Lecture Notes in Mathematics

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Geometry Symposium Utrecht 1980

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Edited by E. Looijenga, D. Siersma, and F. Takens

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Proceedings of a Symposium Held at the University of Utrecht, The Netherlands, August 27–29, 1980

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PREFACE

During August 27-29 1980 a Geometry Symposium was held at the University of Utrecht in honor of Professor Nicolaas H. Kuiper on the occasion of his sixtieth birthday. The topics of the lectures covered much of Nico Kuiper's (research) interests:

- Th.F. Banchoff: Double tangency theorems for pairs of submanifolds,
- N. Desolneux-Moulis: Some new applications of the geometry of infinite dimensional manifolds in variational calculus,
- J. Eells : A conservation law for harmonic maps,
- W.T. van Est: Manifold schemes: motivation and application,
- H. Grauert: Complex Morse singularities,
- W. Pohl: The probability of linking of random curves,
- D. Sullivan: Harmonic functions and geometry of limit sets,
- R. Thom: Generic approximations of collapsing maps,
- J. Tits: Coxeter graphs and incidence geometry: a survey.

The reader will agree that the Geometry Symposium was aptly named. The non-scientific program included a wonderful piano recital by Regina Albrink, a reception offered by the Rector and an enjoyable symposium dinner in one of Utrecht's picturesque places.

To consider organizing a meeting like this there must be at least one good reason. We thank our teacher Nico Kuiper for providing so many of them. Then, to get started certain material conditions have to be fulfilled: we are grateful to the Dutch Board of Education for generous financial support and to the University of Utrecht for offering hospitality and secretarial help. But the most crucial part of such a meeting are the lectures and its participants. We thank the lecturers most heartily for making this event a mathematically inspiring one; we are indebted to the audience for making it succesful. Our final thanks go to the secretariat of the Institut des Hautes Etudes Scientifiques for its beautiful typing of the manuscripts and to Nicole Gaume for acting as a go-between.

Eduard Looijenga
Dirk Siersma
Floris Takens

CONTENTS

Some of the lectures are not represented in this volume because their contents has been (or will be) published elsewhere. We give the following references:

N. Desolneux-Moulis: Orbites périodiques et des systèmes Hamiltoniens autonomes, sém. Bourbaki févr. 1980, Springer Lecture Note 842, 156-173.

W.T. van Est: Sur le groupe fondamental des schémas analytiques de variété à une dimension, Ann. Inst. Fourier 30, 2, 45-77 (1980).

H. Grauert: to appear in Comp. Math.

A CONSERVATION LAW FOR HARMONIC MAPS

P. Baird and J. Eells

1. Motivation and background.

(1.1) Relativity theory has shown that the laws of many stationary aspects of physics should be enlarged to include time. That can be done in such a manner to provide unification of various physical concepts, and to present them in invariant form ; see [19,§3.2] and [33] . For instance,

a) energy and momentum are unified by forming the energy-momentum tensor ;

b) then the conservation of energy is just the time-component of a law which is invariant under the Lorentz group - the other components being the space-components, which express the conservation of momentum.

The case of stationary electromagnetic fields is carried out in [40;pp75,166]. We describe here briefly the case of stress-energy, following the exposition of Feynman [14,II-31-9] .

The stress at a point of an elastic body is described by a 2-tensor (S_{ij}) in $\mathbb{R}^3$, where S_{ij} is the i-component of a force associated to the j-vector in the following way.

Consider a unit area S orthogonal to j at x . The material on the left of S exerts a force on the material on the right and vice versa - these forces are equal and opposite, and we suppose depend only on the j-vector. By choosing one of this pair of forces, we obtain the stress S_{ij} at x corresponding to the j-vector. We assume that S_{ij} behaves like a tensor. Then one can show that the law of conservation of momentum about some origin implies S_{ij} be symmetric, and that the system be in equilibrium implies that S_{ij} be divergence free.

