HANDBOOK OF STRUCTURAL AND MECHANICAL MATRICES

Jan J. Tuma, Ph.D.

HANDBOOK OF STRUCTURAL AND MECHANICAL MATRICES

DEFINITIONS ▪ TRANSPORT MATRICES
STIFFNESS MATRICES ▪ FINITE DIFFERENCES
FINITE ELEMENTS ▪ GRAPHS AND TABLES
OF MATRIX COEFFICIENTS

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PREFACE

This handbook presents in one volume a concise collection of the major matrices encountered in the preparation of micro-, mini-, and main-frame computer programs for the analysis of structural and mechanical systems. It was prepared to serve as a *professional, users-oriented, desktop reference book* for engineers and architects.

The subject matter is divided into four major parts arranged in a logical sequence, each covering a distinct class of structural and mechanical elements.

The first part (Chapters 1–16) presents the matrix models of straight, circular, and parabolic bars and of straight interactive bars subjected to mechanical and thermal causes in a state of static equilibrium.

The second part (Chapters 17–21) shows the lumped- and distributed-mass matrix models of straight bars in a state of free and forced vibration.

The third part (Chapters 22–26) displays the matrix models of circular and rectangular plates subjected to mechanical and thermal causes. The finite-segment, finite-difference, and finite-element matrices are included.

The fourth part (Chapters 27–30) introduces the matrix models of cylindrical, spherical, and conical shells subjected to axisymmetrical causes. Again, the finite-segment, finite-difference, and finite-element matrices are included.

The form of presentation follows the telescopic pattern of this author's *Engineering Mathematics Handbook* and shows the same special features, facilitating an easy and rapid location of the desired information.

1. Each page presents the information in a graphical arrangement pertinent to the specific type of material, designated by a title and section number. Consequently, each page is a table.
2. Left and right pages of the book present related or similar material, with all matrices and their load functions arranged in logical sequences.
3. Matrix coefficients are expressed in analytical forms and, if applicable, their series expansions and graphs are included.
4. A consistent system of symbols is used throughout the book, allowing a rapid familiarization with the material and ensuring an easy use of this material.

Even a casual reader will observe that this book is not just a mechanical compilation of available formulas but represents an organized effort to present the matrix analysis in a new and unified form.

The preparation and organization of the material presented in this book spans a period of 12 years, during which the author has been assisted by many individuals and has relied on an extensive wealth of reference material forming a body of knowledge known as matrix structural and mechanical analysis. The space limitation prevents the inclusion of a complete list of references, yet a great effort was made to credit those sources which were directly used or were used for comparative purposes.

In closing, the author expresses his gratitude to his former assistants Dr. L. A. Hill, Dr. A. J. Celis, Dr. K. S. Havner, Mr. C. Martin, Dr. J. W. Gillespie, Dr. J. W. Harvey, Dr. Ch. O. Heller, Mr. H. C. Boecker, Mr. J. W. Exline, Dr. S. E. French, Mr. T. L. Lassley, Dr. H. S. Yu, Dr. J. T. Oden, Dr. R. K. Munshi, Dr. J. H. Talaba, Dr. E. Citipitioglu, Dr. F. A. Frusti, Dr. M. N. Reddy, Dr. M. E. Kamel, Dr. P. L. Koepsell, Dr. M. M. Douglas, Dr. E. P. Dallam, Dr. J. Ramey, Dr. G. Alberti, Dr. A. Lasker, and Dr. S. M. Aljaweini for their help in the development of this material.

Particular thanks are extended to Dr. C. G. Date for the preparation of graphs in Chapter 10 and for the calculation of tables in Chapter 31, to Dr. N. A. Seyedmadani for the preparation of graphs and tables in Chapters 6, 8, 23, 32, and to Mr. A. A. Mages for the permission to use his tables and graphs in Chapter 12. Although every effort was made to avoid errors, it would be presumptuous to assume that none had escaped detection in a work of this scope. The author earnestly solicits comments and recommendations for improvements and future additions.

