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Mathematical Problems in Image Processing

Partial Differential Equations and the Calculus of Variations

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To Jean-Michel Morel, whose ideas have deeply influenced the mathematical vision of image processing.

Foreword

Image processing, image analysis, computer vision, robot vision, and machine vision are terms that refer to some aspects of the process of computing with images. This process has been made possible by the advent of computers powerful enough to cope with the large dimensionality of image data and the complexity of the algorithms that operate on them.

In brief, these terms differ according to what kind of information is used and output by the process. In image processing the information is mostly the intensity values at the pixels, and the output is itself an image; in image analysis, the intensity values are enriched with some computed parameters, e.g., texture or optical flow, and by labels indicating such things as a region number or the presence of an edge; the output is usually some symbolic description of the content of the image, for example the objects present in the scene. Computer, robot, and machine vision very often use three-dimensional information such as depth and three-dimensional velocity and perform some sort of abstract reasoning (as opposed to purely numerical processing) followed by decision-making and action.

According to this rough classification this book deals with image processing and some image analysis.

These disciplines have a long history that can be traced back at least to the early 1960s. For more than two decades, the field was occupied mostly by computer scientists and electrical engineers and did not attract much interest from mathematicians. Its rather low level of mathematical sophistication reflected the kind of mathematical training that computer scientists and electrical engineers were exposed to and, unfortunately, still are: It is roughly limited to a subset of nineteenth-century mathematics.

This is one reason. Another reason stems from the fact that simple heuristic methods, e.g., histogram equalization, can produce apparently startling results; but these ad hoc approaches suffer from significant limitations, the main one being that there is no precise characterization of why and when they work or don't work. The idea of the proof of correctness of an algorithm under a well-defined set of hypotheses has long been almost unheard of in image processing and analysis despite the strong connection with computer science.

It is clear that things have been changing at a regular pace for some time now. These changes are in my view due to two facts: First, the level of mathematical sophistication of researchers in computer vision has been steadily growing in the last twenty-five years or so, and second, the number of professional mathematicians who develop an interest in this field of application has been regularly increasing, thanks maybe to the examples set by two Fields medallists, David Mumford and Pierre-Louis Lions. As a result of these facts the field of computer vision is going through a crucial mutation analogous to the one that turned alchemy into modern chemistry.

If we now wonder as to the mathematics relevant to image processing and analysis, we come up with a surprisingly long list: Differential and Riemannian geometry, geometric algebra, functional analysis (calculus of variations and partial differential equations), probability theory (probabilistic inference, Bayesian probability theory), statistics (performance bounds, sampling algorithms), and singularity theory (generic properties of solutions to partial differential equations) are all being successfully applied to image processing. It should be apparent that it is, in fact, the whole set of twentieth-century mathematics that is relevant to image processing and computer vision.

In what sense are those branches of mathematics relevant? As I said earlier, many of the original algorithms were heuristic in nature: No proof was in general given of their correctness, and no attempt was made at defining the hypotheses under which they would work or not. Mathematics can clearly contribute to change this state of affairs by posing the problems in somewhat more abstract terms with the benefit of a clarification of the underlying concepts, e.g., what are the relevant functional spaces, and what is the possibility of proving the existence and uniqueness of solutions to these problems under a set of well-defined hypotheses and the correctness of algorithms for computing these solutions? A further benefit of the increase of mathematical sophistication in machine vision may come out of the fact that the mathematical methods developed to analyze images with computers may be important for building a formal theory of biological vision: This was the hope of the late David Marr and should be considered as another challenge to mathematicians, computer-vision scientists, psychophysicists, and neurophysiologists.

Conversely, image processing and computer vision bring to mathematics a host of very challenging new problems and fascinating applications; they contribute to grounding them in the real world just as physics does.

This book is a brilliant "tour de force" that shows the interest of using some of the most recent techniques of functional analysis and the theory of partial differential equations to study several fundamental questions in image processing, such as how to restore a degraded image and how to segment it into meaningful regions. The reader will find early in the book a summary of the mathematical prerequisites as well as pointers to some specialized textbooks. These prerequisites are quite broad, ranging from direct methods in the calculus of variations (relaxation, Gamma convergence) to the theory of viscosity solutions for Hamilton-Jacobi equations and include the space of functions of bounded variations. Lebesgue theory of integration as well as Sobolev spaces are assumed to be part of the reader's culture, but pointers to some relevant textbooks are also provided.

