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Edited by A. Dold and B. Eckmann

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Infinite Matrices of Operators

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CONTENTS

1. Introduction

The classical theory of the transformation of complex sequences by complex infinite matrices is associated largely with the names of Toeplitz, Kojima, and Schur. The basic results of this theory may be conveniently found in the books by Hardy [19], Cooke [9], Maddox [40].

In Hardy's book one also finds detailed accounts of numerous special matrices, or means, e.g. the means of Cesàro, Nörlund, and Borel. Particular attention is given by Hardy to theorms of inclusion and consistency, as well as to theorems of Mercerian and Tauberian type.

Apart from the basic Toeplitz-Kojima-Schur theorems, Cooke, unlike Hardy, tends to deal with some of the more general aspects of the theory of infinite matrices, though like Hardy his treatment is essentially classical. Non-functional analytic methods are employed, and the sequences and matrices considered are restricted to be real or complex.

A decisive break with the classical approach was made by Abraham Robinson [66] in 1950, when he considered the action of infinite matrices of linear operators from a Banach space on sequences of elements of that space.

Our object in the present work is to give an account of some of the main developments which have occurred since Robinson's paper of 1950.

Most of our notation and terminology will be described in Section 2.

In the classical theory of matrix transformations, one of the basic problems is the characterization of matrices which map a sequence space (or merely a set of sequences) E into a sequence space (or set of sequences) F. The first step in this characterization is the determination of the Köthe-Toeplitz dual of E, also called the β-dual of E, where

$$E^{\beta} = \{a \in s : \sum_{k=1}^{\infty} a_k x_k \text{ converges for all } x \in E\}.$$

As usual, s denotes the linear space of all infinite sequences $a = (a_k)$ of complex numbers a_k.

The idea of dual sequence spaces was introduced by Köthe and Toeplitz [28], whose main results concerned α-duals; the α-dual of $E \subset s$ being defined as

$$E^{\alpha} = \{a \in s : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for all } x \in E\}.$$

An account of the theory of α-duals in the scalar case may be found in Köthe [27].

Another dual, the γ-dual, is defined by

$$E^{\gamma} = \{a \in s : \sup_n \left| \sum_{k=1}^{n} a_k x_k \right| < \infty \text{ for all } x \in E\}.$$

Certain topologies on a sequence space, involving β- and α-duality have been examined by Garling [15].

For certain special sequence spaces there are some interesting results given by Lascarides [34].

In Section 3 we investigate several generalized Köthe-Toeplitz duals which arise when the complex sequence (a_k) is replaced by a sequence (A_k) of linear operators. Thus, if X, Y are Banach spaces, each A_k is a linear operator on X into Y, and E is a nonempty set of sequences $x = (x_k)$, with $x_k \in X$, then we define

$$E^{\beta} = \{(A_k) : \sum_{k=1}^{\infty} A_k x_k \text{ converges in the Y-norm, for all } x \in E\}.$$

Section 4 is devoted to the characterization of a number of classes of matrix transformations of linear operators. Inter alia, one finds operator analogues of the theorems of Toeplitz, Kojima, Schur, and of the recent theorem of Crone [11] on infinite scalar matrices which map the Hilbert space ℓ_2 into itself.

In Sections 5, and 6 there is a discussion of Tauberian theorems, and the famous bounded consistency theorem of Mazur-Orlicz-Brudno. Section 7 introduces a new concept of operator Nörlund means and gives some results on the consistency of certain classes of these means.

2. Notation and terminology

By N,R,C we denote the natural, real, and complex numbers, respectively. Some frequently occurring sequence spaces are:

- s, the linear space of complex sequences
- ℓ_o, the space of finite complex sequences,
- c_o, the space of null complex sequences,
- c, the space of convergent complex sequences,
- $[f]$, the space of strongly almost convergent complex sequences,
- f, the space of almost convergent complex sequences,
- ℓ_∞, the space of bounded complex sequences,
- ℓ_p, the space of p-absolutely summable complex sequences, where $0 < p < \infty$,
- w_p, the space of strongly Cesàro summable complex sequences of order 1 and index p, where $0 < p < \infty$.

