ldots, u_p$ alone, as was to be proved.

1·23. Formation of a Partial Differential Equation through the Elimination of an Arbitrary Function.—Let the dependent variable z be related to the independent variables $x_1, \ldots, x_n$ by an equation of the form

$$F(u_1, u_2, \ldots, u_n) = 0,$$

where F is an arbitrary function of its arguments $u_1, u_2, \ldots, u_n$ which, in turn, are given functions of $x_1, \ldots, x_n$ and z. When for z is substituted its value in terms of $x_1, \ldots, x_n$, the equation becomes an identity. If therefore $D_r u_s$ represents the partial derivative of u_s with respect to x_r when z has been replaced by its value, then

$$\begin{vmatrix} D_1 u_1, & \ldots, & D_n u_1 \\ \cdot & \cdot & \cdot \\ D_1 u_n, & \ldots, & D_n u_n \end{vmatrix} = 0.$$

But

$$D_r u_s = \frac{\partial u_s}{\partial x_r} + \frac{\partial u_s}{\partial z} \cdot \frac{\partial z}{\partial x_r},$$

and therefore the partial differential equation satisfied by z is

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial z} \cdot \frac{\partial z}{\partial x_1}, & \ldots, & \frac{\partial u_1}{\partial x_n} + \frac{\partial u_1}{\partial z} \cdot \frac{\partial z}{\partial x_n} \\ \cdot & \cdot & \cdot \\ \frac{\partial u_n}{\partial x_1} + \frac{\partial u_n}{\partial z} \cdot \frac{\partial z}{\partial x_1}, & \ldots, & \frac{\partial u_n}{\partial x_n} + \frac{\partial u_n}{\partial z} \cdot \frac{\partial z}{\partial x_n} \end{vmatrix} = 0.$$

1·231. The Differential Equation of a Surface of Revolution.—The equation

$$F(z, x^2+y^2) = 0$$

represents a surface of revolution whose axis coincides with the z-axis. In the notation of the preceding section,

$$x_1 = x, \quad x_2 = y, \quad u_1 = z, \quad u_2 = x^2+y^2,$$

and therefore z satisfies the partial differential equation :

$$\begin{vmatrix} \frac{\partial z}{\partial x}, & \frac{\partial z}{\partial y} \\ 2x, & 2y \end{vmatrix} = 0$$

or

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$

Conversely, this equation is satisfied by

$$z = \phi(x^2+y^2),$$

where ϕ is an arbitrary function of its argument, and is therefore the differential equation of all surfaces of revolution which have the common axis $x=0$, $y=0$.

1·232. Euler's Theorem on Homogeneous Functions.—Let

$$z=\phi(x, y),$$

where $\phi(x, y)$ is a homogeneous function of x and y of degree n. Then, since $\phi(x, y)$ can be written in the form

$$x^n\psi\left(\frac{y}{x}\right),$$

it follows that

$$x^{-n}z=\psi\left(\frac{y}{x}\right).$$

In the notation of § 1·23,

$$x_1=x, \quad x_2=y, \quad u_1=x^{-n}z, \quad u_2=\psi\left(\frac{y}{x}\right),$$
$$F(u_1, u_2)=u_1-u_2,$$

and therefore z satisfies the partial differential equation :

$$\begin{vmatrix} -nx^{-n-1}z+x^{-n}\dfrac{\partial z}{\partial x}, & x^{-n}\dfrac{\partial z}{\partial y} \\ -yx^{-2}\psi', & x^{-1}\psi' \end{vmatrix}=0,$$

and this equation reduces to

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$$

Similarly, if u is a homogeneous function of the three variables x, y and z, of degree n,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

This theorem can be extended to any number of variables.