Finally, but not least, gratitude is expressed to my wife Hana for her patience, understanding, and encouragement during the preparation of this book.

Tempe, Arizona

Jan J. Tuma

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Part I

Static Analysis of Bars

1

Analysis of Bars Notation, Signs, Basic Relations

STATIC STATE

1.01 INTRODUCTION

(1) Systems considered in Chaps. 1–16 are finite bars of given geometric shapes. Their cross-sectional dimensions are small in comparison to their length ($< 1/10$), and if curved, the radius of curvature of the undeformed bar axis is large in comparison to the largest dimension of the cross section ($> 10/1$). Each bar and its loads are in a state of static equilibrium and a state of time-independent elastic deformation.

(2) Assumptions used in the derivation of analytical relations are

- (*a*) The material of the bar is homogeneous, isotropic, continuous, and follows Hooke's law.
- (*b*) All deformations are small and do not alter (significantly) the initial geometry of the bar and of the load applications.
- (*c*) The initial plane cross section of the bar remains plane during the deformation of the bar.
- (*d*) The material constants (modulus of elasticity and of rigidity, spring constants of connections and of foundations, and volume change coefficients) are known from experiments and are independent of time.

(3) Symbols are defined where they appear first and are all summarized in Appendix A.

1.02 POSITION AND LOAD VECTORS

(1) Position of a point j is given by the position vector s_{0j}, represented symbolically by three coordinates x_{0j}, y_{0j}, z_{0j} and related to the right-handed orthogonal axes x, y, z. In column matrix form,

$$s_{0j} = \{x_{0j}, y_{0j}, z_{0j}\} \qquad (s_{0j} \text{ in m})$$

All coordinates are positive if measured in the positive direction of coordinate axes (Fig. 1.02-1).

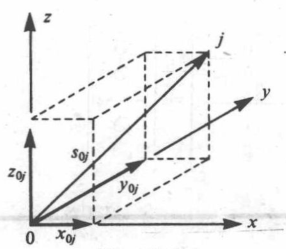


Fig. 1.02-1

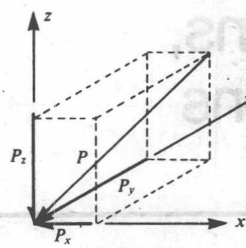


Fig. 1.02-2

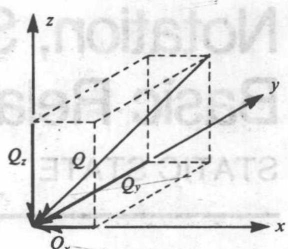


Fig. 1.02-3

(2) Concentrated load and applied moment are single force P and single couple Q represented symbolically by three orthogonal single- and double-headed vector components P_x, P_y, P_z and Q_x, Q_y, and Q_z respectively. In column matrix form,

$$P = \{P_x, P_y, P_z\} \quad (P \text{ in N}) \qquad \text{and} \qquad Q = \{Q_x, Q_y, Q_z\} \quad (Q \text{ in N}\cdot\text{m})$$

All concentrated-load and applied-moment components are positive if acting in the negative direction of coordinate axes (Figs. 1.02-2, 1.02-3).

(3) Intensity of distributed force and distributed moment are force and couple per unit length p and q represented symbolically by three orthogonal single- and double-headed vector components p_x, p_y, p_z and q_x, q_y, q_z respectively. In column matrix form,

$$p = \{p_x, p_y, p_z\} \quad (p \text{ in N/m}) \qquad \text{and} \qquad q = \{q_x, q_y, q_z\} \quad (q \text{ in N}\cdot\text{m/m})$$

All intensities of distributed-force and distributed-moment components are positive if acting in the negative direction of coordinate axes (Figs. 1.02-4, 1.02-5).