Of the above spaces, only $[f]$, f and w_p are not perhaps as standard as the others.

The space f was introduced by Lorentz [36]. We say that $(x_k) \in f$ if and only if there exists $\ell \in C$ such that

$$\frac{1}{r} \sum_{i=p+1}^{p+r} x_i \to \ell \quad (r \to \infty, \text{ uniformly in } p \geq 0).$$

The space $[f]$ was defined by Maddox [48]. We say that $(x_k) \in [f]$ if and only if there exists $\ell \in C$ such that

$$\frac{1}{r} \sum_{i=p+1}^{p+r} |x_i - \ell| \to 0 \quad (r \to \infty, \text{ uniformly in } p \geq 0).$$

We have $c \subset [f] \subset f \subset \ell_\infty$ with strict inclusions, and $c,[f]$, f are closed subspaces of ℓ_∞, which is a Banach space with $||x|| = \sup|x_k|$ for each $x = (x_k) \in \ell_\infty$.

The space w_p has been considered in [39] and [40]. We say that $(x_k) \in w_p$ if and only if there exists $\ell \in C$ such that

$$\frac{1}{n}\sum_{k=1}^{n}|x_k-\ell|^p \to 0 \ (n \to \infty).$$

If $(X,||.||)$ is any Banach space over C then we may define c(X), the convergent X-valued sequences; f(X), the almost convergent X-valued sequences, etc. Thus, e.g. $x = (x_k) \in \ell_\infty(X)$, where $x_k \in X$ for $k \in N$, if $\sup||x_k|| < \infty$. Consequently $\ell_\infty(X)$ becomes a Banach space, with the natural coordinatewise operations, and

$$||x|| = \sup||x_k||, \text{ for } x \in \ell_\infty(X).$$

Similarly, $x = (x_k) \in w_p(X)$, $0 < p < \infty$, if and only if there exists $\ell \in X$ such that

$$\frac{1}{n}\sum_{k=1}^{n}||x_k-\ell||^p \to 0 \ (n \to \infty).$$

Every space of complex sequences listed above may be generalized to a space of X-valued sequences merely by replacing the modulus in C by the norm in X, when appropriate.

If X, Y are Banach spaces then we denote by

$$B(X,Y)$$

the Banach algebra of bounded linear operators on X into Y, with the usual operator norm. Thus, if $T \in B(X,Y)$ the operator norm of T is

$$||T|| = \sup\{||Tx|| : x \in S\},$$

where $S = \{x \in X : ||x|| \le 1\}$ is the closed unit sphere in X.

By U we mean the set of all $x \in X$ such that $||x|| = 1$. The zero element of X, and Y, is denoted by Θ.

The continuous dual of Y, i.e. the space of continuous linear functionals on Y, is B(Y,C), and is written as Y*. If $f \in Y^*$ and $y \in Y$ we use the notation

$$(f,y) = f(y).$$

For each $T \in B(X,Y)$ we denote the adjoint of T by T*, where T* is defined by

$$(f,Tx) = (T^*f,x), \text{ for all } f \in Y^* \text{ and all } x \in X.$$

We shall also write

$$S^* = \{f \in Y^* : ||f|| \le 1\},$$

and make use of the well-known fact that, by the Hahn-Banach extension theorem, for every $y \in Y$ there exists $f \in S^*$ such that $||y|| = f(y)$.

The following concept was introduced by Robinson [66] and was termed the group norm by Lorentz and Macphail [37].

2.1 Definition. Let $(T_k) = (T_1, T_2, \ldots)$ be a sequence in $B(X,Y)$. Then the group norm of (T_k) is

$$||(T_k)|| = \sup \left|\left| \sum_{k=1}^{n} T_k x_k \right|\right|$$

where the supremum is over all $n \in N$ and all $x_k \in S$.

