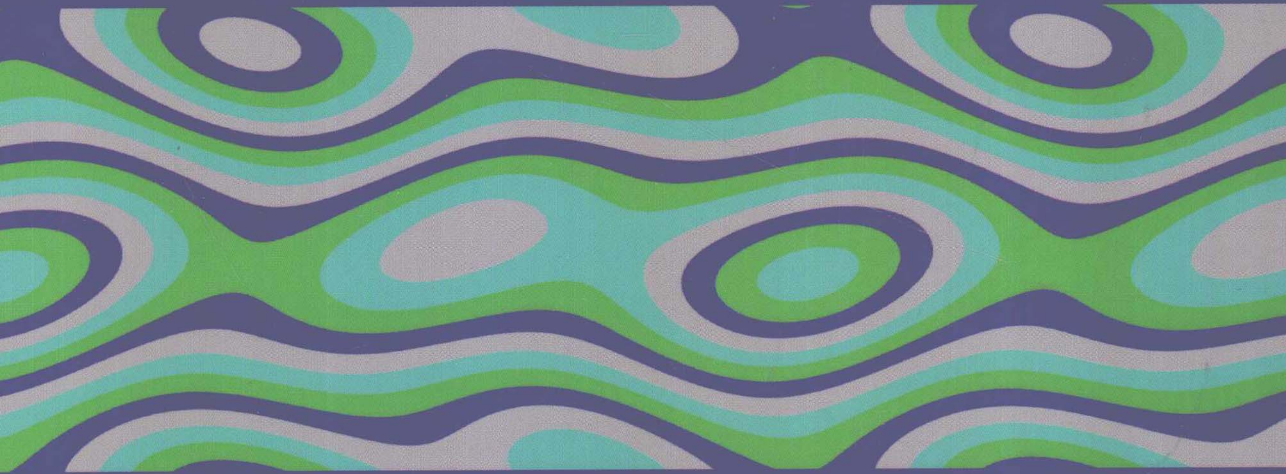
Advances in Mathematical Fluid Mechanics

Contributions to Current Challenges in Mathematical Fluid Mechanics



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Preface

This volume consists of five research articles, each dedicated to a significant topic in the mathematical theory of the Navier–Stokes equations, for compressible and incompressible fluids, and to related questions. All results given here are new and represent a noticeable contribution to the subject.

One of the most famous predictions of the Kolmogorov theory of turbulence is the so-called Kolmogorov–Obukhov five-thirds law. As is known, this law is heuristic and, to date, there is no rigorous justification. The article of A. Biryuk deals with the Cauchy problem for a multi-dimensional Burgers equation with periodic boundary conditions. Estimates in suitable norms for the corresponding solutions are derived for "large" Reynolds numbers, and their relation with the Kolmogorov–Obukhov law are discussed. Similar estimates are also obtained for the Navier–Stokes equation.

In the late sixties J. L. Lions introduced a "perturbation" of the Navier–Stokes equations in which he added in the linear momentum equation the hyper-dissipative term $(-\Delta)^{\beta}u$, $\beta \geq 5/4$, where Δ is the Laplace operator. This term is referred to as an "artificial" viscosity. Even though it is not physically motivated, artificial viscosity has proved a useful device in numerical simulations of the Navier–Stokes equations at high Reynolds numbers. The paper of of D. Chae and J. Lee investigates the global well-posedness of a modification of the Navier–Stokes equation similar to that introduced by Lions, but where now the original dissipative term $-\Delta u$ is replaced by $(-\Delta)^{\alpha}u$, $0 \leq \alpha < 5/4$. Existence, uniqueness and stability of solutions is proved in appropriate Besov spaces depending on the parameter α.

Space averaged Navier–Stokes equations are the basic equations for large eddy simulation of turbulent flows. In deriving these equations it is tacitly understood that differentiation and averaging operations can be interchanged. Actually, this procedure introduces a "commutation error" term that is typically ignored. The main objective in the paper of A. Dunca, V. John and W. L. Layton is to furnish a characterization of this term to be neglected. The authors go on to provide a justification for neglecting this term if and only if the Cauchy stress vector of the underlying flow is identically zero on the boundary of the domain. In other words, neglecting the commutation error is reasonable only for flows in which the boundary exerts no influence on the flow.

Since the appearence of the paper of J. G. Heywood in the mid-seventies, the problem of a flow through an aperture (the "aperture domain" problem) has attracted the attention of many researchers. But even now, a number of basic questions remain unresolved.The article of T. Hishida provides a further, significant contribution. Specifically, the author proves $L^q - L^r$ estimates for the Stokes semigroup in an aperture domain of $\mathbb{R}^n$, $n \geq 3$. These estimates are then used to show the existence, uniqueness and asymptotic behavior in time of strong solutions of the Navier–Stokes equations having "small" initial data in L^n and zero flux through the aperture.

The mathematical analysis of the well-posedness of the Navier–Stokes equations in the case of a compressible viscous fluid is a relatively new branch of mathematical fluid mechanics, its first contribution dating back to a paper of J. Nash in the early sixties. Many problems remain to be solved in this area, despite the significant contributions of many mathematicians. In particular, there remain very interesting problems concerning steady flow in an exterior domain, especially regarding the asymptotic behavior of solutions. This latter problem is analyzed in the paper of T. Leonavičienė and K. Pileckas, in the case when the velocity of the fluid is zero at large distances and the body force is the sum of an "arbitrary large" potential term and a "small" non-potential term.

We would like to express our warm thanks to Professors H. Beirão da Veiga, A. Fursikov and Y. Giga who recommended the publication of these articles.

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Contents

Advances in Mathematical Fluid Mechanics, 1–30

On Multidimensional Burgers Type Equations with Small Viscosity

Andrei Biryuk

Abstract. We consider the Cauchy problem for a multidimensional Burgers type equation with periodic boundary conditions. We obtain upper and lower bounds for derivatives of solutions for this equation in terms of powers of the viscosity and discuss how these estimates relate to the Kolmogorov–Obukhov spectral law. Next we use the estimates obtained to get certain bounds for derivatives of solutions of the Navier–Stokes system.