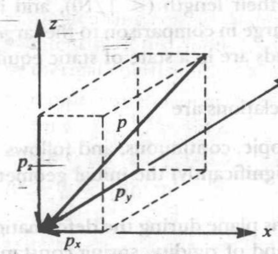


Fig. 1.02-4

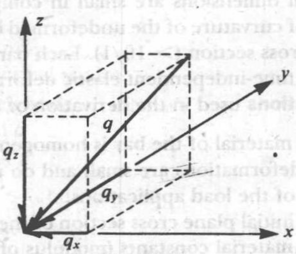


Fig. 1.02-5

(4) Sign convention of loads introduced above offers simplification in numerical calculations since many physical situations call for loads acting in these directions.

1.03 STRESS AND REACTIVE VECTORS

(1) **Stress-resultant force vector** is a single force acting at the centroid of the section and represented symbolically by three orthogonal single-headed vector components along the principal axes of the section. They are

U = normal force (N) $\quad V, W$ = shearing forces (N)

(2) **Stress-resultant moment vector** is a single moment acting at the centroid of the section and represented symbolically by three orthogonal double-headed vector components along the principal axes of the section. They are

X = torsional moment (N·m) $\quad Y, Z$ = bending moments (N·m)

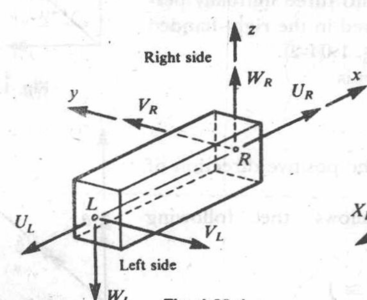


Fig. 1.03-1

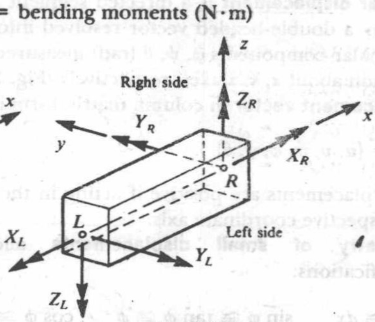


Fig. 1.03-2

(3) **Stress-resultant vectors** in column matrix form are

$$S_L = \{U_L, V_L, W_L, X_L, Y_L, Z_L\} \qquad S_R = \{U_R, V_R, W_R, X_R, Y_R, Z_R\}$$

where the subscripts L and R identify the left and right ends, respectively. All stress-resultant components acting on the right side (far side) of the section are positive if acting in the positive direction of the principal axes. For their left-side (near-side) counterparts, the opposite is true (Figs. 1.03-1, 1.03-2).

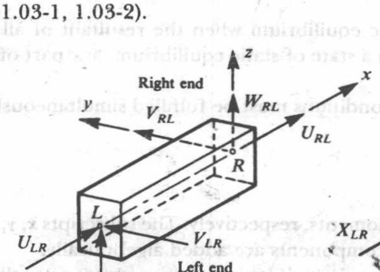


Fig. 1.03-3

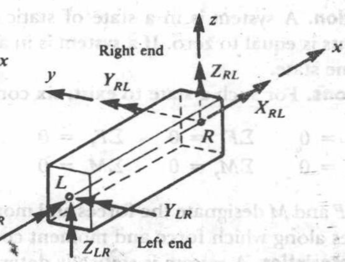


Fig. 1.03-4

(4) **Reactions** are force and moment vectors developed by loads and/or other causes at the points of support. In column matrix form they are

$$S_{LR} = \{U_{LR}, V_{LR}, W_{LR}, X_{LR}, Y_{LR}, Z_{LR}\} \qquad S_{RL} = \{U_{RL}, V_{RL}, W_{RL}, X_{RL}, Y_{RL}, Z_{RL}\}$$

where the first and second subscripts identify the near and far ends, respectively. All reactions are positive if acting in the positive direction of the respective axes (Figs. 1.03-3, 1.03-4).