Now a force is a time-rate of change of momentum , so we could as well describe S_{ij} as the rate of flow of the i-component of momentum through a unit area orthogonal to j. Thus $(S_{ij})_{1\le i,j\le 3}$ are the space components of a 2-tensor in four dimensional Minkowski space with components $(S_{ij})_{0\le i,j\le 3}$; thus the 0-components S_{io} are those of energy flow, and S_{oo} is the energy density. The tensor field $S = (S_{ij})_{0\le i,j\le 3}$ is traditionally called the <u>stress-energy tensor</u> of the system.

In intrinsic terms, we shall interpret a symmetric 2-covariant tensor field S as a stress-energy tensor, as follows : for any timelike vector v at a point, we interpret

a) $S(v,v)$ as the energy density as measured by v ;

b) $S(v,-)$ as the momentum density (of the mass/energy distribution) as measured by v ;

c) $S|_{v^\perp}$ as the stress tensor as measured by v.

(1.2) If the field equations of the physical system are derivable from a variational principle

(1.3) $I(s) = \int L(j^k s)dx$,

then by restricting attention to special variations we proceed to define the stress-energy tensor S ; at an extremal s of I it can be shown that S is conservative :

(1.4) $\operatorname{div} S \equiv 0$.

That result is due to Hilbert [20] ; for an exposition, see [19,§3.3] .

(1.5) During a most instructive conversation many years ago (in April 1963), Professor A. H. Taub suggested that the stress-energy tensor should be useful in the theory of harmonic maps. Although that prospect has lain dormant in the meantime, recent developments have confirmed Taub's prediction.

Indeed, if $\phi : (M,g) \to (N,h)$ is a map between Riemannian manifolds (here and henceforth we shall use the notation and terminology of [10]), then its <u>energy density</u> $e_\phi : M \to \mathbb{R}(\ge 0)$ is defined at each point $x \in M$ by

(1.6) $e_\phi(x) = \frac{1}{2}\,|d\phi\,(x)|^2$,

where the vertical bars denote the Hilbert-Schmidt norm in the space $L(T_x(M), T_{\phi(x)}(N))$.
For any compact domain M' in M we define the <u>energy of</u> ϕ <u>in</u> M' by

(1.7) $E(\phi, M') = \int_{M'} e_\phi(x)\,dx$.

The Euler-Lagrange operator associated with E is called the <u>tension field of</u> ϕ

(1.8) $\tau_\phi = \operatorname{div} d\phi$,

where div is the divergence operator of the Riemannian vector bundle $T^*(M) \otimes \phi^{-1}T(N)$.
And <u>the stress energy tensor of</u> ϕ <u>is found to be</u>

(1.9) $S_\phi = e_\phi\, g - \phi^* h$.

A map $\phi : (M,g) \to (N,h)$ is <u>harmonic</u> if $\tau_\phi \equiv 0$ on M . Such a map then satisfies the conservation law

(1.10) $\operatorname{div} S_\phi \equiv 0$

Here $\operatorname{div} S_\phi$ is alternative notation for $\nabla^* S_\phi$, where ∇^* is the adjoint of the covariant differential $\nabla : C(T^*(M) \otimes \odot^2 T^*(M)) \to C(T^*(M))$.

The purpose of this paper is to derive that simple law (Theorem 2.9 below), and to show how it unifies and simplifies various properties (both old and new) of harmonic maps.

2. Derivation of the stress-energy tensor.

Let us first consider the effect of variations induced by a vector field $X \in C(T(M))$. If $(\xi(t))$ denotes its trajectories, set $g(t) = \xi^*(t)g$.

We derive two standard facts.

(2.1) Lemma. $\left.\frac{\partial \det g(t)}{\partial t}\right|_{t=0} = \operatorname{Trace}(L_X g) \det g$.

Proof. First of all, in charts we have

$$(2.2) \quad \left.\frac{\partial \det g(t)}{\partial t}\right|_{t=0} = \left.\frac{g^{ij}\partial g_{ij}}{\partial t} \det g(t)\right|_{t=0} = \operatorname{Trace} \left.\frac{\partial g}{\partial t} \det g(t)\right|_{0} .$$

Let $m = \dim M$. If we take an orthomormal base $(e_i)_{1 \leq i \leq m}$ with respect to $g = g(o)$ on $T_x(M)$, let $\eta \in \Lambda^m T_x^*(M)$ be the m-covector dual to $e_1 \wedge \ldots \wedge e_m$, so $\eta(e_1,\ldots,e_m) = 1$. Then

$$\det g(t) = \eta(g(t)e_1,\ldots,g(t)e_m) , \quad \text{with} \quad g(o) = I .$$

