Systems of Nonlinear Partial Differential Equations

edited by **J. M. Ball**

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Systems of Nonlinear Partial Differential Equations

edited by

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A. J. Majda (University of California, Berkeley), *The Design and Numerical Analysis of Vortex Methods*

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PREFACE

This volume contains the proceedings of a NATO/London Mathematical Society Advanced Study Institute held in Oxford from 25 July - 7 August 1982. The institute concerned the theory and applications of systems of nonlinear partial differential equations, with emphasis on techniques appropriate to systems of more than one equation. Most of the lecturers and participants were analysts specializing in partial differential equations, but also present were a number of numerical analysts, workers in mechanics, and other applied mathematicians.

The organizing committee for the institute was J.M. Ball (Heriot-Watt), T.B. Benjamin (Oxford), J. Carr (Heriot-Watt), C.M. Dafermos (Brown), S. Hildebrandt (Bonn) and J.S. Pym (Sheffield).

The programme of the institute consisted of a number of courses of expository lectures, together with special sessions on different topics.

It is a pleasure to thank all the lecturers for the care they took in the preparation of their talks, and S.S. Antman, A.J. Chorin, J.K. Hale and J.E. Marsden for the organization of their special sessions.

The institute was made possible by financial support from NATO, the London Mathematical Society, the U.S. Army Research Office, the U.S. Army European Research Office, and the U.S. National Science Foundation.

The lectures were held in the Mathematical Institute of the University of Oxford, and residential accommodation was provided at Hertford College.

Valuable assistance with the organization of the institute and with the typing of the proceedings was given by Maureen Gardiner, Anne Hodgson and Janice McClelland.

J.M. BALL

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PART I

EXPOSITORY LECTURES

ALGEBRAIC AND TOPOLOGICAL INVARIANTS FOR REACTION-DIFFUSION EQUATIONS

Charles C. Conley and Joel A. Smoller[1]

University of Wisconsin and University of Michigan

§ 1 INTRODUCTION

In the last few years, a good deal of progress has been made in the understanding of the qualitative properties of solutions of reaction-diffusion equations. This has been due to the introduction of new topological techniques into the field; in particular, we mention the concept of an isolated invariant set, and its index, as developed in [2].

This index is a homotopy invariant associated with an isolated invariant set. It is a generalization of the classical Morse index of an isolated rest point for a gradient flow in $\mathbb{R}^n$, but it is applicable to a far wider class of invariant sets; in fact, it is even relevant for partial differential equations, as we shall show here.

Under some very reasonable and natural hypotheses, one can decompose an isolated invariant set into its so-called Morse sets. This immediately implies the existence of certain exact sequences of cohomology groups defined on these Morse sets. Thus one obtains algebraic invariants associated with the given isolated invariant set. As is the usual case with algebraic invariants, they are discrete, and since they "depend continuously on the topology", a change in the algebraic structure reflects a change in the underlying topology - e.g., a bifurcation can occur. Moreoever, the exact sequence can also be used to compute indices. Finally, we point out that the aforementioned index actually has a "dynamic" aspect to it, in that it carries stability information. That is, one can use it to obtain precise statements on the dimensions of unstable manifolds.

J. M. Ball (ed.), Systems of Nonlinear Partial Differential Equations, 3–24.

In this paper, we want to point out the relevance of these ideas in the study of reaction-diffusion equations. The reader will see that there is involved only a bare minimum of algebraic topology, and that the "rewards" one obtains far exceed the effort involved in learning the topology.

§ 2 THE ABSTRACT FRAMEWORK

A. Algebraic Topology

We begin with some (very elementary) ideas from algebraic topology. The reader can consult any standard textbook in the field for motivation, background, and details.

We denote by (A,B) a pair of topological spaces where $A \supset B$. To any such pair, there is associated a sequence of Abelian groups $\{H^n(A,B): n = 0,1,2,\dots\}$, and $H^n(A,B)$ is called the nth (integral) cohomology group of the pair (A,B). If S^k is a k-sphere then it turns out that

$$H^n(S^k, pt.) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases}$$

If A, B, and C are topological spaces, and $A \supset B \supset C$, then there is a so called exact sequence of cohomology groups:

$$\dots \to H^{n-1}(B,C) \xrightarrow{\phi_{n-1}} H^n(A,B) \xrightarrow{\psi_n} H^n(A,C) \xrightarrow{\theta_n} H^n(B,C) \xrightarrow{\phi_n} \dots \quad .$$

This means that the above maps are all linear, and the image of any map is the kernel of the following map. For example,

$$\ker \psi_n = \operatorname{im} \phi_{n-1} \quad .$$

This is basically all of the algebraic topology that we shall need.