The book can be read by professional mathematicians (who are, I think, its prime target) as an example of the application of different parts of modern functional analysis to some attractive problems in image processing. These readers will find in the book most of the proofs of the main theorems (or pointers to these in the literature) and get a clear idea of the mathematical difficulty of these apparently simple problems. The proofs are well detailed, very clearly written, and, as a result, easy to follow. Moreover, since most theorems can also be turned into algorithms and computer programs, their conclusions are illustrated with spectacular results of processing performed on real images. Furthermore, since the authors provide examples of several open mathematical questions, my hope is that this book will attract more mathematicians to their study.

It can also be read by the mathematically inclined computer-vision researcher. I do not want to convey the idea that I underestimate the amount of work necessary for such a person to grasp all the details of all the proofs, but I think that it is possible at a first reading to get a general idea of the methods and the main results. Hopefully, this person will then want to learn in more detail the relevant mathematics, and this can be done by alternating reading the textbooks that are cited and studying the proofs in the book. My hope is that this will convince more image-processing scientists that this mathematics must become part of the tools they use.

This book, written by two mathematicians with a strong interest in images, is a wonderful contribution to the mutation I was alluding to above, the transformation of image processing and analysis as well as computer, robot, and machine vision into formalized fields, based on sets of competing scientific theories within which predictions can be performed and methods (algorithms) can be compared and evaluated. This is hopefully a step in the direction of understanding what it means to see.

Sophia Antipolis, France OLIVIER FAUGERAS

Preface

It is surprising when we realize just how much we are surrounded by images. Images allow us not only to perform complex tasks on a daily basis, but also to communicate, transmit information, and represent and understand the world around us. Just think, for instance, about digital television, medical imagery, and video surveillance. The tremendous development in information technology accounts for most of this. We are now able to handle more and more data. Many day-to-day tasks are now fully or partially accomplished with the help of computers. Whenever images are involved we are entering the domains of computer vision and image processing. The requirements for this are reliability and speed. Efficient algorithms have to be proposed to process these digital data. It is also important to rely on a well-established theory to justify the well-founded nature of the methodology.

Among the numerous approaches that have been suggested, we focus on partial differential equations (PDEs), and variational approaches in this book. Traditionally applied in physics, these methods have been successfully and widely transferred to computer vision over the last decade. One of the main interests in using PDEs is that the theory behind the concept is well established. Of course, PDEs are written in a continuous setting referring to analogue images, and once the existence and the uniqueness have been proven, we need to discretize them in order to find a numerical solution. It is our conviction that reasoning within a continuous framework makes the understanding of physical realities easier and stimulates the intuition necessary to propose new models. We hope that this book will illustrate this idea effectively.

The message we wish to convey is that the intuition that leads to certain formulations and the underlying theoretical study are often complementary. Developing a theoretical justification of a problem is not simply "art for art's sake." In particular, a deep understanding of the theoretical difficulties may lead to the development of suitable numerical schemes or different models.

This book is concerned with the mathematical study of certain image-processing problems. Thus we target two audiences:

- The first is the mathematical community, and we show the contribution of mathematics to this domain by studying classical and challenging problems that come from computer vision. It is also the occasion to highlight some difficult and unsolved theoretical questions.

- The second is the computer vision community: we present a clear, self-contained, and global overview of the mathematics involved for the problems of image restoration, image segmentation, sequence analysis, and image classification.

We hope that this work will serve as a useful source of reference and inspiration for fellow researchers in applied mathematics and computer vision, as well as being a basis for advanced courses within these fields.

This book is divided into six main parts. Chapter 1 introduces the subject and gives a *detailed plan of the book*. In chapter 2, most of the mathematical notions used therein are recalled in an educative fashion and illustrated in detail. In Chapters 3 and 4 we examine how PDEs and variational methods can be successfully applied in the restoration and segmentation of one image. Chapter 5 is more applied, and some challenging computer vision problems are described, such as sequence analysis or classification. Since the final goal of any approach is to compute a numerical solution, we propose an introduction to the method of finite differences in the Appendix.