Mathematics Subject Classification (2000). 35B10, 35A30, 76D05.

Keywords. Kolmogorov–Obukhov spectral law, bounds for derivatives, degenerate state.

1. Introduction

We study the dynamics of m-dimensional vector field $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ on the n-dimensional torus $\mathbb{T}^n = {}^{\mathbb{R}^n}\!/_{(\ell\mathbb{Z})^n}$ described by the equation

$$\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{f}(\boldsymbol{u})} \boldsymbol{u} = \nu \Delta \boldsymbol{u} + \boldsymbol{h}(t, \boldsymbol{x}) \,. \tag{1.1}$$

Here ν is a positive parameter ("the viscosity"), $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map, $\boldsymbol{h}$ is a smooth forcing term and $\nabla_{\boldsymbol{f}(\boldsymbol{u})}$ is the derivative along the vector $\boldsymbol{f}(\boldsymbol{u})$, i.e., $\nabla_{\boldsymbol{f}(\boldsymbol{u})} \boldsymbol{u} = (\boldsymbol{f}(\boldsymbol{u}) \cdot \nabla) \boldsymbol{u}$.

If $m = n$ and $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$, we have the usual forced Burgers equation. In a potential case (i.e., if the initial state $\boldsymbol{u}_0(\boldsymbol{x}) = \boldsymbol{u}(0, \boldsymbol{x})$ and the field $\boldsymbol{h}$ are gradients of some functions) this equation can be reduced to a linear parabolic equation.

As it is shown in [1], [12], appropriate bounds for derivatives imply estimates for averaged spectral characteristics of the flow. The purpose of this work is to obtain such bounds for solutions of the Cauchy problem for the generalised m-n multidimensional Burgers equation (1.1).

We describe notations used in this article. If $\boldsymbol{v}$ is a vector in $\mathbb{R}^s$, $\mathbb{Z}^s$ or $\mathbb{C}^s$, then $|\boldsymbol{v}|$ denotes its Euclidean (Hermitian) norm $|\boldsymbol{v}|^2 = \sum_{i=1}^{s} |v_i|^2$. If we have to stress the dimension, we denote the norm in $\mathbb{R}^s$ as $|\boldsymbol{v}|_{\mathbb{R}^s}$, etc. By $B(r)$ we denote the ball of radius r centered at the origin. If $A : \mathbb{R}^{s_1} \to \mathbb{R}^{s_2}$ is a linear map then $\|A\|$ denotes the operator-norm of this map associated with the Euclidean norms $|\cdot|$ on $\mathbb{R}^{s_1}$ and $\mathbb{R}^{s_2}$. If $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$, then we write

$$|\boldsymbol{v}| = \sup_{\boldsymbol{x}} |\boldsymbol{v}(\boldsymbol{x})| = \sup_{\boldsymbol{x}} \Big(\sum |v_i(\boldsymbol{x})|^2\Big)^{1/2}. \tag{1.2}$$

Sometimes we will denote this norm by $|\cdot|_{L_\infty}$. If $\boldsymbol{v} = \boldsymbol{v}(t,\boldsymbol{x})$, then $|\boldsymbol{v}| = \sup_{\boldsymbol{x}} |\boldsymbol{v}(t,\boldsymbol{x})|$ is a function of t. For a multi-index α we denote $|\alpha| = \sum |\alpha_i|$.

We set

$$H(t) = \int_0^t \sup_{\boldsymbol{x}\in\mathbb{T}^n} |\boldsymbol{h}(\tau,\boldsymbol{x})|_{\mathbb{R}^m}\, d\tau\,. \tag{1.3}$$

We also denote

$$[\boldsymbol{f}]_{C^k(r)} = \max_{|\beta|=k} \sup_{\{\boldsymbol{u}\in\mathbb{R}^m : |\boldsymbol{u}|\leqslant r\}} \Big(\sum_{j=1}^n \Big|\frac{\partial^k}{\partial \boldsymbol{u}^\beta} f_j\Big|^2\Big)^{1/2}, \tag{1.4}$$

and

$$\|\boldsymbol{u}\|_k^2 = \int_{\mathbb{T}^n} \sum_{i=1}^m u_i(-\Delta)^k u_i d\boldsymbol{x} = \sum_{i=1}^m \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \big|D^\alpha u_i\big|^2_{L_2(\mathbb{T}^n)}$$

$$= \sum_{i=1}^m \sum_{j_1,..,j_k=1}^n \Big|\frac{\partial^k u_i}{\partial x_{j_1}\cdots\partial x_{j_k}}\Big|^2_{L_2(\mathbb{T}^n)}. \tag{1.5}$$

Here $k \geqslant 0$ is an integer and $\binom{|\alpha|}{\alpha} = \binom{|\alpha|}{\alpha_1,\dots,\alpha_n} = \frac{(\alpha_1+\cdots+\alpha_n)!}{\alpha_1!\cdots\alpha_n!}$ are coefficients in the generalised binomial expansion $(x_1 + x_2 + \ldots + x_n)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} \boldsymbol{x}^\alpha$. If $\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x})$, then $\|\boldsymbol{u}\|_k = \|\boldsymbol{u}(t,\cdot)\|_k$ is a function of t.

Our main results are stated in the following two theorems, where $\boldsymbol{u}(t,\boldsymbol{x})$, $t \geqslant 0$, is any smooth solution for the equation (1.1):

Theorem 1. *For any $k \geqslant 0$, $t \geqslant 0$, and $\nu > 0$ we have*

$$\|\boldsymbol{u}(t,\cdot)\|_k \leqslant R_k(t) \max\Big\{\frac{1}{\nu^k}, \frac{\|\boldsymbol{u}_0\|_k}{R_k(0)}, \frac{\sup_{[0,t]}\|\boldsymbol{h}\|_{k-1}}{R_k(0)}\Big\}\,. \tag{1.6}$$

Here $\|\boldsymbol{h}\|_{-1} = 0$, $R_0(t) = (|\boldsymbol{u}_0| + H(t))\ell^{n/2}$ and

$$R_k(t) = \big(1 + C_{k,m,n} \max_{s=0\ldots k-1}\{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}_0|+H(t))}\,(|\boldsymbol{u}_0| + H(t))^s\}\big)^k (|\boldsymbol{u}_0| + H(t))\ell^{n/2},$$

where the constant $C_{k,m,n}$ depends on k, m, n only.