Then using (2.1)

$$\frac{d}{dt}\Big[\det g(t)\Big]_{t=0} = \sum_{k=1}^{m} \left.\eta\left(g(t)e_1,\ldots,\frac{\partial g(t)}{\partial t} e_k,\ldots,g(t)e_m\right)\right|_{t=0}$$

$$= \sum_{k=1}^{m} \eta\left(e_1,\ldots,\frac{\partial g(o)}{\partial t} e_k,\ldots,e_m\right)$$

$$= \sum_{k=1}^{m} \frac{\partial g_{kk}(o)}{\partial t} = \operatorname{Trace} \frac{\partial g(o)}{\partial t} ;$$

thus (2.2) follows at once.

Secondly, by definition, the Lie derivative

$$L_X g = \left.\frac{\partial g(t)}{\partial t}\right|_{o} = \lim_{t \to o} \frac{\xi^*_t g - g}{t} ,$$

so (2.1) follows from (2.2).

(2.3) Lemma. If $\eta(t) = [\det g(t)]^{1/2} dx^1 \wedge \ldots \wedge dx^m$ is the volume element of $g(t)$, then $\eta(t) = \xi^*(t)\eta$;

$$(2.4) \quad \left.\frac{\partial \eta(t)}{\partial t}\right|_o = L_X \eta = \frac{1}{2} \operatorname{Trace}(L_X g)\,\eta \; ;$$

$$= \frac{1}{2} \langle L_X g, g\rangle \eta \; ;$$

$$= \frac{1}{2} \operatorname{Trace} \frac{\partial g(o)}{\partial t} \eta$$

Proof. At $t = 0$,

$$\frac{\partial \eta(t)}{\partial t} = \frac{1}{2} [\det g(t)]^{1/2} \frac{\partial \det g(t)}{\partial t} \; dx^1 \wedge \ldots \wedge dx^m$$

$$= 1/2 \; \operatorname{Trace}\,(L_X g)\,[\det g(t)]^{1/2} \; dx^1 \wedge \ldots \wedge dx^m \; .$$

Now, for any vector field $X \in C(T(M))$ let $\phi_* X \in C(\phi^{-1} T(N)$ be that variation of ϕ given by $x \to \phi_*(x)\, X(x)$, for all $x \in M$.

(2.5) Lemma.

$$L_X e\phi = \langle d\phi , \nabla(\phi_* X)\rangle - 1/2 \langle L_X g, \phi^* h\rangle \; .$$

Proof. $L_X e_\phi = (de_\phi)(X) = \langle \nabla_X(d\phi), d\phi\rangle$.

A direct calculation gives (2.5), using the standard identity (in any chart)

$$(2.6) \quad (L_X g)_{ij} = X_{i,j} + X_{j,i} \; . \quad (1 \le i,j \le m) \; .$$

(2.7) Lemma. For any map $\phi : (M,g) \to (N,h)$ and vector field $X \in C(T(M))$ we have $L_X (e_\phi \eta) = \langle d\phi , \nabla(\phi_* X)\rangle \eta + \frac{1}{2} \langle L_X g, S_\phi\rangle \eta$, where

$$(2.8) \quad S_\phi = e_\phi \, g - \phi^* h \in C(\odot^2 T^*(M)) .$$

S_ϕ is the stress-energy tensor of ϕ .

Proof. Apply (2.5) and (2.4) to

$$L_X(e_\phi \eta) = (L_X e_\phi)\eta + e_\phi L_X \eta = \langle d\phi , \nabla(\phi_* X)\rangle \eta - \frac{1}{2} \langle L_X g, \phi^* h\rangle \eta + \frac{1}{2} e_\phi \langle L_X g, g\rangle \eta .$$

We shall denote the divergence of S_ϕ by $\operatorname{div} S_\phi$ or by $\nabla^* S_\phi$. In a chart,

$(\mathrm{div}\, S_\phi)_i = (S_\phi)_{ij,j}$. Thus $\mathrm{div}\, S_\phi \in C(T^*M)$.