We would like to express our deep gratitude to the following people for their various contributions:

- The ARIANA group (joint project CNRS–INRIA–UNSA) from INRIA Sophia Antipolis and in particular Laure Blanc-Féraud and Christophe Samson for providing results regarding the classification problem.

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Guide to the Main Mathematical Concepts and Their Application

This book is principally organized by image processing problems and not by mathematical concepts. The aim of this guide is to highlight the different concepts used and especially to indicate where they are applied.

Direct Method in the Calculus of Variations (Section 2.1.2)

This terminology is used when the problem is to minimize an integral functional, for example

$$inf \left\{ F(u) = \int_{\Omega} f(x, u(x), \nabla u(x))\, dx,\ u \in V \right\}. \qquad (\mathcal{F})$$

The classical (or direct) method consists in defining a minimizing sequence $u_n \in V$, bounding u_n uniformly in V, and extracting a subsequence converging to an element $u \in V$ (compactness property, Section 2.1.1) and proving that u is a minimizer (lower semicontinuity property, Section 2.1.2). This technique has been applied in two cases:

- Image restoration (Section 3.2.3, Theorem 3.2.2)
- Sequence segmentation (Section 5.1.3)

Relaxation (Section 2.1.3)

When the direct method does not apply to a minimization problem $(\mathcal{F})$ (because the energy is not lower semicontinuous (l.s.c.) or the space is not reflexive, for example) it is then a classical approach to associate with $(\mathcal{F})$ another problem called $(R\mathcal{F})$ (relaxed problem), that is, another functional RF (relaxed functional). Then $(R\mathcal{F})$ is related to $(\mathcal{F})$ thanks to the following two properties: The first is that $(R\mathcal{F})$ is well posed that is $(R\mathcal{F})$ has solutions and $\min\{RF\} = inf\{F\}$. The second is that we can extract from minimizing sequences of $(\mathcal{F})$ subsequences converging to a solution of $(R\mathcal{F})$. We have used this concept for the following:

- Image restoration for which the initial formulation was mathematically ill posed (Section 3.2.3, Theorem 3.2.1 and Section 3.3.1, Proposition 3.46).

Γ-convergence (Section 2.1.4)

The notion of Γ-convergence relates to convergence for functionals. It is particularly well adapted to deal with free discontinuity problems. Roughly speaking, if the sequence of functionals F_h Γ-converges to another functional F, and if u_h is a sequence of minimizers of F_h and u a minimizer of F, then (up to sequence) $\lim\limits_{h\to 0} F_h(u_h) = F(u)$ and $\lim\limits_{h\to 0} u_h = u$. This notion is illustrated in the following two cases:

- Approximation of the Mumford–Shah segmentation functional (Section 4.2.4, Theorem 4.2.8).
- Image classification (Section 5.2.1).

Viscosity Solutions (Section 2.3)

The theory of viscosity solutions aims at proving the existence and uniqueness of a solution for fully nonlinear PDEs of the form

$$\frac{\partial u}{\partial t} + F(x, u(x), \nabla u(x), \nabla^2 u(x)) = 0.$$

This is a very weak notion because solutions are expected to be only continuous. We have used this theory for the following:

- The Alvarez–Guichard–Lions–Morel scale space theory (Section 3.3.1, Theorem 3.3.2).
- Geodesic active contours and level-set methods (Section 4.3.3, Theorem 4.3.2).

Notation and Symbols

About Functionals

For Ω an open subset of R^N we define the following real-valued function spaces:

$\mathcal{B}(\Omega)$	Borel subset of Ω.
S^{N-1}	Unit sphere in R^N.
dx	Lebesgue measure in R^N.
$\mathcal{H}^{N-1}$	Hausdorff measure of dimension $N-1$.
$BV(\Omega)$	Space of bounded variation.
BV−w*	The weak* topology of $BV(\Omega)$.
$C_0^p(\Omega)$	Space of real-valued functions, p continuously differentiable with compact support.
$C_0^\infty(\Omega)$	Space of real-valued functions, infinitely continuously differentiable with compact support.
$C^{0,\gamma}(\Omega)$	For $0 < \gamma \leq 1$: space of continuous functions f on Ω such that $\|f(x) - f(y)\| \leq C\|x - y\|^\gamma$, for some constant C, x, $y \in \Omega$. It is called the space of Hölder continuous functions with exponent γ.
$C^{k,\gamma}(\Omega)$	Space of k-times continuously differentiable functions whose kth partial derivatives belong to $C^{0,\gamma}(\Omega)$.