Definition 1. The vector field $\boldsymbol{u}_0$ is degenerate with respect to equation (1.1) if the matrix $\frac{\partial \boldsymbol{f}(\boldsymbol{u}_0)}{\partial \boldsymbol{x}}$ (this is an $n \times n$ matrix, which depends on $\boldsymbol{x}$) is everywhere nilpotent, i.e., for each point $\boldsymbol{x}$ some power of this matrix is equal to 0.

Theorem 2. *Suppose that the initial state $\boldsymbol{u}_0$ is a non-degenerate vector field. Then there exist ν-independent positive real constants T, c and r_2, r_3, r_4, ... such that:*
If $H(T) < \frac{c}{2}$ then $\forall \nu > 0$ and $\forall k \geqslant 2$, we have:

$$\max_{j=1\dots n} \frac{1}{T} \int_0^T \sup_{\boldsymbol{x} \in \mathbb{T}^n} \left| \frac{\partial^k \boldsymbol{u}}{\partial x_j{}^k}(t, \boldsymbol{x}) \right|_{\mathbb{R}^m} dt \geqslant \frac{r_k}{\nu^{k/2}} . \tag{1.7}$$

The constants T, c, r_2, r_3,... depend on the non-degeneracy of the initial state $\boldsymbol{u}_0$ in quite complicated way (see (3.11), (3.16), (3.19), and (3.33)). The nearer is $\boldsymbol{u}_0$ to set of degenerate vector functions, the bigger is T and the smaller are c, r_2, r_3,... . If $\boldsymbol{u}_0 = \lambda \boldsymbol{v_0}$ and $\lambda \to 0$, then $T \propto \lambda^{-1}$, $r_k \propto \lambda^{k/2}$, c does not depend on λ.

In Section 3 we give an example of a degenerate non-constant initial state for which derivatives of the solution are bounded by ν-independent constants for all $t \geqslant 0$. Moreover, for the two dimensional case ($m = n = 2$) and for $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$, $\boldsymbol{h} \equiv 0$ we show that any solution with a degenerate initial state retains bounded derivatives. This fact is based on a result from the classical geometry due to Pogorelov–Hartman–Nirenberg, known as the "Cylinder Theorem". In Section 3 we show that in the case $m = n$ and $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{u}$, any non-constant *potential* initial state is non-degenerate.

The exponents of viscosity ν in inequalities (1.6) and (1.7) are not sharp. In the one-dimensional case ($m = n = 1$) sharp values for the exponents can be obtained. Namely, it is shown in [1] that for $k \geqslant 1$ we have

$$\|\boldsymbol{u}\|_k \leqslant C_k \left(\tfrac{1}{\nu}\right)^{k-1/2} , \quad \left(\frac{1}{T}\int_0^T \|\boldsymbol{u}\|_k^2 dt\right)^{1/2} \geqslant c_k \left(\tfrac{1}{\nu}\right)^{k-1/2}.$$

As a consequence of these inequalities one can get bounds for magnitudes of the derivatives:

$$\left|\frac{d^k u}{dx^k}\right|_{L_\infty} \leqslant \frac{C'_k}{\nu^k} , \quad \frac{1}{T}\int_0^T \left|\frac{d^k u}{dx^k}\right|_{L_\infty} dt \geqslant \frac{c'_k}{\nu^k} .$$

The first inequality for $k = 0$ follows by the maximum principle and for $k \geqslant 1$ – by the inequality $|v|_{L_\infty} \leqslant |v|_{L_2}^{1/2} |v_x|_{L_2}^{1/2}$ which holds for any periodic function v with zero meanvalue, see [1], Sect. 3. To derive the second inequality (see [1] and formula (3.7) there) we use the well-known fact that for periodic solutions of 1D Burgers-type equations the quantity $\left|\frac{du}{dx}\right|_{L_1}$ is bounded uniformly in t (see e.g. the appendix in [1]). Then for $k = 1$ the second estimate follows by the Hölder inequality $|u_x|_{L_2}^2 \leqslant |u_x|_{L_\infty} |u_x|_{L_1}$, while for $k > 1$ it follows by interpolation with the upper bound for $k = 0$.

This article is organised as follows. In Section 2 we prove the upper estimates (1.6) (theorem 1). Section 3 is devoted to proving the lower bounds (1.7) (theorem 2). In Section 4 we obtain some results on behaviour of Fourier coefficients of solutions that can be extracted from the bounds (1.6) and (1.7). Assuming that there is a Kolmogorov–Obukhov type spectral asymptotics for the Fourier coefficients of solutions of (1.1), we get bounds for the exponents of the spectral law and for the

Kolmogorov dissipation scale. In Section 5, treating the Navier–Stokes system as a partial case of (1.1), we derive lower bounds for derivatives of its solutions.

The author is grateful to Professor S. Kuksin for constant attention to this work.

2. Upper estimates

In this section we prove Theorem 1. The componentwise representation of (1.1) is

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} f_j(u_1,\dots,u_m)\frac{\partial u_i}{\partial x_j} = \nu\Delta u_i + h_i(t,x_1,\dots,x_n)\,, \tag{2.1}$$

where $i = 1,\dots,m$.