It may happen that the group norm is not finite, though we are usually concerned with problems which give rise to finite group norms.

2.2 Summation convention. A sum Σx_k without limits will always be over $k \in N$, i.e.

$$\Sigma x_k = \sum_{k=1}^{\infty} x_k.$$

Some elementary properties of group norms are given in:

2.3 Proposition. (i) If $(A_k) \in s(C^*)$ then the A_k may be identified with complex numbers a_k and

$$||(A_k)|| = \Sigma |a_k|,$$

whence the group norm is finite if and only if $a \in \ell_1$.

(ii) If (T_k) is a sequence in $B(X,Y)$ and we write

$$R_n = (T_n, T_{n+1}, T_{n+2}, \ldots)$$

then

(a) $||T_m|| \le ||R_n||$ for all $m \ge n$,

(b) $||R_{n+1}|| \le ||R_n||$ for all $n \in N$,

(c) $||\sum_{k=n}^{n+p} T_k x_k|| \le ||R_n||.\ \max\{||x_k|| : n \le k \le n+p\}$,

for any x_k and all $n \in N$, and all non-negative integers p.

(iii) If (T_k) is a sequence in $B(X,Y)$ then $\Sigma||T_k|| < \infty$ implies $||(T_k)|| < \infty$. Also, $||(T_k)|| < \infty$ implies $\sup_k ||T_k|| < \infty$.

(iv) If Z denotes the set of all sequences $T = (T_k)$ such that each group norm $||T||$ is finite then Z becomes a Banach space, with the natural operations, under the norm $||T||$.

Proof. (i) There exist complex numbers a_k such that $A_k z = a_k z$ for all $z \in C$. Now for all $n \in N$ and all $x_k \in S$,

$$|\sum_{k=1}^{n} a_k x_k| \le \sum_{k=1}^{n} |a_k|.$$

Also, if $n \in N$, and we define $\text{sgn } z = |z|/z\ (z \neq 0)$, $\text{sgn } 0 = 1$, then

$$|\sum_{k=1}^{n} a_k z_k| = \sum_{k=1}^{n} |a_k|$$

for $z_k = \text{sgn } a_k$ $(1 \le k \le n)$. It follows that $||(A_k)|| = \Sigma|a_k|$, with the understanding that the group norm is not finite when $\Sigma|a_k|$ diverges.

(ii) Let $x \in S$ and define $x_m = x$, $x_k = \Theta$ $(n \le k < m)$. Then

$$||T_m x|| = ||\sum_{k=n}^{m} T_k x_k|| \le ||R_n||,$$

which yields (a).

Now take $x_k \in S$ for $n + 1 \le k \le m$, so that

$$||\sum_{k=n+1}^{m} T_k x_k|| = ||T_n\Theta + \sum_{k=n+1}^{m} T_k x_k|| \le ||R_n||,$$

which yields (b).

Let M denote the max in (c). The case $M = 0$ is trivial. If $M > 0$ then x_k/M is in S for $n \le k \le n + p$, and (c) follows.

(iii) If $\Sigma||T_k|| < \infty$ and $n \in N$, $x_k \in S$, then

$$||\sum_{k=1}^{n} T_k x_k|| \le \sum_{k=1}^{n} ||T_k||\ ||x_k|| \le \Sigma||T_k||$$

whence $||(T_k)|| \le \Sigma||T_k||$. We note that the converse implication is generally false. For example, define $T_k \in B(\ell_\infty, \ell_\infty)$ by

$$T_k x = (0, 0, \ldots, x_1, 0, 0, \ldots)$$

with x_1 in the k-position, where $x = (x_k) \in \ell_\infty$. Then it is clear that $||T_k|| = 1$ for all $k \in N$, so $\Sigma||T_k||$ diverges. However, if

$$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots) \in S$$

for $k \in N$ then $|x_k^{(n)}| \le ||x^{(n)}|| \le 1$ for all n and k, and so for any $n \in N$,

$$||\sum_{k=1}^{n} T_k x^{(k)}|| = ||(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)}, 0, 0, \ldots)|| \le 1$$

whence $||(T_k)|| \le 1$. Moreover, on choosing $x^{(k)} = (1, 1, 1, \ldots)$ for $k \in N$ we see that $||(T_k)|| = 1$.