(2.9) Theorem. The stress-energy tensor $S_\phi \in C(\odot^2 T^*(M))$ of any map $\phi : (M,g) \to (N,h)$ has divergence

(2.10) $\mathrm{div}\, S_\phi = -\langle \tau_\phi , d\phi \rangle$.

Consequently,

a) if ϕ is harmonic, then S_ϕ is conservative (i.e, $\mathrm{div}\, S_\phi \equiv 0$) .

b) if ϕ is a map which is a differentiable submersion almost everywhere on M, and if $\mathrm{div}\, S_\phi \equiv 0$, then ϕ is harmonic.

Proof. From (2.6) we obtain

$$\frac{1}{2} \langle L_X g, S_\phi \rangle = \langle \nabla X, S_\phi \rangle .$$

Applying the divergence theorem and integration by parts to (2.7), assuming that X has compact support we obtain

$$0 = \int_M L_X(e_\phi \eta) = \int_M \langle d\phi, \nabla(\phi_* X)\rangle \eta + \int_M \langle \nabla X, S_\phi \rangle \eta = -\int_M (\langle \tau_\phi, d\phi\rangle + \nabla^* S_\phi) X\eta .$$

Because that is true for all compact X , we find (2.10) satisfied ; the rest of the Proposition follows immediately.

(2.11) Remark. In case b) it suffices to assume that ϕ is C^2 . If ϕ is a C^1-diffeomorphism between compact surfaces, then $\mathrm{div}\, S_\phi \equiv 0$ insures that ϕ is harmonic [34, Chapter 5]. In view of the basic regularity theorem [10,§3.5] , it seems natural to pose the

(2.12) Problem. If ϕ is a continuous L^2_1-map satisfying the hypotheses of b) above, then is ϕ harmonic ?

(2.13) Corollary. Let X be a Killing field of (M,g), and S^ϕ the contravariant representation of the stress-energy tensor of a harmonic map $\phi : (M,g) \to (N,h)$. Then the contraction $Y = \langle S^\phi , X \rangle$ is a vector field with $\mathrm{div}\, Y \equiv 0$.

In particular, the total flux over the boundary of any closed domain M' in M of the X-component of S^ϕ is 0 :

$\int_{\partial M'} \langle Y,\nu\rangle\, dx' = \int_{M'} \operatorname{div} Y\, dx = 0$, where ν is the unit outward normal field of $\partial M'$.

<u>Proof</u>. Killing fields X are characterised by $L_X g \equiv 0$. Thus from (2.6) we get $\operatorname{div} Y = \langle \operatorname{div} S^\phi, X\rangle + \frac{1}{2} \langle S^\phi, L_X g\rangle \equiv 0$.

(2.14) There are various instances where stress-energy appears in the variational theory of Riemannian fibre bundles. For example,

a) in the derivation of extremal Riemannian metrics ; that is in the spirit of Hilbert's work [20]; see [25] and [28].

b) in the study of the extremals of the elastic-energy functional (for fixed $\lambda, \mu \in \mathbb{R}$)

$$EL(\phi) = \int_M \left[\frac{\sigma e_\phi^2}{2} + \frac{\mu |\phi^* h|^2}{4} \right] dx ,$$

as given in [35] .

c) in the theory of functionals of the elementary symmetric functions σ_k of the eigenvalues of $\phi^* h$ with respect to g [41] . If

$$E_k(\phi) = \int_M \sigma_k(g^{-1}\phi^* h)\, dx,$$

then its Euler-Lagrange equation is

$$\operatorname{Trace} \nabla[d\phi \circ T_{k-1}(g^{-1}\phi^* h)] = 0 ;$$

and its stress-energy tensor

$$S_k(\phi) = \frac{1}{2} \sigma_k(g^{-1}\phi^* h) g - \phi^* h \circ T_{k-1}(g^{-1}\phi^* h) ,$$

where T_{k-1} is the Newton tensor field [29,30,41] .

d) in recent work of Tóth [36] , using the stress-energy tensor to study geodesic variations of harmonic maps into locally symmetric Riemanian manifolds.