Lemma 1. *Let T and ν be any positive numbers. Let $v = v(t,\boldsymbol{x})$ be a continuous function on $[0,T]\times\mathbb{T}^n$ with the continuous derivatives v_t, v_{x_j} and $v_{x_jx_j}$ for any $j = 1,\dots,n$. Let $V_j = V_j(t,\boldsymbol{x})$, $j = 1,\dots,n$ and $g = g(t,\boldsymbol{x})$ be continuous functions on $[0,T]\times\mathbb{T}^n$. Suppose that on $[0,T]\times\mathbb{T}^n$ we have the following partial differential inequality:*

$$v_t + \sum_{j=1}^{n} V_j \frac{\partial v}{\partial x_j} \leqslant \nu\Delta v + g(t,x_1,\dots,x_n).$$

Then for any $(t,\boldsymbol{x}) \in [0,T]\times\mathbb{T}^n$ we have

$$v(t,\boldsymbol{x}) \leqslant \max_{\boldsymbol{y}\in\mathbb{T}^n} v(0,\boldsymbol{y}) + \int_0^t \max_{\boldsymbol{y}\in\mathbb{T}^n} g(\tau,\boldsymbol{y})d\tau.$$

Proof. Making the substitution $v(t,\boldsymbol{x}) = \tilde{v}(t,\boldsymbol{x}) + q(t)$, where $q{:}[0,T]\to\mathbb{R}$ is a function such that $q'(t) = \max_{\boldsymbol{y}\in\mathbb{T}^n} g(t,\boldsymbol{y})$, we reduce this lemma to the case $g\equiv 0$. Now the statement of the lemma becomes a classic maximum principle, see e.g. [4]. □

Applying this lemma for $v(t,\boldsymbol{x}) = \sum a_i u_i(t,\boldsymbol{x})$ and $g(t,\boldsymbol{x}) = \sum a_i h_i(t,\boldsymbol{x})$ with appropriate unit vector $\boldsymbol{a}\in\mathbb{R}^m$ we obtain

$$|\boldsymbol{u}(t,\cdot)| \leqslant |\boldsymbol{u}_0| + H(t)\,. \tag{2.2}$$

Here the norm $|\cdot|$ is defined by (1.2) and $H(t)$ is defined by (1.3).

Since $\|\boldsymbol{u}\|_0 \leqslant \ell^{n/2}\,|\boldsymbol{u}|$, we have

$$\|\boldsymbol{u}(t,\cdot)\|_0 \leqslant \ell^{n/2}(|\boldsymbol{u}_0| + H(t))\,. \tag{2.3}$$

This proves (1.6) for $k=0$. Next, we multiply (2.1) by $(-\Delta)^k u_i$, take the sum over $i = 1,\dots,m$, and integrate over the period (over the torus):

$$\tfrac{1}{2}\tfrac{d}{dt}\|\boldsymbol{u}\|_k^2 - b(\boldsymbol{f},\boldsymbol{u},(-\Delta)^k\boldsymbol{u}) = -\nu\|\boldsymbol{u}\|_{k+1}^2 + \Upsilon_2^k\,.$$

Here we denote

$$b(\boldsymbol{f},\boldsymbol{u},\boldsymbol{v}) = -\int_{\mathbb{T}^n} \sum_{\substack{j=1\dots n\\ i=1\dots m}} f_j \frac{\partial u_i}{\partial x_j} v_i d\boldsymbol{x}\,. \tag{2.4}$$

and

$$\Upsilon_2^k = \int_{\mathbb{T}^n} \sum_{i=1\dots m} h_i (-\Delta)^k u_i d\boldsymbol{x}\,.$$

Lemma 2. *For the functional b introduced above we have*

$$b(\boldsymbol{f}(\boldsymbol{u}), \boldsymbol{u}, \boldsymbol{u}) \leqslant [\boldsymbol{f}]_{C^0(|\boldsymbol{u}|)} \|\boldsymbol{u}\|_1 \|\boldsymbol{u}\|_0\,, \tag{2.5}$$

$$b(\boldsymbol{f}(\boldsymbol{u}), \boldsymbol{u}, (-\Delta)\boldsymbol{u}) \leqslant [\boldsymbol{f}]_{C^0(|\boldsymbol{u}|)} \|\boldsymbol{u}\|_1 \|\boldsymbol{u}\|_2\,, \tag{2.6}$$

and for any $k \geqslant 2$ we have

$$b(\boldsymbol{f}(\boldsymbol{u}), \boldsymbol{u}, (-\Delta)^k\boldsymbol{u}) \leqslant C_{k,m,n} \max_{s=0,\dots,k-1} \{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}|)} |\boldsymbol{u}|^s_{L_\infty}\} \|\boldsymbol{u}\|_k \|\boldsymbol{u}\|_{k+1}\,. \tag{2.7}$$

Proof. First we prove a general inequality on $b(\cdot,\cdot,\cdot)$:

$$|b(\boldsymbol{f}, \boldsymbol{u}, \boldsymbol{v})| \leqslant |\boldsymbol{f}|\, \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_0\,, \tag{2.8}$$

where $|\boldsymbol{f}| = \sup\big(\sum_{j=1}^n f_j^2\big)^{1/2}$ and the norms $\|\cdot\|_s$ are defined in (1.5). Up to a constant factor this inequality is obvious. Below we show that for the chosen norm the constant is equal to 1. By the definition (2.4) of $b(\cdot,\cdot,\cdot)$ and the Cauchy–Schwartz inequality we have

$$|b(\boldsymbol{f}, \boldsymbol{u}, \boldsymbol{v})| \leqslant \int \Big(\sum_{j=1}^n f_j^2\Big)^{1/2} \Big(\sum_{j=1}^n \Big(\sum_{i=1}^m \tfrac{\partial u_i}{\partial x_j} v_i\Big)^2\Big)^{1/2} dx$$

now we again use the Cauchy–Schwartz inequality $(\sum a_i b_i)^2 \leqslant (\sum a_i^2)(\sum b_i^2)$ to continue as follows:

$$\leqslant |\boldsymbol{f}| \int \Big(\sum_{j=1}^n \Big(\sum_{i=1}^m \big(\tfrac{\partial u_i}{\partial x_j}\big)^2\Big)\Big(\sum_{i=1}^m v_i^2\Big)\Big)^{1/2} dx$$

$$= |\boldsymbol{f}| \int \Big(\sum_{j=1}^n \sum_{i=1}^m \big(\tfrac{\partial u_i}{\partial x_j}\big)^2\Big)^{1/2} \Big(\sum_{i=1}^m v_i^2\Big)^{1/2} dx$$

$$\leqslant |\boldsymbol{f}| \Big(\int \sum_{j=1}^n \sum_{i=1}^m \big(\tfrac{\partial u_i}{\partial x_j}\big)^2 d\boldsymbol{x}\Big)^{1/2} \Big(\int \sum_{i=1}^m v_i^2 dx\Big)^{1/2} = |\boldsymbol{f}|\, \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_0\,.$$

The inequality (2.8) and therefore (2.5) are proved. Using $\|\Delta \boldsymbol{u}\|_0 = \|\boldsymbol{u}\|_2$ we arrive at (2.6).