Now suppose that $||(T_k)|| < \infty$. By (ii)(a) above we have $||T_m|| \le ||R_1|| = ||(T_k)||$ for all $m \in N$, whence $\sup_m||T_m|| \le ||(T_k)||$. The converse implication is always false in $B(X,X)$, where X is a non-trivial Banach space, since we may take T_k as the identity operator for every k.

(iv) With the natural operations $T + T' = (T_k + T_k')$ and $\lambda T = (\lambda T_k)$, for

$\lambda \in C$, it is routine to check completeness. The proof uses the fact that $B(X,Y)$ is a Banach space with the usual norm.

2.4 Definition (Generalized Köthe-Toeplitz duals). Let X and Y be Banach spaces and (A_k) a sequence of linear, but not necessarily bounded, operators A_k on X into Y. Suppose E is a nonempty subset of $s(X)$. Then the α-dual of E is defined as

$$E^{\alpha} = \{(A_k) : \Sigma||A_k x_k|| \text{ converges for all } (x_k) \in E\}.$$

The β-dual of E is defined as

$$E^{\beta} = \{(A_k) : \Sigma A_k x_k \text{ converges for all } (x_k) \in E\}.$$

We remark that convergence is in the norm of Y, in the definition of E^{β}.

In case $X = Y = C$ and the A_k are identified with complex numbers a_k, then $E \subset s$ and

$$E^{\alpha} = \{a : \Sigma|a_k x_k| < \infty \text{ for all } (x_k) \in E\},$$

$$E^{\beta} = \{a : \Sigma a_k x_k \text{ converges for all } (x_k) \in E\}.$$

The α and β-duals of the commonly occuring sequence spaces are all well-known, e.g. $c_o^{\alpha} = c_o^{\beta} = c^{\alpha} = c^{\beta} = \ell_{\infty}^{\alpha} = \ell_{\infty}^{\beta} = \ell_1$.

2.5 Definition. Let X and Y be Banach spaces and $A = (A_{nk})$ an infinite matrix of linear, but not necessarily bounded, operators A_{nk} on X into Y.

Suppose E is a nonempty subset of $s(X)$ and F is a nonempty subset of $s(Y)$. Then we define the matrix class (E,F) by saying that $A \in (E,F)$ if and only if, for every $x = (x_k) \in E$,

$$A_n(x) = \Sigma A_{nk} x_k = \sum_{k=1}^{\infty} A_{nk} x_k$$

<u>converges in the norm of</u> Y, <u>for each</u> n, <u>and the sequence</u>

$$Ax = (\Sigma A_{nk}x_k)_{n\in N}$$

<u>belongs to</u> F.

In case X = Y = C, and the A_{nk} are identified with complex numbers a_{nk} we shall make use of the following conditions in order to characterize some of the important matrix classes. It is to be understood that a condition such as (2.1) involves the convergence of $\Sigma|a_{nk}|$ for each n. As usual, a summation without limits is over $k \in N$. unless otherwise indicated.

(2.1) $$\sup_n \Sigma|a_{nk}| < \infty,$$

(2.2) $$\sup_n \Sigma|\Delta a_{nk}| < \infty, \text{ where } \Delta a_{nk} = a_{nk} - a_{n,k+1},$$

(2.3) $\Sigma|a_{nk}|$ converges uniformly in n,

(2.4) $$\sup_{n,k}|a_{nk}| < \infty,$$

(2.5) $$\lim_n \Sigma|a_{nk}| = 0,$$

(2.6) $$\sup_k \sum_{n=1}^{\infty} |a_{nk}|^p < \infty, \text{ where } p \geq 1,$$