1.04 DISPLACEMENTS

(1) Elastic curve. The deformation of a bar is defined as the change in its shape caused by loads and/or volume changes. As the bar deforms, its centroidal axis takes on a new shape, called the elastic curve. The coordinates and the slopes of this curve measured from the initial axis of the undeformed bar are designated as the linear displacements and angular displacements, respectively.

(2) Linear displacement is a directed segment represented symbolically by a single-headed vector resolved into three mutually perpendicular components u, v, w (m) measured along x, y, z axes, respectively (Fig. 1.04-1).

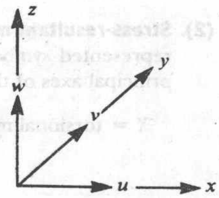


Fig. 1.04-1

(3) Angular displacement is a directed segment represented symbolically by a double-headed vector resolved into three mutually perpendicular components ϕ, ψ, θ (rad) measured in the right-handed direction about x, y, z axes, respectively (Fig. 1.04-2).

(4) Displacement vector in column matrix form is

$$\Delta = \{u, v, w, \phi, \psi, \theta\}$$

All displacements are positive if acting in the positive direction of the respective coordinate axis.

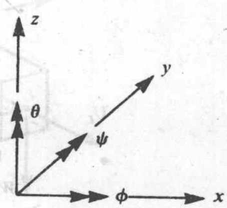


Fig. 1.04-2

(5) Geometry of small displacements allows the following simplifications:

$$ds \cong dx \qquad \sin\phi \cong \tan\phi \cong \phi \qquad \cos\phi \cong 1$$
$$\sin\psi \cong \tan\psi \cong \psi \qquad \cos\psi \cong 1$$
$$\sin\theta \cong \tan\theta \cong \theta \qquad \cos\theta \cong 1$$

where ds is the elemental length of the elastic curve and dx is the corresponding elemental length of undeformed bar axis.

1.05 STATIC EQUILIBRIUM

(1) Definition. A system is in a state of static equilibrium when the resultant of all forces and moments is equal to zero. If a system is in a state of static equilibrium, any part of it is also in the same state.

(2) Equations. For such a state to exist, six conditions must be fulfilled simultaneously:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0$$
$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0$$

where F and M designate the forces and moments, respectively. The subscripts x, y, z represent the axes along which force and moment components are added algebraically.

(3) Characteristics. A system is statically determinate if its reactions and stress resultants can be computed from the conditions of static equilibrium alone. A system is statically indeterminate if its reactions and stress resultants cannot be computed from the conditions of static equilibrium alone and deformation conditions must be considered. The superfluous forces and moments (which are not necessary for static equilibrium) are called redundants, and their number defines the degree of static indeterminacy of the system.

1.06 STATIC DETERMINACY AND INDETERMINACY

(1) Beams, arches, rings, and frames. For a system of bars connected together by joints, some or all of which are rigid, the total number of independent conditions of static equilibrium is

$$e = 6b + 6j + s$$

where b = number of bars, j = number of joints, and s = number of releases (number of zero forces and/or moments at supports and in connections). Since there are 12 unknown end stress resultants in each bar and r unknown reactions at supports, the number of redundants is

$$n = 12b + r - e = 6b + r - 6j - s > 0$$

If $n = 0$, the system is statically determinate, and if $n < 0$, the system is geometrically unstable. In these equations the number of joints j includes all internal joints and all points of support.

(2) Trusses. For a space system of bars connected together by frictionless hinges, the total number of independent conditions of static equilibrium is

$$e = b + 3j + s$$

where b, j, and s have the same meaning as in (1). Since there are only two unknown end stress resultants in each bar and r unknown reactions at supports, the number of redundants reduces to

$$n = 2b + r - e = b - 3j - s > 0$$

If $n = 0$, the truss is statically determinate, and if $n < 0$, the truss is geometrically unstable. As in (1), the number of joints j in both equations includes all internal joints and all points of support.