Consider the case $k \geqslant 2$. By (2.4) we have

$$b(\boldsymbol{f}, \boldsymbol{u}, (-\Delta)^k \boldsymbol{u}) = (-1)^{k-1} \int \sum_{j_0,\dots,j_k=1}^n \sum_{i=1}^m f_{j_0}(u_1,\dots,u_m) \frac{\partial u_i}{\partial x_{j_0}} \frac{\partial^2}{\partial x_{j_1}^2} \cdots \frac{\partial^2}{\partial x_{j_k}^2} u_i d\boldsymbol{x}.$$

Integrating by parts $k-1$ times we obtain

$$b(\boldsymbol{f}, \boldsymbol{u}, (-\Delta)^k \boldsymbol{u})$$

$$= \int \sum_{j_0,\dots,j_k=1}^n \sum_{i=1}^m \frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_{k-1}}} \Big(f_{j_0}(u_1,\dots,u_m) \frac{\partial u_i}{\partial x_{j_0}}\Big) \frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_{k-1}}} \frac{\partial^2}{\partial x_{j_k}^2} u_i\, d\boldsymbol{x}.$$

Using the identity

$$\|\boldsymbol{u}\|_{k+1}^2 = \int \sum_{j_1,\ldots,j_{k-1}=1,\ldots,n} \sum_{i=1}^{m} \Big(\sum_{j_k=1}^{n} \frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_{k-1}}} \frac{\partial^2}{\partial x_{j_k}^2} u_i\Big)^2 d\boldsymbol{x}$$

we get

$$|b(\boldsymbol{f}, \boldsymbol{u}, (-\Delta)^k \boldsymbol{u})|$$
$$\leqslant \Big(\int \sum_{j_0,\ldots,j_{k-1}=1}^{n} \sum_{i=1}^{m} \Big(\frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_{k-1}}} \big(f_{j_0}(u_1,\ldots,u_m) \frac{\partial u_i}{\partial x_{j_0}}\big)\Big)^2 d\boldsymbol{x}\Big)^{1/2} \|\boldsymbol{u}\|_{k+1}$$

Now to prove the lemma it suffices to verify the inequality

$$\int \Big| \frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_{k-1}}} \big(f_{j_0}(u_1,\ldots,u_m) \frac{\partial u_i}{\partial x_{j_0}}\big)\Big| dx$$
$$\leqslant C'_{k,m,n} \max_{s=0,\ldots,k-1} \{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}|)} |\boldsymbol{u}|_{L_\infty}^s\} \|\boldsymbol{u}\|_k . \quad (2.9)$$

(Indeed, (2.9) implies (2.7) with $C_{k,m.n} = (mn^{k-1})^{1/2} C'_{k,m,n}$.) Expanding the brackets in (2.9) we get no more than $(m+1)(m+2)\cdots(m+k-1)$ terms of the form

$$\int_{\mathbb{T}^n} D_{\boldsymbol{u}}^{\beta} f_{j_0} D_{\boldsymbol{x}}^{\alpha_0} u_i D_{\boldsymbol{x}}^{\alpha_1} u_{i_1} \cdots D_{\boldsymbol{x}}^{\alpha_{|\beta|}} u_{i_{|\beta|}} d\boldsymbol{x} ,$$

where $|\alpha_0| + |\alpha_1| + \cdots + |\alpha_{|\beta|}| = k$ and the indexes i_s (where $s = 1, \ldots, |\beta|$) vary between 1 and m. The modulus of this integral is not bigger than

$$\mathfrak{A} = \big|D_{\boldsymbol{u}}^{\beta} f_{j_0}\big|_{L_\infty(B(|\boldsymbol{u}|))} \big|D_{\boldsymbol{x}}^{\alpha_0} u_i\big|_{L_{\frac{2k}{|\alpha_0|}}} \big|D_{\boldsymbol{x}}^{\alpha_1} u_{i_1}\big|_{L_{\frac{2k}{|\alpha_1|}}} \cdots \big|D_{\boldsymbol{x}}^{\alpha_{|\beta|}} u_{i_{|\beta|}}\big|_{L_{\frac{2k}{|\alpha_{|\beta|}|}}} .$$

Here $B(r)$ denotes the ball in $\mathbb{R}^m$ of radius r in the Euclidean norm, centered at the origin. Using the Gagliardo–Nirenberg inequality (see [9], pp. 106–107)

$$|D_{\boldsymbol{x}}^{\alpha_s} u_{i_s}|_{L_{\frac{2k}{|\alpha_s|}}} \leqslant 4^{|\alpha_s|(k-|\alpha_s|)} |u_{i_s}|_{L_\infty}^{1-\frac{|\alpha_s|}{k}} \|u_{i_s}\|_k^{\frac{|\alpha_s|}{k}} ,$$

and the inequality $\sum_{s=0}^{|\beta|}(|\alpha_s|k - |\alpha_s|^2) \leqslant \sum_{s=0}^{|\beta|}(|\alpha_s|k - |\alpha_s|) = k^2 - k$ we obtain

$$\mathfrak{A} \leqslant 4^{k^2-k} |D_{\boldsymbol{u}}^{\beta} f_j|_{L_\infty(B(|\boldsymbol{u}|))} |\boldsymbol{u}|_{L_\infty}^{|\beta|} \|\boldsymbol{u}\|_k .$$

Now using the fact that left hand side of (2.9) $\leqslant (m+1)\cdots(m+k-1)\max\{\mathfrak{A}\}$, we arrive at (2.9) with $C'_{k,m,n} = 4^{k^2-k}(m+1)(m+2)\cdots(m+k-1)$. □

Corollary 1. *For $k \geqslant 1$ we have:*

$$b(\boldsymbol{f}(\boldsymbol{u}), \boldsymbol{u}, (-\Delta)^k \boldsymbol{u}) \leqslant B_k(t) \|\boldsymbol{u}\|_k \|\boldsymbol{u}\|_{k+1} ,$$

where $B_k(t) = C_{k,m,n} \max_{s=0\ldots k-1} \{[\boldsymbol{f}]_{C^s(|\boldsymbol{u}_0|+H(t))} (|\boldsymbol{u}_0| + H(t))^s\}$.