(2.7) $\lim_n a_{nk}$ exists for each k,

(2.8) $\lim_n a_{nk} = 0$ for each k,

(2.9) $\lim_n \Sigma a_{nk}$ exists,

(2.10) $$\lim_n \Sigma a_{nk} = 1$$

(2.11) $$\sup_n \sum_{r=0}^{\infty} 2^{r/p} \max\{|a_{nk}| : 2^r \leq k < 2^{r+1}\} < \infty,$$

where $0 < p < 1$,

(2.12) $$\sup_n \sum_{r=0}^{\infty} 2^{r/p}(\Sigma_r |a_{nk}|^q)^{1/q} < \infty,$$

where $p \geq 1$, $1/p + 1/q = 1$, and Σ_r is over $2^r \leq k < 2^{r+1}$.

If $p = 1$ in (2.12) we interpret Σ_r as $\max\{|a_{nk}| : 2^r \leq k < 2^{r+1}\}$.

2.6 Theorem. $(\ell_\infty,\ell_\infty) = (c,\ell_\infty) = (c_0,\ell_\infty)$, and $A \in (\ell_\infty,\ell_\infty)$ if and only if (2.1) holds.

A proof, along classical lines, may be found in Petersen [62].

2.7 Theorem (KOJIMA-SCHUR). $A \in (c,c)$ if and only if (2.1), (2.7), (2.9) hold.

See Schur [68], or Hardy [19], or Cooke [9].

2.8 Definition. If $A \in (c,c)$ we say that A is conservative. The characteristic of a conservative A is defined to be

$$\chi(A) = \lim_n \Sigma a_{nk} - \Sigma(\lim_n a_{nk}).$$

If $\chi(A) = 0$, we say that A is conull, whilst if $\chi(A) \neq 0$, we say that A is coregular.

2.9 Theorem (TOEPLITZ). $A \in (c,c)$, leaving the limit of every convergent sequence invariant, if and only if (2.1), (2.8), (2.10) hold.

See Toeplitz [76], or [9], [19], [40].

2.10 Theorem. $A \in (\gamma,c)$, where $\gamma = \{x : \Sigma x_k$ converges$\}$, if and only if (2.2), (2.7) hold.

See Cooke [9].

2.11 Theorem (SCHUR). $A \in (\ell_\infty,c)$ if and only if (2.3), (2.7) hold.

See Schur [68], or Maddox [40], p.169. Also, one sees from the proof in Maddox [40], p.169 that $A \in (\ell_\infty, c)$ if and only if (2.1), (2.7) and

$$\lim_n \Sigma |a_{nk} - \lim_n a_{nk}| = 0. \tag{2.13}$$

We remark that another set of necessary and sufficient conditions for $A \in (\ell_\infty, c)$ is (2.7) and

$$\lim_n \Sigma |a_{nk}| = \Sigma |\lim_n a_{nk}|.$$

2.12 Theorem. $A \in (\ell_\infty, c_0)$ *if and only if (2.5) holds.*

See, for example, Maddox [40], p.169. An interesting consequence of Theorem 2.11 is that strong and weak convergence of sequences coincide in ℓ_1 (Maddox [40], p.170).

2.13 Theorem.

(i) $A \in (\ell_1, \ell_\infty)$ *if and only if (2.4) holds.*

(ii) $A \in (\ell_1, \ell_p)$, *where* $p \geq 1$, *if and only if (2.6) holds.*

See Hahn [16] for (i) of Theorem 2.13, and Maddox [40], p.167 for (ii). The condition for $A \in (\ell_1, \ell_1)$ was first given by Knopp and Lorentz [26].

2.14 Theorem. $A \in (\ell_\infty, \ell_1)$ *if and only if*

$$\sup \sum_k \Big| \sum_{n \in E} a_{nk} \Big| < \infty, \tag{2.14}$$

where the supremum is taken over all finite subsets E *of* N.

See Zeller [82], Mehdi [54].

Some remarks on Theorem 2.14 may be of interest. The condition

$$\sum_n \sum_k |a_{nk}| < \infty \tag{2.15}$$