1·24. Formation of a Total Differential Equation in Three Variables.—The equation

$$\phi(x, y, z)=c$$

represents a family of surfaces, and it will be supposed that to each value of c corresponds one, and only one, surface of the family. Now let (x, y, z) be a point on a particular surface and $(x+\delta x,\ y+\delta y,\ z+\delta z)$ a neighbouring point on the same surface, then

$$\phi(x+\delta x,\ y+\delta y,\ z+\delta z)-\phi(x, y, z)=0.$$

Assuming that the partial derivatives

$$\frac{\partial\phi}{\partial x}, \quad \frac{\partial\phi}{\partial y}, \quad \frac{\partial\phi}{\partial z}$$

exist and are continuous, this equation may be written in the form

$$\left\{\frac{\partial\phi(x, y, z)}{\partial x}+\epsilon_1\right\}\delta x+\left\{\frac{\partial\phi(x, y, z)}{\partial y}+\epsilon_2\right\}\delta y+\left\{\frac{\partial\phi(x, y, z)}{\partial z}+\epsilon_3\right\}\delta z=0,$$

where $\epsilon_1, \epsilon_2, \epsilon_3\to 0$, as $\delta x, \delta y, \delta z\to 0$.

Now let ϵ_1, ϵ_2 and ϵ_3 be made zero and let dx, dy and dz be written for δx, δy and δz respectively. Then there results the total differential equation

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=0,$$

which has been derived from the primitive by a consistent and logical process.

If the three partial derivatives have a common factor μ, and if

$$\frac{\partial\phi}{\partial x}=\mu P, \quad \frac{\partial\phi}{\partial y}=\mu Q, \quad \frac{\partial\phi}{\partial z}=\mu R,$$

then if the factor μ is removed, the equation takes the form

$$Pdx+Qdy+Rdz=0.$$

That there is no inconsistency in the above use of the differentials dx, etc., may be verified by considering a particular equation in two variables, namely,

$$y-f(x)=c.$$

The above process gives rise to the total differential equation

$$dy-f'(x)dx=0,$$

and thus the quotient of the differentials dy, dx is in fact the differential coefficient dy/dx.

Example.—The primitive

$$\frac{(x+z)(y+z)}{x+y}=c$$

gives rise to the total differential equation

$$\frac{y^2-z^2}{(x+y)^2}dx+\frac{x^2-z^2}{(x+y)^2}dy+\frac{2z+x+y}{x+y}dz=0,$$

which, after multiplication by $(x+y)^2$, becomes

$$(y^2-z^2)dx+(x^2-z^2)dy+(2z+x+y)(x+y)dz=0.$$

1·3. The Solutions of an Ordinary Differential Equation.—When an ordinary differential equation is known to have been derived by the process of elimination from a primitive containing n arbitrary constants, it is evident that it admits of a solution dependent upon n arbitrary constants. But since it is not evident that any ordinary differential equation of order n can be derived from such a primitive, it does not follow that if the differential equation is given *a priori* it possesses a general solution which depends upon n arbitrary constants. In the formation of a differential equation from a given primitive it is necessary to assume certain conditions of differentiability and continuity of derivatives. Likewise in the inverse problem of integration, or proceeding from a given differential equation to its primitive, corresponding conditions must be assumed to be satisfied. From the purely theoretical point of view the first problem which arises is that of obtaining a set of conditions, as simple as possible, which when satisfied ensure the existence of a solution. This problem will be considered in Chapter III., where an *existence theorem*, which for the moment is assumed, will be proved, namely, that when a set of conditions of a comprehensive nature is satisfied an equation of order n does admit of a unique solution dependent upon n arbitrary initial conditions. From this theorem it follows that the most general solution of an ordinary equation of order n involves n, and only n, arbitrary constants.

It must not, however, be concluded that no solution exists which is not a mere particular case of the general solution. To make this point clear, consider the differential equation obtained by eliminating the constant c from between the primitive,

$$\phi(x, y, c)=0,$$

and the derived equation,

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}p=0 \qquad \left(p\equiv\frac{dy}{dx}\right).$$

The derived equation in general involves c; let the primitive be solved for c and let this value of c be substituted in the derived equation. The derived equation then becomes the differential equation

$$\left[\frac{\partial\phi}{\partial x}\right]+\left[\frac{\partial\phi}{\partial y}\right]p=0,$$