Integrating Υ_2^k by parts, we obtain $\Upsilon_2^k \leqslant \|\boldsymbol{h}\|_{k-1} \|\boldsymbol{u}\|_{k+1}$, So we have

$$\tfrac{1}{2}\tfrac{d}{dt}\|\boldsymbol{u}\|_k^2 \leqslant \|\boldsymbol{u}\|_{k+1}\left(-\nu\|\boldsymbol{u}\|_{k+1} + B_k(t)\|\boldsymbol{u}\|_k + \|\boldsymbol{h}\|_{k-1}\right).$$

Now using the interpolation inequality in the form $\|\boldsymbol{u}\|_{k+1} \geqslant \|\boldsymbol{u}\|_k \left(\frac{\|\boldsymbol{u}\|_k}{\|\boldsymbol{u}\|_0}\right)^{1/k}$, we have

$$\tfrac{1}{2}\tfrac{d}{dt}\|\boldsymbol{u}\|_k^2 \leqslant \|\boldsymbol{u}\|_{k+1}\left(\|\boldsymbol{u}\|_k\left(-\nu\left(\tfrac{\|\boldsymbol{u}\|_k}{\|\boldsymbol{u}\|_0}\right)^{1/k} + B_k(t)\right) + \|\boldsymbol{h}\|_{k-1}\right).$$

It follows from this relation that

$$\text{if } \|\boldsymbol{u}\|_k > \frac{(B_k(t)+1)^k\|\boldsymbol{u}\|_0}{\nu^k} \text{ and } \|\boldsymbol{u}\|_k > \|\boldsymbol{h}\|_{k-1}, \text{ then } \|\boldsymbol{u}\|_k \text{ is decreasing.} \tag{2.10}$$

We denote the right hand side of (1.6) by $F_k(t)$. It is clear from the definition of the function F_k that

$$\|\boldsymbol{u}(0,\cdot)\|_k \leqslant F_k(0).$$

Using (2.3) we see that if $\|\boldsymbol{u}\|_k > F_k(t)$ then $\|\boldsymbol{u}\|_k$ is decreasing by argument (2.10). Since $F_k(t)$ is a non-decreasing function we obtain that $\|\boldsymbol{u}\|_k$ never can be greater than $F_k(t)$. We arrive at (1.6). Theorem 1 is proven.

3. Lower estimates

In this section we prove Theorem 2. Throughout this section we use standard facts from linear algebra about linear transformations. For the convenience of the reader we very briefly outline the proofs. See reference [7], for an elegant, coordinate free presentation. We start from brief discussing of the notion of the degeneracy of a vector field.

3.1. Degeneracy condition

Using the fact that an $n \times n$ matrix A is nilpotent iff $A^n = 0$, we can give a definition of degeneracy that is equivalent to the previous one, but more robust.

Definition 2. The vector field $\boldsymbol{u}_0$ is degenerate iff $\left(\frac{\partial \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))}{\partial \boldsymbol{x}}\right)^n \equiv 0.$

Let $\tilde{\boldsymbol{f}}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$. Consider the characteristic polynomial of the matrix $\frac{\partial \tilde{\boldsymbol{f}}}{\partial \boldsymbol{x}}$:

$$\chi_{\boldsymbol{x}}(\lambda) = \det\left(\tfrac{\partial \tilde{\boldsymbol{f}}}{\partial \boldsymbol{x}} - \lambda \mathbb{1}\right) = (-\lambda)^n + (-\lambda)^{n-1} I_1(\boldsymbol{x}) + \cdots + I_n(\boldsymbol{x}).$$

Expanding the determinant, we obtain

$$I_k(\boldsymbol{x}) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \det \begin{pmatrix} \frac{\partial \tilde{f}_{i_1}}{\partial x_{i_1}} & \cdots & \frac{\partial \tilde{f}_{i_1}}{\partial x_{i_k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{f}_{i_k}}{\partial x_{i_1}} & \cdots & \frac{\partial \tilde{f}_{i_k}}{\partial x_{i_k}} \end{pmatrix}. \tag{3.1}$$

Using the Jordan form of the matrix we see that if $\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x})$ is nilpotent then all $I_j(\boldsymbol{x})$ are zero numbers. From the Hamilton–Cayley identity (any matrix is a root of its characteristic polynomial) we get the converse. Thus we have that the matrix $\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x})$ is nilpotent iff $I_1(\boldsymbol{x}) = \cdots = I_n(\boldsymbol{x}) = 0$. We got another equivalent definition of degeneracy which we will use subsequently:

Definition 3. The vector field $\boldsymbol{u}_0$ is degenerate iff $I_k(\boldsymbol{x}) \equiv 0$ for all $k \in \{1, \ldots, n\}$.

We also note that if $m < n$, then the matrix $\frac{\partial \tilde{f}_i}{\partial x_j}$ has rank $\leqslant m$, so for $k \in [m+1, n]$ we have $I_k(\boldsymbol{x}) \equiv 0$.

Lemma 3. *For each* $k = 1, \ldots, n$ *we have* $\int_{\mathbb{T}^n} I_k(\boldsymbol{x}) d\boldsymbol{x} = 0$.