3. Various illustrations.

(3.1) Example. If $\dim M = 1$, then $S_\phi = -\frac{1}{2}|\phi'|^2$ and $\operatorname{div} S_\phi = -\langle \frac{D\phi'}{dt}, \phi' \rangle$.

(3.2) Example. If $N = \mathbb{R}$, then $S_\phi = \frac{1}{2}|d\phi|^2 g - d\phi \otimes d\phi$ and $\operatorname{div} S_\phi = -\langle \Delta\phi, d\phi \rangle$.

(3.3) Example. Suppose that $\phi : (M,g) \to (N,h)$ is a nonconstant map. Then $S_\phi \equiv 0$ iff $m = 2$ and ϕ is weakly conformal (i.e. there is a function $\mu : M \to \mathbb{R}(\geq 0)$ such that $\phi^*h = \mu g$). Indeed, if $S_\phi \equiv 0$ then ϕ is weakly conformal with $\mu = e_\phi$; and $0 = \operatorname{Trace} S_\phi = (m-2)e_\phi$, so $m = 2$. Conversely, if $\phi^*h = \mu g$, then $2e_\phi = m\mu$, so

(3.4) $S_\phi = \frac{m-2}{2}\mu g$.

Furthermore, if $m > 2$ and $\phi : (M,g) \to (N,h)$ is harmonic and weakly conformal, then ϕ is homothetic (i.e. μ is constant). For Theorem 2.9. asserts that $\operatorname{div} S_\phi \equiv 0$, and from (3.4) we find $0 = \frac{m-2}{2} \mu,_j g_{ij}$ $(1 \leq i \leq m)$, whence $d\mu \equiv 0$ on M.

(3.5) Remark. We first learned of that property in a letter from Professor J.H. Sampson in 1975. Special cases can be found in the literature ; e.g., if $m = n$ see [15, Theorem 8b] and [23, Theorem 5.7]. And [21] for the general case with the requirement that μ has isolated zeros.

(3.6) Example. If $\phi : (M,g) \to (N,h)$ is a totally geodesic map (i.e., $\nabla d\phi \equiv 0$), then ϕ^*h is parallel. Consequently, e_ϕ is constant and S_ϕ is parallel :

$$\nabla S_\phi \equiv 0 .$$

Proof. For any $X,Y,Z \in C(T(M))$ we have

(3.7) $\nabla_X[(\phi^*h)(Y,Z)] = (\nabla_X\phi^*h)(Y,Z) + (\phi^*h)(\nabla_X Y,Z) + (\phi^*h)(Y,\nabla_X Z)$;

(3.8) $\nabla_X[(d\phi)Y] = (\nabla d\phi)(X,Y) + (d\phi)(\nabla_X Y) = (d\phi)(\nabla_X Y)$,

because ϕ is totally geodesic. Now specialize X,Y,Z so that

(3.9) $\nabla_X Y = 0 = \nabla_X Z$

at a prescribed point $x \in M$. Then from (3.7) evaluated at x we obtain

$$(\nabla_X\phi^*h)(Y,Z) = \nabla_X <d\phi(Y),d\phi(Z)> = <\nabla_X(d\phi)Y,(d\phi)Z> + <(d\phi)Y,\nabla_X(d\phi)Z> = 0$$

by (3.8). We conclude that $\nabla(\phi^*h) \equiv 0$ on M .

Now $2e_\phi = <g,\phi^*h>$, so $\nabla(2e_\phi) = \nabla<g,\phi^*h> \equiv 0$; and consequently $\nabla S_\phi \equiv 0$ too.

(3.10) Example. If $\phi:(M,g) \hookrightarrow (N,h)$ is an isometric immersion, then $S_\phi = \frac{m-2}{2} g$, whence

$$\nabla S_\phi \equiv 0 \equiv \nabla^* S_\phi ,$$

whether or not ϕ is harmonic (i.e., is a minimal immersion).

