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Joseph Lipman

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61 **Residues and traces of differential forms via Hochschild homology**, Joseph Lipman

Residues and Traces of Differential Forms via Hochschild Homology

Contents

§0. INTRODUCTION

The *residue symbol* introduced by Grothendieck [**RD**, pp. 195-199] has been found useful in various contexts: duality theory of algebraic varieties, Gysin homomorphisms of manifolds with vector fields having isolated zeros, integral representations in several complex variables, just to mention a few (cf. for example [**L**], [**AC**], [**AY**], and their bibliographies).

However, in spite of its broad interest, the theory of the residue symbol does not seem to have been written down in a really satisfactory manner. One difficulty is that Grothendieck's approach depends on the global duality machinery developed in [**RD**]; and furthermore proofs are not given there. (A more detailed version is presented in [**Bv**]; and for a complete treatment of the case of algebraic varieties, with a somewhat different slant, cf. [**L**].) Grothendieck considers a smooth map $f\colon X \to Y$ of locally noetherian schemes, with q-dimensional fibres, and a closed subscheme Z of X defined by an ideal I which is locally generated by q elements, and such that Z is finite over Y. With $i\colon Z \to X$ the inclusion, and $g = f \circ i\colon Z \to Y$, there is a *residue isomorphism* $i^!f^! \tilde{\to} g^!$, or, more concretely, a sheaf isomorphism:

$$g_*(\mathcal{H}om_{\mathcal{O}_Z}(\Lambda^q(I/I^2), i^*\Omega^q_{X/Y})) \tilde{\to} \mathcal{H}om_{\mathcal{O}_Y}(g_*\mathcal{O}_Z, \mathcal{O}_Y)$$

($\Omega^q_{X/Y}$ = relative differential q-forms) upon which the theory of the residue symbol is built.

But in fact the residue symbol can be viewed as a formal algebraic construct, which can be defined and studied directly with only the elements of ring theory and homological algebra. Indeed, while duality theory may provide the primary motivation for residues,[(1)] eliminating it from their theoretical foundation results not only in greater simplicity, but also in greater generality, and ultimately, one hopes, in more

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(1) and that is why [**L**] appeared before this paper. (The relation of this paper to [**L**] is made explicit in Appendix A of §3 below.) My own interest in the subject was inspired by p. 81 of [**S**], and by §§10 and 15 of [**Z**].

interconnections with other areas (see the end of this Introduction). In any case, the purpose of this paper is *to provide an elementary development of the theory of residues* .

The possibility of carrying out such a development of residues was known long ago to Cartier. He proposed a local definition, which could, in principle, be used to establish the properties listed in [**RD**], just as an exercise. It turned out to be quite a long exercise [**L**, p. 137]. In print, a beginning along these lines was made by Hopkins in [**H**]. The definition in [**H**], somewhat simpler than Cartier's, uses Koszul complexes, Ext functors, etc. I personally was uncomfortable with this definition, because Koszul complexes seem somehow too specialized; but I knew of no alternative. Then, around 1980, in an attempted proof of the "exterior differentiation" formula (R9) of [**RD**, p. 199] (given here in Appendix B of §3), the formalism of Hochschild homology began to extrude itself. It quickly became clear that this formalism provided a very convenient and surprisingly natural framework for the whole theory. Such, in brief, is the background of this paper.

* * *

The basic situation considered is the following: A is a commutative ring, R is an A-algebra (not necessarily commutative), and there is given a *representation* of R, i.e. an A-algebra homomorphism $R \to \mathrm{Hom}_A(P, P)$, where P is a finitely generated projective A-module. For each $q \geq 0$, there is then a natural R^c-linear pairing (R^c = center of R):

$$H^q(R, \mathrm{Hom}_A(P, P)) \otimes_{R^c} H_q(R, R) \to H_0(R, \mathrm{Hom}_A(P, P))$$

where H^q and H_q denote Hochschild cohomology and homology (reviewed at the beginning of §1). The usual trace map $\mathrm{Hom}_A(P, P) \to A$ factors through

$$H_0(R, \mathrm{Hom}_A(P, P)) = \mathrm{Hom}_A(P, P)/\{\text{commutators}\},$$

and composing with the preceding pairing we obtain the *residue homomorphism* (cf. (1.5)):

$$\mathrm{Res}^q\colon H^q(R, \mathrm{Hom}_A(P, P)) \otimes_{R^c} H_q(R, R) \to A$$

which is our basic object of study.