Proof. We need to show that

$$\int_{\mathbb{T}^n} \det \left(\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x}) + \lambda \mathbf{1}\right) d\boldsymbol{x} \equiv (\ell\lambda)^n. \tag{3.2}$$

Since both the left hand side and the right hand side are polynomials in λ, it is sufficient to prove this equality for all integer λ. We write $\frac{\partial \tilde{f}_i}{\partial x_j}(\boldsymbol{x}) + \lambda \mathbf{1} = \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{x}}$, where the vector valued function $\boldsymbol{\Psi}$ is defined by the formula $\boldsymbol{\Psi}(\boldsymbol{x}) = \tilde{\boldsymbol{f}}(\boldsymbol{x}) + \lambda \boldsymbol{x}$. Since λ is an integer, then this function defines a map from $\mathbb{T}^n$ to $\mathbb{T}^n$. Since $\boldsymbol{\Psi}$ is homotopic to the map $\boldsymbol{x} \mapsto \lambda \boldsymbol{x}$ on torus $\mathbb{T}^n$ (a homotopy is given by $\boldsymbol{\Psi}_t(\boldsymbol{x}) = t\tilde{\boldsymbol{f}}(\boldsymbol{x}) + \lambda \boldsymbol{x}$), we have $\deg \boldsymbol{\Psi} = \deg\{\boldsymbol{x} \mapsto \lambda \boldsymbol{x}\}$ and hence $\deg \boldsymbol{\Psi} = \lambda^n$. Using the formula

$$\int_{\mathbb{T}^n} \det \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{x}} d\boldsymbol{x} = \deg \boldsymbol{\Psi} \int_{\mathbb{T}^n} d\boldsymbol{x}$$

(see [3], II, chpt. 3) we arrive at (3.2) (since $\int_{\mathbb{T}^n} d\boldsymbol{x} = \ell^n$). □

It follows that any potential degenerate initial state is constant. Indeed, if $\boldsymbol{f}(\boldsymbol{u}_0) = \nabla U$ then the function $U : \mathbb{R}^n \to \mathbb{R}$ necessarily has no more than linear growth (because $\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$ is periodic) and is also a harmonic function (because $\Delta U(\boldsymbol{x}) = \operatorname{div} \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x})) = I_1(\boldsymbol{x}) \equiv 0$); so $U(\boldsymbol{x}) = B\boldsymbol{x} + \boldsymbol{c}$ where B and $\boldsymbol{c}$ are a constant matrix and a constant vector, respectively.

Consider the case $n = 2$, i.e., $\dim \boldsymbol{x} = 2$.

Theorem 3. *Let* $n = 2$. *Then the vector field* $\boldsymbol{u}_0$ *is degenerate iff there exist a function* $\varphi_0 \colon \mathbb{R} \to \mathbb{R}$ *and real numbers* b_1, b_2, c_1, *and* c_2 *such that*

$$\begin{aligned} \{\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))\}_1 &= \quad b_2\varphi_0(b_1x_1 + b_2x_2) + c_1, \\ \{\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))\}_2 &= -b_1\varphi_0(b_1x_1 + b_2x_2) + c_2. \end{aligned} \tag{3.3}$$

Proof. The sufficiency is trivial. Indeed, if the vector field $\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$ has the form (3.3), then the Jacobi matrix

$$\begin{pmatrix} \partial f_1(\boldsymbol{u}_0)/\partial x_1 & \partial f_1(\boldsymbol{u}_0)/\partial x_2 \\ \partial f_2(\boldsymbol{u}_0)/\partial x_1 & \partial f_2(\boldsymbol{u}_0)/\partial x_2 \end{pmatrix} = \begin{pmatrix} b_1b_2\varphi_0' & b_2^2\varphi_0' \\ -b_1^2\varphi_0' & -b_2b_1\varphi_0' \end{pmatrix}$$

is nilpotent.

Necessity. Let c_1 and c_2 be the mean values of $f_1(\boldsymbol{u}_0(\boldsymbol{x}))$ and $f_2(\boldsymbol{u}_0(\boldsymbol{x}))$, respectively. Since $\operatorname{div} \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x})) = 0$, there exists a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{rot}\psi = \boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x})) - \binom{c_1}{c_2}$, where $\operatorname{rot}\psi = \binom{\partial\psi/\partial x_2}{-\partial\psi/\partial x_1}$. We note that the function $\psi(x_1, x_2)$ is $\mathbb{T}^2$-periodic, and hence bounded. Since the determinant of the Jacobi matrix of $\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{x}))$ is zero, we have that determinant of the Hessian of ψ is zero. Consider the graph of the function ψ in $\mathbb{R}^3$. The Gaussian curvature of this surface is given by the formula (see [3], I, chpt. 2)

$$K = \frac{\psi_{xx}\psi_{yy} - \psi_{xy}^2}{(1 + \psi_x^2 + \psi_y^2)^2}.$$

Therefore, $K = 0$. Now we use the fact that any complete surface of constant zero Gaussian curvature is a cylinder over a flat curve (see [13]; [14], chpt. 5). Since the function ψ is bounded, every generator of this cylinder is a horizontal line, hence it's equation can be written in the form

$$\begin{cases} b_1 x_1 + b_2 x_2 = const, \\ z = \tilde{\psi}(const). \end{cases}$$

We conclude that

$$\psi(x_1, x_2) = \tilde{\psi}(b_1 x_1 + b_2 x_2)$$

and (3.3) follows with $\varphi_0 = \tilde{\psi}'$. □

Corollary 2. *Suppose, $m = n = 2$, $\boldsymbol{f}(\boldsymbol{u}) \equiv \boldsymbol{u}$, and $\boldsymbol{h} \equiv 0$; then the solution of the Cauchy problem* (1.1), (3.3) *remains of the form* (3.3)*:*

$$\boldsymbol{u}(t, \boldsymbol{x}) = \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix} \varphi(t, b_1 x_1 + b_2 x_2) + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where the function φ satisfies the equation

$$\varphi_t + (b_1 c_1 - b_2 c_2)\varphi' = (b_1^2 + b_2^2)\nu\varphi''.$$

In this case we have ν-independent upper bounds for derivatives of the solution.