(3.11) Let $\phi:(M,g) \to (N,h)$ be a Riemannian submersion. Then $S_\phi = \frac{n}{2} g - \phi^*h$; and

a) $\nabla^* S_\phi \equiv 0$ iff the fibres of ϕ are minimal ; that reaffirms [38, Prop. 4D] . Such a ϕ is an example of a harmonic morphism, of which more will be said in §5 below.

b) $\nabla S_\phi \equiv 0$ iff the fibres of ϕ are totally geodesic iff the second fundamental form $\nabla d\phi$ of ϕ vanishes on pairs of vertical vectors [38 §3] .

Proof b), the Proof a) being similar. Use indices

$1 \le a,b,c \le m$, $1 \le i,j \le n$, $n+1 \le r,s \le m$.

Take a local orthonormal frame field (X_a) with (X_i) horizontal and (X_r) vertical. Then $(\nabla_{X_b} S_\phi)(X_c,X_a) = (\phi^*h)(\nabla_{X_b} X_c,X_a) + (\phi^*h)(X_c,\nabla_{X_b} X_a)$. Taking $a = r$ and $c = i$ gives $(\nabla_{X_b} S_\phi)(X_i,X_r) = (\phi^*h)(X_i,\nabla_{X_b} X_r)$. Thus $\nabla S_\phi \equiv 0$ implies that the horizontal component $(\nabla X_r)^H \equiv 0$.

Conversely, if $(\nabla X_r)^H \equiv 0$, then $(\nabla_{X_b} S_\phi)(X_i,X_r) = (\phi^*h)(X_i,\nabla_{X_b} X_r) = 0$. Similarly, $(\nabla_{X_b} S_\phi)(X_s,X_r) = 0$. Finally,

$$(\nabla_{X_b} S_\phi)(X_j,X_i) = (\phi^*h)(\nabla_{X_b} X_j,X_i) + (\phi^*h)(X_j,\nabla_{X_b} X_i) = g((\nabla_{X_b} X_j)^H,X_i) + g(X_j,(\nabla_{X_b} X_i)^H)$$

$$= \nabla_{X_b} g(X_j,X_i) \equiv 0 .$$

In summary, $(\nabla X_r)^H \equiv 0$ implies $\nabla_{X_b} S_\phi \equiv 0$ for all $1 \le b \le m$.

To prove the second equivalence in b, take $y \in N$ and let $F_y = \phi^{-1}(y)$, and $i_y : (F_y, k|_{F_y}) \to (M,g)$ the isometric inclusion map. From the composition law [11,(4.1)] we find

$$0 = \nabla d(\phi \cdot i_y) = \phi_*(\nabla d i_y) + \nabla d\phi(i_{y*}, i_{y*}) ,$$

whence for $X, Y \in C(T(F_y))$, $\nabla d\phi(X,Y) = -\phi_*(\nabla d i_y)(X,Y)$. Since $(\nabla d i_y)(X,Y)$ is horizontal and ϕ_* is an isomorphism on horizontal vectors, the right member vanishes iff $\nabla d i_y \equiv 0$. I.e., $\nabla d\phi$ vanishes on pairs of vertical vectors iff the fibres are totally geodesic.

(3.12) Example. Let $\phi : (M,g) \to (V,h)$ be an isometric immersion of (M,g) into a Euclidean space V . Let G denote the Grassmannian of m-planes in V through the origin - and endow G with its standard Riemannian metric k . If $\gamma : M \to G$ is the Gauss map of ϕ , then

a) the second fundamental form β_ϕ of ϕ can be identified (using the representation of the tangent vector bundle $T(G) = K^* \otimes K^\perp$, where $K \to G$ is the vector bundle whose fibre over $L \in G$ is L itself) with the differential of γ :

$$(3.13) \quad \beta_\phi = \nabla d\phi = d\gamma ;$$

b) the third fundamental form of ϕ is $\gamma^* k$. Then we have the basic interrelationship [27] .

$$(3.14) \quad \gamma^* k = \langle \beta_\phi, \tau_\phi \rangle - \text{Ricci}^{\,g} ;$$

i.e.

$$\gamma_i^a \gamma_j^b k_{ab} = \beta_{ij}^\alpha \tau^\beta h_{\alpha\beta} - R_{ij} .$$

If $R^g = g^{ij} R_{ij}$ is the scalar curvature of (M,g) then we calculate $2e_\gamma = |\tau_\phi|^2 - R^g$. Consequently, the stress-energy tensor of γ is

$$(3.15) \quad S_\gamma = \frac{|\tau_\phi|^2 - R^g}{2} g - \beta_\phi . \tau_\phi + \text{Ricci}^{\,g} .$$

<u>If the immersion has constant mean curvature, then</u> $\operatorname{div} S_\gamma \equiv 0$. That is an application of the theorem of Ruh-Vilms characterising such immersions via harmonicity of their Gauss maps [31] .