B. Isolated Invariant Sets and Morse Decompositions

Let Γ be a metric space, and let F be a flow on Γ; $F: \Gamma \times \mathbb{R} \to \Gamma$. We denote $F(\gamma,t)$ by $\gamma \cdot t$. For $N \subset \Gamma$, set

$$I(N) = \{\gamma \in N: \gamma \cdot \mathbb{R} \subset N\} \;;$$

$I(N)$ is called the invariant set in N. For $\gamma \in \Gamma$, we define the alpha- and omega-limit sets of γ by

$$\alpha(\gamma) = \bigcap_{t<0} \mathrm{cl}\{\gamma\cdot(-\infty,t)\}, \quad \omega(\gamma) = \bigcap_{t>0} \mathrm{cl}\{\gamma\cdot(t,\infty)\}.$$

Definition 1. (Morse Decomposition): Let I be a compact invariant set in Γ. A Morse decomposition of I is a finite collection $\{M_i\}_1^n$ of disjoint, compact invariant sets in I which can be ordered $(M_1, M_2, \ldots, M_n)$ such that if $\gamma \in I\setminus \bigcup_1^n M_i$, then there are indices $i<j$ with $\alpha(\gamma) \in M_j$ and $\omega(\gamma) \in M_i$. The above ordering will be termed admissible.

For example, if I consists of two rest points M_1 and M_2 which are connected by an orbit (see figure 1), then (M_1, M_2) is a Morse decomposition of I.

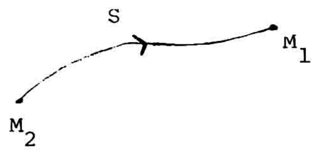


FIGURE 1

The reason that Morse decompositions are of interest to us is that many "sensible" properties of the equation are reflected in the Morse decomposition of its (isolated) invariant sets. Here sensible means stable under small perturbations.

In order to make our abstract theory applicable to partial differential equations, we introduce the concept of a local flow.

Definition 2. (Local Flow). A locally compact subset $X \subset \Gamma$ is called a local flow if for each $\gamma \in X$ there exists an $\varepsilon>0$, and a neighborhood U of γ such that $(X \cap U)\cdot[0,\varepsilon) \subset X$.

For example, we may think of X as a Sobolev space W^s; then our definition means that if the data is in W^s, the same is true for the corresponding solutions for small time.

From now on, we will always assume that X is a local flow in Γ.

Definition 3. (Isolated Invariant Set): Let N be a compact subset of X. If $I(N) \subset \mathrm{int}\,(N)$, relative to X, then $I(N)$ is called an isolated invariant set and N is called an isolating neighborhood.

The proof of the next lemma follows easily from the definitions, see [5].

<u>Lemma</u> 4. If S is an isolated invariant set, and $\{M_i\}_1^n$ is a Morse decomposition of S, then each M_i is an isolated invariant set.

For a given isolated invariant set S, we can associate to it an "index". To this end we require the following notion. Recall that by a pair of subsets (A,B) we mean $A \supset B$.

<u>Definition</u> 5. (<u>Index</u> <u>Pair</u>): Let S be an isolated invariant set and N an isolating neighborhood for S. A compact pair of subsets (N_1, N_0) of N is called an index pair for S provided the following three conditions hold:

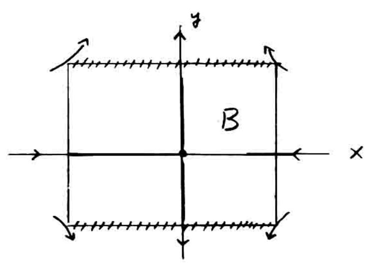
a) $cl(N_1 \setminus N_0)$ is an isolating neighborhood of S .

b) N_0 is positively invariant relative to N_1 ; ie if $\gamma \in N_0$ and $\gamma \cdot [0,t] \subset N_1$, then $\gamma \cdot [0,t] \subset N_0$.

c) If $\gamma \in N_1$ and $\gamma \cdot \mathbb{R}_+ \not\subset N$, then there is a t>0 with $\gamma \cdot [0,t) \subset N_1$, and $\gamma \cdot t \in N_0$.