$(C_0^\infty(\Omega))'$	Dual of $C_0^\infty(\Omega)$, i.e. the space of distributions on Ω.
$L^p(\Omega)$	Space of Lebesgue measurable functions f such that $\int_\Omega \|f\|^p \, dx < \infty$.
$L^\infty(\Omega)$	Space of Lebesgue measurable functions f such that there exists a constant c with $\|f(x)\| \leq c$, a.e. $x \in \Omega$.
$\mathcal{M}(\Omega)$	Space of Radon measures.
$SBV(\Omega)$	Space of special functions of bounded variation.
$LSC(\Omega)$	Space of lower semicontinuous functions on Ω.
$USC(\Omega)$	Space of upper semicontinuous functions on Ω.
$W^{1,p}(\Omega)$	With $1 \leq p \leq \infty$: Sobolev space of functions $f \in L^p(\Omega)$ such that all derivatives up to order p belong to $L^p(\Omega)$. $W^{1,\infty}(\Omega)$ is identified with the space of locally Lipschitz functions.
$W_0^{1,p}(\Omega)$	$\{u \in W^{1,p}(\Omega) : \ u\|_{\partial\Omega} = 0\}$.

(Vector-valued spaces will be written in boldface: $\mathbf{BV(\Omega)}$, $\mathbf{C}_0^p(\Omega)$, $\mathbf{L^p(\Omega)}$, $\mathbf{M}(\Omega)$, $\mathbf{W^{1,p}}(\Omega)$, $\mathbf{SBV(\Omega)}$).

For X a Banach space with a norm $|.|_X$ and $v : (0,T) \to X$:

$C^m(0,T;X)$	With $m \geq 0$, $0 < T < \infty$: space of functions from $[0,T]$ to X m-times continuously differentiable. It is a Banach space with the norm $\|v\|_{C^m(0,T;X)} = \max_{0 \leq l \leq m} \left(\sup_{0 \leq t \leq T} \left\| \frac{\partial^l v}{\partial t^l}(t) \right\|_X \right)$.
$L^p(0,T;X)$	With $1 \leq p < \infty$: space of functions $v \to v(t)$ measurable on $(0,T)$ for the measure dt (i.e., the scalar functions $t \to \|v\|_X$ are dt-measurable). It is a Banach space with the norm $\|v\|_{L^p(0,T;X)} = \left(\int_0^T \|v\|_X^p \, dt \right)^{1/p} < +\infty$.
$L^\infty(0,T;X)$	Space of functions v such that $\|v\|_{L^\infty(0,T;X)} = \inf_c \ \{\|v\|_X \leq c, \ \text{a.e. } t\}$.

For a functional $F : X \to]-\infty, +\infty]$ where X is a Banach space:

$Argmin\ F$	$\{u \in X : F(u) = inf_X F\}$.
$R_\tau(F)$, RF, $\overline{F}$	Relaxed functional of F (for the τ-topology).

l.s.c.(sequentially)	Lower semicontinuous: F is called l.s.c. if for every sequence (u^n) converging to u we have $\underline{lim}_{n\to+\infty} F(u^n) \geq F(u)$.
u.s.c.(sequentially)	Upper semicontinuous: F is called u.s.c. if for every sequence (u^n) converging to u we have $\overline{lim}_{n\to+\infty} F(u^n) \leq F(u)$.

About Measures

For μ and ν two Radon measures:

$\|\mu\|$	Total variation of the measure μ. If μ is vector-valued, we also denote $\|\mu\| = \|\mu_1\| + \cdots + \|\mu_N\|$.
$\nu \ll \mu$	ν is absolutely continuous with respect to μ if $\mu(A) = 0 \Rightarrow \nu(A) = 0$ for all Borel set $A \in R^N$.
$\nu \perp \mu$	ν is singular with respect to μ if there exists a Borel set $B \subset R^N$ such that $\mu(R^N - B) = \nu(B) = 0$.