Let us now interpret that : First of all, Einstein's field tensor [19,p.74] $Ricci^g - \frac{R^g}{2} g$ is divergence free :

$$R_{ki},_k - \frac{R,_i}{2} g \equiv 0 ,$$

as a consequence of Bianchi's second identity. Secondly therefore,

(3.16) $S_{ij,k} = \langle \tau,_k \tau \rangle g_{ij} - \beta^{\alpha}_{ij,k} \tau^{\lambda} h_{\alpha\lambda} - \beta^{\alpha}_{ij} \tau^{\lambda},_k h_{\alpha\lambda} = - \beta^{\alpha}_{ij,k} \tau^{\lambda} h_{\alpha\lambda}$

since ϕ has constant mean curvature. The interpretation (3.13) gives $\nabla\beta_{\phi} = \beta_{\gamma}$, the second fundamental form of the map γ , so (3.16) becomes

$$\nabla S_{\gamma} = - \langle \beta_{\gamma} , \tau_{\phi} \rangle .$$

Therefore, with the interpretation $T(G) = K^* \otimes K^{\perp}$,

$$\operatorname{div} S_{\gamma} = - \langle \tau_{\gamma} , \tau_{\phi} \rangle \equiv 0 .$$

(3.17) <u>Remark</u> : For any space form (V,h) of constant curvature c , the analogue of (3.14) is [27].

$\gamma^* k = \langle \beta_{\phi} , \tau_{\phi} \rangle - Ricci^g + c(m-1)g$, and we can proceed with that as above.

(3.18) <u>Remark</u>. Harmonicity of Gauss maps $\gamma_{\mathcal{F}}$ of a Riemannian foliation $\mathcal{F}$ is studied in [39] . That should be taken into account in consideration (5.7) below.

4. Maps from Kähler manifolds.

(4.1) Let (M,g) be a Kähler manifold of $\dim_{\mathbb{C}} M = m$. Then the complex structure induces a decomposition of its complexified tangent bundle

$$T^{\mathbb{C}}(M) = T'(M) \oplus T''(M) ,$$

and hence a type decomposition of all tensor fields on M. In particular, if $\phi : (M,g) \to (N,h)$ is a map into a Riemannian manifold, then its stress-energy tensor has the decomposition

(4.2) $S_\phi = S^{(2,0)} + S^{(1,1)} + S^{(0,2)}$

and

$$S^{(2,0)} = \overline{S^{(0,2)}} \in C(\odot^2 T'^*(M)) .$$

Similarly, the complex extension of the covariant differential of (M,g), treated as a Riemannian manifold now, decomposes :

(4.3) $\nabla^{\mathbb{C}} = \nabla' + \nabla''$,

where $\nabla' : C(T'(M)) \times C(\otimes T^*M) \to C(\otimes T^*M)$, and similarly for ∇'' .

These decompositions provide greater precision in the assertion of Theorem 2.9 ; indeed, write out

$$\nabla^* S_\phi = (\nabla'^* + \nabla''^*)(S^{(2,0)} + S^{(1,1)} + S^{(0,2)})$$

and compare types, noting that ∇'^* carries (p,q)-types into $(p-1,q)$-types ; and similarly for ∇''^* . We conclude that $\nabla^* S_\phi \equiv 0$ iff

(4.4) $\nabla'^* S^{(2,0)} + \nabla''^* S^{(1,1)} \equiv 0$, and/or

$$\nabla'^* S^{(1,1)} + \nabla''^* S^{(0,2)} \equiv 0 .$$

Thus we obtain the

(4.5) Proposition. *If $\phi : (M,g) \to (N,h)$ is a harmonic map of a Kähler manifold into a Riemannian manifold, then equations (4.4) are satisfied.*