(3) Internal releases included in (1) and (2) are results of special conditions imposed on the system. Five typical releases at point j are:

(*a*) *Free end*

$$U_j = 0 \quad V_j = 0 \quad W_j = 0 \quad X_j = 0 \quad Y_j = 0 \quad Z_j = 0$$
$$u_j \neq 0 \quad v_j \neq 0 \quad w_j \neq 0 \quad \phi_j \neq 0 \quad \psi_j \neq 0 \quad \theta_j \neq 0$$

(*b*) *Spherical roller normal to x, y plane*

$$U_j = 0 \quad V_j = 0 \quad W_j \neq 0 \quad X_j = 0 \quad Y_j = 0 \quad Z_j = 0$$
$$u_j \neq 0 \quad v_j \neq 0 \quad w_j = 0 \quad \phi_j \neq 0 \quad \psi_j \neq 0 \quad \theta_j \neq 0$$

(*c*) *Spherical hinge*

$$U_j \neq 0 \quad V_j \neq 0 \quad W_j \neq 0 \quad X_j = 0 \quad Y_j = 0 \quad Z_j = 0$$
$$u_j = 0 \quad v_j = 0 \quad w_j = 0 \quad \phi_j \neq 0 \quad \psi_j \neq 0 \quad \theta_j \neq 0$$

(*d*) *Cylindrical hinge along x axis*

$$U_j \neq 0 \quad V_j \neq 0 \quad W_j \neq 0 \quad X_j = 0 \quad Y_j \neq 0 \quad Z_j \neq 0$$
$$u_j = 0 \quad v_j = 0 \quad w_j = 0 \quad \phi_j \neq 0 \quad \psi_j = 0 \quad \theta_j = 0$$

1.06 STATIC DETERMINACY AND INDETERMINACY
(Continued)

(e) *Linear guide along x axes*

$$U_j = 0 \quad V_j \neq 0 \quad W_j \neq 0 \quad X_j = 0 \quad Y_j \neq 0 \quad Z_j \neq 0$$
$$u_j \neq 0 \quad v_j = 0 \quad w_j = 0 \quad \phi_j \neq 0 \quad \psi_j = 0 \quad \theta_j = 0$$

This summary shows that the number of releases equals six in all cases. The conditions stated in the first row are used in (1.06-1, 2), and the conditions stated in the second row are used in (1.07-2, 3).

1.07 KINEMATIC DETERMINACY AND INDETERMINACY

(1) Definitions. Any system of bars can be always visualized as a system of joints connected together by elastic springs. Since each joint may have as many as six degrees of freedom (three linear and three angular), the kinematics of the system is defined by the displacements of each joint, which may be introduced as the kinematic redundants. The degree of kinematic indeterminacy of the system is then equal to the number of admissible, independent, and unknown joint displacements.

(2) Beams, arches, rings, and frames. For a system of b bars connected together by j joints (some or all of which are rigid), with s internal releases, g internal constraints, and r reactive constraints, the total number of admissible and independent joint displacements is

$$d = 6j + s - g - r$$

where the internal releases are those defined in (1.06-3). The total number of joints includes all internal joints and all points of support. The internal constraints are given by the type of connections, and the reactive constraints equal the number of reactions.

(3) Trusses. In a space truss with b bars and j frictionless hinges connecting these bars, the number of admissible and independent joint linear displacements is

$$d = 3j + s - g - r$$

where s, g, r have the same meaning as in (2) above.

1.08 CLASSIFICATION OF SYSTEMS

(1) System of order one is a coplanar system of bars acted upon by loads in the system plane. Typical examples of systems of order one are planar trusses, beams loaded in the plane of symmetry of the cross section, planar frames, and arches loaded in their plane. Chapters 4–10 show the matrices of straight and curved bars of order one.

(2) System of order two is a coplanar system of bars acted upon by loads normal to the system plane. The most typical systems in this category are arches and rings loaded normal to their plane, grids, planar frames loaded normal to their plane, and plane curved (bent) bars loaded normal to their plane. Chapters 11–15 show the matrices of straight and curved bars of order two.