About Functions

For a function $f : \Omega \subset R^N \to R$ and a sequence of functions $(f^n)_{n\in N}$ belonging to a Banach space X:

$f^n \xrightarrow[X]{} f$	The sequence (f^n) converges strongly to f in X.
$f^n \underset{X}{\rightharpoonup} f$	The sequence (f^n) converges weakly to f in X.
$f^n \overset{*}{\underset{X}{\rightharpoonup}} f$	The sequence (f^n) converges to f for the weak* topology of X.
$\overline{lim}_{n\to+\infty} f^n$	$\overline{lim}_{n\to+\infty} f^n(x) = inf_k\, sup \{f^k(x), f^{k+1}(x), \ldots\}$.
$\underline{lim}_{n\to+\infty} f^n$	$\underline{lim}_{n\to+\infty} f^n(x) = -\overline{lim}_{n\to+\infty} f^n(x) = sup_k\, inf\, \{f^k(x), f^{k+1}(x), \ldots\}$.
$\|f\|_X$	Norm of f in X.
spt(f)	For a measurable function $f : \Omega \subset R^N \to R$, let $(w_i)_{i\in I}$ be the family of all open subsets such that $w_i \in \Omega$ and for each $i \in I$, $f = 0$ a.e. on w_i. Then spt (the support of f) is defined by spt$f = \Omega - \bigcup_i w_i$.
Df	Distributional gradient of f.
$D^2 f$	Hessian matrix of f (in the distributional sense).
∇f	Gradient of f in the classical sense. It corresponds to the absolutely continuous part of Df with respect to the Lebesgue measure dx.

$\mathrm{div}(f)$	Divergence operator: $\mathrm{div}(f) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}$.
$\nabla^2 f$	Hessian matrix of f in the classical sense: $(\nabla^2 f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.
Δf	Laplacian operator: $\Delta f = \sum_{i=1}^{N} \frac{\partial^2 f}{\partial {x_i}^2}$.
$⨍_\Omega f\, dx$	Mean value of f over Ω: $⨍_\Omega f\, dx = \frac{1}{\|\Omega\|} \int_\Omega f\, dx$.
$\overset{\bullet}{f}$	Precise representation of f.
$\mathcal{P}^{\pm}_{\Omega}$	"superjets."

For a function $\phi : R^N \to R$:

$\phi^*(.)$	The Fenchel–Legendre conjugate.
$\phi^\infty(.)$	The recession function defined by $\phi^\infty(z) = \lim\limits_{s\to+\infty} \phi(sz)/s$.

Miscellaneous notation

$A \underset{\text{strong}}{\hookrightarrow} B$	A is relatively compact in B.
$A \underset{\text{weak}}{\hookrightarrow} B$	A is weakly relatively compact in B.
O^*	The adjoint operator of O.
$\|.\|$	Euclidean norm in R^N.
G_σ	The Gaussian kernel defined by $G_\sigma(x) = 1/(2\pi\,\sigma^2)\ \exp\left(-\|x\|^2/(2\,\sigma^2)\right)$.
$B(x,r) \subset R^N$	Ball of center x and radius r.
$\mathcal{S}(N)$	Subset of $N \times N$ symmetric matrices.
$SNR(I_1/I_2)$	*Signal-to-noise ratio*: used to estimate the quality of an image I_2 with respect to a reference image I_1. It is defined by $SNR(I_1/I_2) = 10\log_{10}\left[\frac{\sigma^2(I_2)}{\sigma^2(I_1 - I_2)}\right]$, where σ is the variance.
$\alpha \vee s \wedge \beta$	Truncature function equal to α if $s \le \alpha$, β if $s \ge \beta$, s otherwise.
$\mathrm{sign}(s)$	Sign function equal to 1 if $s > 0$, 0 if $s = 0$, and -1 if $s < 0$.
χ_R	Characteristic function of R: $\chi_R(x) = \begin{cases} 1 \text{ if } x \in R, \\ 0 \text{ otherwise.} \end{cases}$
$\mathrm{Per}_\Omega(R)$	Perimeter of R in Ω defined as the total variation of χ_R.

Symbols for the Reader's Convenience

👁	Indicates general references, books, reviews, or other parts of the book where related information can be found.
☛	Summary of an *important idea.*
■	Symbol marking the *end* of a proof, example or remark.
✱	This symbol indicates *unsolved or challenging unsolved problems* that need to be investigated further.

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