To get the *residue symbol* , we need to relate H^q and H_q to more concrete objects. Suppose for simplicity that R is commutative. Assume further that $P = R/I$ for some ideal I in R, and set

$$(I/I^2)^* = \mathrm{Hom}_P(I/I^2, P).$$

There are then natural homomorphisms of graded R-algebras

(1.8.3) $$\bigoplus_{n\geq 0} \otimes^{n}[(I/I^2)^*] \to \bigoplus_{n\geq 0} H^n(R, \mathrm{Hom}_A(P, P))$$

(1.10.2) $$\bigoplus_{n\geq 0} \Omega^n{}_{R/A} \to \bigoplus_{n\geq 0} H_n(R, R)$$

so that, via Res^q, we get a natural map

$$t^q: \otimes^q[(I/I^2)^*] \otimes_R \Omega^q{}_{R/A} \to A$$

(equal, when $q = 0$, to the trace map $P \to A$). For $\nu \in \Omega^q$, and $\alpha_1, \ldots, \alpha_q \in (I/I^2)^*$, we set

$$\mathrm{Res}\begin{bmatrix} \nu \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = t^q(\alpha_1 \otimes \cdots \otimes \alpha_q \otimes \nu).$$

Finally, if I/I^2 is free over P, with basis, say, $(f_i + I^2)_{1\leq i\leq q}$ $(f_i \in I)$, and if $(\alpha_1, \ldots, \alpha_q)$ is the dual basis of $(I/I^2)^*$, then we set

$$\mathrm{Res}\begin{bmatrix} \nu \\ f_1, \ldots, f_q \end{bmatrix} = \mathrm{Res}\begin{bmatrix} \nu \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}.$$

Details are worked out in §1, which culminates with the "determinant formula" (1.10.5) and its corollaries.

Sections 2, 3, and 4 are more or less independent of each other.

In section 2, we study the behavior of Res^q when the data (A, R, P) vary. In particular we prove a "base-change" formula relative to a ring homomorphism $\psi: A \to A'$:

$$\mathrm{Res}'\begin{bmatrix} \nu' \\ \alpha_1', \ldots, \alpha_q' \end{bmatrix} = \psi\left(\mathrm{Res}\begin{bmatrix} \nu \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}\right)$$

where "$'$" means "apply the functor $\otimes_A A'$ to everything in sight". (Cf. (2.4) for an exact formulation.) We also show how the residues in this paper lead to the residues in **[H]**; and then deduce the "transition formula" (2.8):

$$\mathrm{Res}\begin{bmatrix} \nu \\ g_1, \ldots, g_q \end{bmatrix} = \mathrm{Res}\begin{bmatrix} \det(r_{ij})\nu \\ f_1, \ldots, f_q \end{bmatrix}$$

for *regular* sequences $\mathbf{g} = (g_1, \ldots, g_q)$, $\mathbf{f} = (f_1, \ldots, f_q)$ in R, with

$$f_i = \sum_{j=1}^{q} r_{ij} g_j \qquad\qquad r_{ij} \in R, \quad 1 \leq i \leq q,$$

and such that $R/\mathbf{g}R$ and $R/\mathbf{f}R$ are finite and projective over A. (For this formula, at least, Koszul complexes remain unavoidable.)

At this point, we will have, among other things, reworked and extended most of the material in [**H**].

The first "hard" result appears in §3 (Corollary (3.7)): it is a formula for residues with respect to powers of the members of a quasi-regular sequence $\mathbf{f} = (f_1, \ldots, f_q)$ in the A-algebra R, with $R/\mathbf{f}R$ finite and projective over A. Such a formula in the case of power series rings is well-known; and we relate our "formally Cohen-Macaulay" situation to this case by embedding R into a power series ring in $(f_1, \ldots, f_q)$, with coefficients in the (usually) non-commutative finite projective A-algebra $\mathrm{Hom}_A(R/\mathbf{f}R, R/\mathbf{f}R)$. As a corollary we obtain in (3.10) a relation between Jacobian determinants, traces, and residues, which enables us, in particular, to derive the residues defined in [**L**] from those in this paper (cf. Appendix A). We also use (3.7) in Appendix B, to obtain the "exterior differentiation" formula alluded to above.