Further on we shall use the polynomial

$$P_{\boldsymbol{x}}(t) = t^n \chi_{\boldsymbol{x}}(\tfrac{-1}{t}) = \det\bigl(\delta_{ij} + \tfrac{\partial \bar{f}_i}{\partial x_j} t\bigr) = 1 + I_1(\boldsymbol{x})t + I_2(\boldsymbol{x})t^2 + \cdots + I_n(\boldsymbol{x})t^n, \quad (3.4)$$

rather than a characteristic polynomial.

3.2. General idea

In this subsection we present an auxiliary theorem from which we then derive Theorem 2. This auxiliary theorem is technically complicated. Here we deal mainly with general ideas, and postpone the technicalities to the next subsection.

We denote the right hand side of (2.1) by g_i:

$$\frac{\partial}{\partial t} u_i + \sum_{j=1}^{n} f_j(u_1, \ldots, u_m) \frac{\partial u_i}{\partial x_j} = g_i(t, x_1, \ldots, x_n)\,. \quad (3.5)$$

Theorem 4. *Let $\boldsymbol{f}\colon \mathbb{R}^m \to \mathbb{R}^n$ be a C^1-smooth map and let $\boldsymbol{u}_0\colon \mathbb{T}^n \to \mathbb{R}^m$ be a C^1-smooth vector field.*

1) *If $\boldsymbol{u}_0$ is non-degenerate, then there exist $T = T(\boldsymbol{f}, \boldsymbol{u}_0) < \infty$ and $c = c(\boldsymbol{f}, \boldsymbol{u}_0) > 0$ such that for any C^1-smooth vector field $\boldsymbol{u}\colon [0,T] \times \mathbb{T}^n \to \mathbb{R}^m$ with $\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})$ we have*

$$\int_0^T \sup_{\boldsymbol{x}\in\mathbb{T}^n} |\boldsymbol{g}(\tau,\boldsymbol{x})| d\tau \geqslant c\,, \tag{3.6}$$

where $\boldsymbol{g}$ is given by (3.5).

2) *If $\boldsymbol{u}_0$ is degenerate, then there is a C^1-smooth vector field $\boldsymbol{u}\colon [0,+\infty) \times \mathbb{T}^n \to \mathbb{R}^m$ such that $\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})$ and $g_i(t,\boldsymbol{x}) \equiv 0$.*

Proof. **1)** Without loss of generality it can be assumed that $\boldsymbol{u}(t,\boldsymbol{x})$ is defined for all $t \geqslant 0$. Consider the flow on the cylinder $\mathbb{T}^n \times [0,\infty)$ generated by the vector field $\boldsymbol{f}(\boldsymbol{u})$. In other words we consider the Cauchy problem

$$\tfrac{d}{dt}\boldsymbol{\gamma}(t,\boldsymbol{\xi}) = \boldsymbol{f}(\boldsymbol{u}(t,\boldsymbol{\xi}))$$

with the initial state $\boldsymbol{\gamma}(0,\boldsymbol{\xi}) = \boldsymbol{\xi}$. Here $\boldsymbol{\xi}$ is the Lagrange coordinate of the flow $\boldsymbol{\gamma}$.

For any fixed time t we have a map $\boldsymbol{\gamma}(t,\cdot) : \mathbb{T}^n \to \mathbb{T}^n$. Since $\boldsymbol{\gamma}(t,\cdot)$ is a continuous family of diffeomorphisms, equal identity for $t = 0$, then its Jacobian is everywhere positive.

Combining the chain rule and (3.5), we obtain

$$\tfrac{d}{dt}\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi})) = \boldsymbol{g}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi}))\,.$$

Suppose $\boldsymbol{g}(\cdot) \equiv 0$; then $\boldsymbol{f}\big(\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi}))\big) \equiv \boldsymbol{f}\big(\boldsymbol{u}(0,\boldsymbol{\gamma}(0,\boldsymbol{\xi}))\big)$ and $\boldsymbol{\gamma}(t,\boldsymbol{\xi}) = \boldsymbol{\gamma}^{\mathbf{0}}(t,\boldsymbol{\xi})$, where

$$\boldsymbol{\gamma}^{\mathbf{0}}(t,\boldsymbol{\xi}) = \boldsymbol{\xi} + t\boldsymbol{f}(\boldsymbol{u}_0(\boldsymbol{\xi}))\,. \tag{3.7}$$

It follows that if the function $\boldsymbol{g}$ is small, then the flow $\boldsymbol{\gamma}(t,\boldsymbol{\xi})$ is close (in the C^0-norm) to the map (3.7). For a detailed proof of this fact, we refer to the next subsection. For the time being we simply note that this is a consequence of the following inequality:

$$\big|\boldsymbol{u}(t,\boldsymbol{\gamma}(t,\boldsymbol{\xi})) - \boldsymbol{u}(0,\boldsymbol{\gamma}(0,\boldsymbol{\xi}))\big| \leqslant \int_0^t \sup_{\boldsymbol{x}\in\mathbb{T}^n} |\boldsymbol{g}(\tau,\boldsymbol{x})| d\tau\,.$$

Since for each $k = 1,\ldots,n$, we have $\int_{\mathbb{T}^n} I_k(\boldsymbol{x})d\boldsymbol{x} = 0$ (see lemma 3) and since some of the I_k are not identically zero (due to the non-degeneracy of $\boldsymbol{u}_0$), we obtain that there exists a point $\boldsymbol{x}^\star \in \mathbb{T}^n$ and a number $l \in [1,\ldots,n]$ such that $I_l(\boldsymbol{x}^\star) < 0$ and $I_k(\boldsymbol{x}^\star) = 0$ for $k > l$.

The Jacobian of (3.7) is expressed by polynomial (3.4). For the time $t = T$ at the point $\boldsymbol{\xi} = \boldsymbol{x}^\star$, we have

$$\det\Big(\frac{\partial\boldsymbol{\gamma}^0}{\partial\boldsymbol{\xi}}\Big|_{\substack{t=T\\ \boldsymbol{\xi}=\boldsymbol{x}^\star}}\Big) = P_{\boldsymbol{x}^\star}(T) = 1 + I_1(\boldsymbol{x}^\star)T + I_2(\boldsymbol{x}^\star)T^2 + \cdots + I_l(\boldsymbol{x}^\star)T^l\,.$$