The definition takes into account that we have only a local flow. Note too that c) implies that the orbit through γ exits N only through N_0 .

To illustrate this definition, consider the picture in the figure below. Here we can take the square B



<u>FIGURE 2</u>

to be N_1 and the "top" and "bottom" sides to be N_0 . Of course S is the hyperbolic rest point at the origin.

The desired topological invariants will be described in terms of index pairs. Before giving this description, we need to recall a few standard topological notions. First, we say that two topological spaces X and Y are <u>homotopic</u> if we can continuously deform one into the other. Next, if A is a closed subset of X, the quotient space X/A is the topological space obtained from X by identifying all points $a \in A$ to a single point. We can now give the main theorem.

Theorem 6. i) Index pairs exist; ie, given an isolated invariant set S, one can always prove the existence of an index pair for S .

ii) For any index pair (N_1, N_0) of S, the homotopy type of the quotient space N_1/N_0 is independent of N, N_1, N_0, and depends only on S . It is thus a topological invariant associated with the isolated invariant set S .

In view of ii), we may unambiguously define the (homotopy) index of S, by

$$h(S) = [N_1/N_2] .$$

where the brackets denote the homotopy equivalence class.[2] Observe that for the above example $N_1/N_0 = \Sigma^1$, a (pointed) 1-sphere.

Now it is not difficult to check that this index extends the Morse index in the sense that if x_0 is a non-degenerate rest point of a gradient system of ordinary differential equations, and k is the dimension of the unstable manifold at x_0, then $h(x_0) = \Sigma^k$. Furthermore, there is an addition formula for the index. Namely, if S_1 and S_2 are disjoint isolated invariant sets, then $S_1 \cup S_2$ is an isolated invariant set, and

$$h(S) = h(S_1) \vee h(S_2) , \qquad (*)$$

where by v, we denote the operation of "glueing" the two (quotient) spaces together at their distinguished point.

Note that if $S = \phi$, then it is easy to see $h(S) = \bar{0}$, the one point (pointed) space. Thus if $h(S) \neq \bar{0}$, then $S \neq \phi$. This is how the index is used as an existence tool.

Next, we turn to a useful result which can be viewed as a generalization of the first part of the last theorem.

Theorem 7. ([2,5]). Let S be an isolated invariant set and let $(M_1, M_2, \ldots, M_n)$ be an admissible ordering of a Morse decomposition of S . Then there exists an increasing sequence of compact sets $N_0 \subset N_1 \subset \ldots \subset N_n$ such that i) (N_n, N_o) is an index pair for S, and ii) (N_k, N_{k-1}) is an index pair for M_K .

We say that the sets N_0, N_1, ..., N_n form a Morse filtration of S .

Now if $N_0 \subset \ldots \subset N_n$ is a Morse filtration of S, we consider the triple (N_{p+1}, N_p, N_{p-1}). As we have seen above, there is an exact sequence,

$$\ldots \to H^{n-1}(N_p, N_{p-1}) \to H^n(N_{p+1}, N_p) \to H^n(N_{p+1}, N_{p-1}) \to \ldots \quad .$$

Now under reasonable hypotheses, there is an isomorphism $H^n(A,B) \approx H^n(A/B, \phi) \equiv H^n(A/B)$, provided B is closed in A. Applying this to the above exact sequence gives

$$\ldots \to H^{n-1}(N_p/N_{p-1}) \to H^n(N_{p+1}/N_p) \to H^n(N_{p+1}/N_{p-1}) \to \ldots \quad .$$

In particular, referring back to our previous example of an orbit connecting to rest points (see figure 1), we have the important exact sequence in terms of indices:

$$\ldots \to H^{n-1}(h(M_1)) \to H^n(h(M_2)) \to H^n(h(S)) \to \ldots \quad . \qquad (1)$$

It is useful to have a criterion for deciding when Morse decompositions exist, as well as one for being able to find isolated invariant sets. Both of these are difficult problems, in general, but here is one useful result. Before stating it recall that a differential equation is called <u>gradient-like</u> in a region if there is a function which strictly increases on orbits in the region.