(3) System of order three is a nonplanar system of bars acted upon by loads of arbitrary directions. A space truss, space frame, and circular helix girder are typical systems of order three. Chapters 2 and 3 show the matrices of straight and curved bars of order three.

(4) Resolution. A planar system of bars acted on by loads of arbitrary direction can always be resolved into a system of order one by taking the in-plane load components and into a system of order two by taking the normal-to-plane load components.

1.09 GEOMETRIC TRANSFORMATIONS

(1) Two coordinate systems are introduced in the analysis:

(*a*) *Reference system* (datum, global system, 0 system) is an arbitrarily selected set of orthogonal axes x^o, y^o, z^o whose direction is fixed and common for all parts of the structure.

(*b*) *Member system* (local system, *S* system) is given by the principal axes x^s, y^s, z^s at the station of investigation. The position coordinates of a joint, support, or cross section *j* related to these systems are directed segments, so that

$$x_{0j}^o = -x_{j0}^o \qquad y_{0j}^o = -y_{j0}^o \qquad z_{0j}^o = -z_{j0}^o$$
$$x_{0j}^s = -x_{j0}^s \qquad y_{0j}^s = -y_{j0}^s \qquad z_{0j}^s = -z_{j0}^s$$

where the superscript and first and second subscripts identify the system, origin, and position, respectively. They form the position vectors defined in (1.02-1) and are related to each other by the angular transformation matrices.

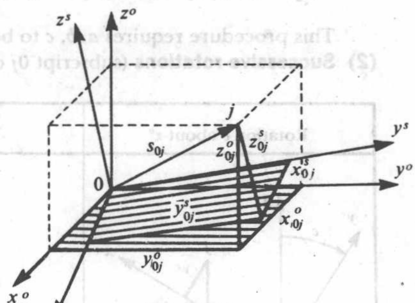


Fig. 1.09-1

(2) Angular transformation matrix equations are

$$\underbrace{\begin{bmatrix} x_{0j}^o \\ y_{0j}^o \\ z_{0j}^o \end{bmatrix}}_{s_{0j}^o} = \underbrace{\begin{bmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{bmatrix}}_{R^{os}} \underbrace{\begin{bmatrix} x_{0j}^s \\ y_{0j}^s \\ z_{0j}^s \end{bmatrix}}_{s_{0j}^s} \qquad \underbrace{\begin{bmatrix} x_{0j}^s \\ y_{0j}^s \\ z_{0j}^s \end{bmatrix}}_{s_{0j}^s} = \underbrace{\begin{bmatrix} \alpha_x & \beta_x & \gamma_x \\ \alpha_y & \beta_y & \gamma_y \\ \alpha_z & \beta_z & \gamma_z \end{bmatrix}}_{R^{so}} \underbrace{\begin{bmatrix} x_{0j}^o \\ y_{0j}^o \\ z_{0j}^o \end{bmatrix}}_{s_{0j}^o}$$

where s_{0j}^o and s_{0j}^s are the position vectors of *j* in the 0 and *S* systems, respectively, and R^{os}, R^{so} are the angular transformation matrices whose coefficients are the direction cosines of angles between the respective axes (Fig. 1.09-1).

(3) Relations of angular transformation matrices are

$$R^{so} = R^{os)T} = R^{os)-1} \qquad R^{os} = R^{so)T} = R^{so)-1}$$

where the superscripts)*T* and)−1 identify the transpose and inverse of the matrix, respectively.

(4) Relations of direction cosines are summarized below (Ref. 5, p. 17).