<u>Theorem</u> 8. Let X be a gradient-like local flow.

1) If the flow has a finite rest point set C contained in an isolating neighborhood N, then the rest points form a Morse decomposition for I(N).

2) If γ_0 is an isolated rest point, then γ_0 is an isolated invariant set.

<u>Proof</u>. 1) Let ϕ be the gradient-like function. If $\gamma \in I(N) \setminus C$ and $t>0$, then $\phi(\gamma \cdot t) > \phi(\gamma)$, and so ϕ is constant on $\omega(\gamma)$. Consider the flow on the compact set I(N). The flow is gradient-like on I(N); thus $\omega(\gamma)$ and $\alpha(\gamma)$ both must be rest points. If we now order $C = \{\gamma_{i_1}, \ldots, \gamma_{i_n}\}$ in such a way that if $k<j$,

$F(x_{i_k}) > F(x_{i_j})$, then 1) easily follows. Let N be a compact neighborhood of γ_0 such that γ_0 is the only rest point in N. Let U be any neighborhood of γ_0, $U \subsetneq N$. Since $N \setminus U$ is compact, there is a $\delta>0$ such that $F(\gamma \cdot 1) - F(\gamma) \geq \delta > 0$ for all $\gamma \in N \setminus U$. If $\gamma \in N$ and $\gamma \cdot \mathbb{R} \subset N$, then $\omega(\gamma)$ and $\alpha(\gamma)$ are rest points in I(N);

thus $\omega(\gamma) = \gamma_o = \alpha(x_o)$, and $y \cdot \mathbb{R} \subset U$. Since U was arbitrary, $\gamma \cdot \mathbb{R} = \{\gamma_o\}$. Thus $\{\gamma_o\}$ is the only invariant set in N, and this proves (2).

We turn now to the last general notion, that of continuation. Like every "index" worthy of the name, our index is a homotopy invariant. Thus, consider a parametrised family of equations $\dot{u} = f(\lambda, u)$. Suppose that N is an isolating neighborhood for all parameter values λ in a closed connected set, say an arc. Then the various so determined isolated invariant sets are said to be related by continuation. By definition, if S' is so related to S, and S to S", then S' is so related to S". In this case one can prove the following theorem.

Theorem 9. If S and S' are isolated invariant sets which are related by continuation, then $h(S) = h(S')$.

Of course this result is often used to compute indices; namely one continues the given equation to one in which it is easier to compute indices.

§ 3 LOCAL FLOWS AND REACTION-DIFFUSION EQUATIONS

In order to be able to apply these general topological methods to partial differential equations, it is necessary to show that solutions generate local flows. We shall prove a general theorem on the existence of local flows, and we then shall show that the hypotheses can be verified for systems of reaction-diffusion equations.

Again let Γ be a metric space with metric ρ; by a curve on Γ is meant a continuous function $\gamma: I \to \Gamma$, where $I = \mathrm{dom}(\gamma)$ is an open interval in $\mathbb{R}$. A curve is called regular if its graph is closed in $\mathbb{R} \times \Gamma$. $C(I', \Gamma')$ denotes the set of regular curves γ on Γ such that $I' \subset \mathrm{dom}(\gamma)$ and $\gamma(I') \subset \Gamma'$.

Let F be a flow on Γ. The local flow X is called two sided if for all $\gamma \in X$, there is an $\varepsilon > 0$ and a neighborhood U of γ such that $(U \cap X) \cdot (-\varepsilon, \varepsilon) \subset X$. If $\gamma \in X$ implies $\gamma \cdot \mathbb{R}_+ \subset X$, X is called a semi-flow. If $\gamma \in X$ implies $\gamma \cdot \mathbb{R} \subset X$, X is called an invariant set; or more consistently, a sub flow. We give some examples. Let $C(\Gamma)$ denote the regular curves on Γ.

1. For any $t_o \in \mathbb{R}$, $C(t_o, \Gamma)$ is a two-sided local flow.
2. $C(\mathbb{R}_-, \Gamma)$ is a semi-flow.
3. Let C^- consist of those curves which have an interval of the form $(-\infty, t)$ in their domain. Then C^- is an invariant set.