Diagonal terms of matrix product $R^{os}R^{so}$	Off-diagonal terms of matrix product $R^{os}R^{so}$	Off-diagonal terms of matrix product $R^{so}R^{os}$	Diagonal terms of matrix product $R^{so}R^{os}$
$\alpha_x^2 + \alpha_y^2 + \alpha_z^2 = +1$	$\alpha_x\beta_x + \alpha_y\beta_y + \alpha_z\beta_z = 0$	$\alpha_x\alpha_y + \beta_x\beta_y + \gamma_x\gamma_y = 0$	$\alpha_x^2 + \beta_x^2 + \gamma_x^2 = +1$
$\beta_x^2 + \beta_y^2 + \beta_z^2 = +1$	$\beta_x\gamma_x + \beta_y\gamma_y + \beta_z\gamma_z = 0$	$\alpha_y\alpha_z + \beta_y\beta_z + \gamma_y\gamma_z = 0$	$\alpha_y^2 + \beta_y^2 + \gamma_y^2 = +1$
$\gamma_x^2 + \gamma_y^2 + \gamma_z^2 = +1$	$\gamma_x\alpha_x + \gamma_y\alpha_y + \gamma_z\alpha_z = 0$	$\alpha_z\alpha_x + \beta_z\beta_x + \gamma_z\gamma_x = 0$	$\alpha_z^2 + \beta_z^2 + \gamma_z^2 = +1$

1.10 DIRECTION COSINES BY ROTATION

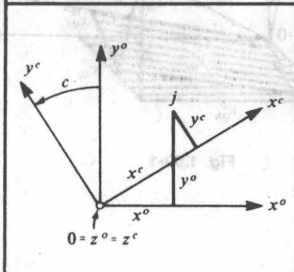
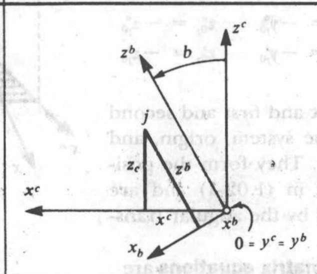
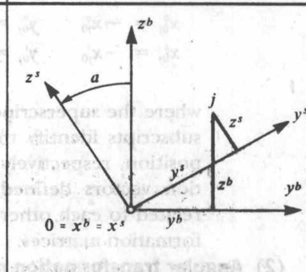
(1) Procedure. The numerical values of the direction cosines in (1.09-2) can be obtained by successive rotation of coordinate axes. Symbols used in this procedure are

$$a, b, c = \text{right-handed angles (rad)}$$

$$S_a = \sin a,\ S_b = \sin b,\ S_c = \sin c,\ C_a = \cos a,\ C_b = \cos b,\ C_c = \cos c$$

This procedure requires a, b, c to be given angles.

(2) Successive rotations (subscript $0j$ omitted in x, y, z)

Rotation c about z^o	Rotation b about y^c	Rotation a about x^b
		
$R^{oc} = \begin{bmatrix} C_c & -S_c & 0 \\ S_c & C_c & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R^{cb} = \begin{bmatrix} C_b & 0 & S_b \\ 0 & 1 & 0 \\ -S_b & 0 & C_b \end{bmatrix}$	$R^{bs} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_a & -S_a \\ 0 & S_a & C_a \end{bmatrix}$
$R^{co} = \begin{bmatrix} C_c & S_c & 0 \\ -S_c & C_c & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R^{bc} = \begin{bmatrix} C_b & 0 & -S_b \\ 0 & 1 & 0 \\ S_b & 0 & C_b \end{bmatrix}$	$R^{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_a & S_a \\ 0 & -S_a & C_a \end{bmatrix}$

(3) Space angular transformation matrices introduced in (1.09-2) are equal to the chain product of the component matrices (2) executed in the order of rotation. Thus

$$R^{os} = \begin{bmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{bmatrix} = \underbrace{\begin{bmatrix} C_bC_c & -C_aS_c + S_aS_bC_c & S_aS_c + C_aS_bC_c \\ S_bS_c & C_aC_c + S_aS_bS_c & -S_aC_c + C_aS_bS_c \\ -S_b & S_aC_b & C_aC_b \end{bmatrix}}_{R^{oc}R^{cb}R^{bs}}$$

By (1.09-2),

$$R^{so} = R^{(os)T} = R^{sb}R^{bc}R^{co}$$

(4) Planar angular transformation matrices are the component matrices in (2) above.