The second "hard" result is the *trace formula* (4.7.1), expressing a kind of adjointness relation between certain "trace" and "cotrace" maps in the Hochschild formalism. In terms of residue symbols, one consequence is the following.

We consider as above a commutative A-algebra R, and an ideal $I \subset R$ such that $P = R/I$ is finite and projective over A. We consider further a finite projective commutative R-algebra R', and set $I' = I R'$ (so that $P' = R'/I'$ is also finite and projective over A). Then, for any $\alpha \in \mathrm{Hom}_P(I/I^2, P)$ there exists a unique $\alpha' \in \mathrm{Hom}_{P'}(I'/I'^2, P')$ (the "cotrace" of α) making the following diagram (with horizontal arrows representing obvious maps) commute:

$$\begin{array}{ccc} I/I^2 & \longrightarrow & I'/I'^2 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ P & \longrightarrow & P' \end{array}$$

Furthermore, under suitable hypotheses (e.g. R smooth over A, or R' étale over R) there is a "trace map"

$$\tau_q : \Omega^q_{R'/A} \to \Omega^q_{R/A}\,;$$

and we have, for any $\nu \in \Omega^q_{R'/A}$:

$$\operatorname{Res}\begin{bmatrix} \nu \\ \alpha_1', \ldots, \alpha_q' \end{bmatrix} = \operatorname{Res}\begin{bmatrix} \tau_q(\nu) \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}.$$

The problem of defining a trace map τ_q for differential forms is indicated in [**RD**, p. 188]. Considerable work has been done on this problem, best documented in [**K**,§16]. A novel definition was discovered by Angéniol [**A**, pp. 108 ff]. His approach was computational; but it turned out that the definition could best be understood via Hochschild homology (cf. (4.6.5)). In fact, with R and R′ as above, and $H = \operatorname{Hom}_R(R', R')$, there is a trace map on homology, defined to be the composition

$$H_q(R', R') \xrightarrow{\text{natural}} H_q(H, H) \to H_q(R, R), \tag{0.1}$$

where the second arrow comes from "Morita equivalence". (We give a different description in §4.5). D. Burghelea has informed me that this type of composition also arose independently in work on Chern classes in cyclic homology. Differential forms are brought into the picture through the natural map $\Omega^q{}_{R/A} \to H_q(R, R)$ (cf. (1.10.2)); but since this map is not fully understood, several hard questions concerning conditions for the existence of a trace map for differential forms remain (cf. §(4.6)). Anyway, once residues and traces are both defined via Hochschild homology, the road to the "trace formula" in (4.7) is open.

* * *

The constructions in §4 suggest some tantalizing possibilities with respect to recent developments in other areas. One connection with cyclic homology has been indicated above (following (0.1)). Secondly, there is a natural homotopy class of maps, C, defined in (4.1), which underlies both the trace and the cotrace. A concrete – but highly non-canonical – representative of this class is described in (4.2). From this description, one can see that the "intermediate fundamental classes" recently defined by Angéniol and Lejeune-Jalabert [**AL**] could conveniently (i.e. with little or no computation) be formulated in terms of homotopy classes like C.

Further connections with cyclic homology might come out of arguments in Appendix B of §3; but I am unable to say more.

This Introduction began with the claim that there has not yet appeared a really satisfactory exposition of residues, a situation which this paper is meant to remedy, at least in part. The preceding remarks indicate that there might well be a more fundamental approach to the subject, encompassing a great deal more than we have dealt with here. If this paper helps someone toward such a discovery, it will have served its purpose.

* * *

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§1. THE RESIDUE HOMOMORPHISM

The general definition of residues, due basically to Cartier, has numerous formulations in terms of homological products. In this section we give a concrete description, more or less self-contained, of one such formulation (Definition (1.5.1)) via Hochschild homology and cohomology of associative algebras. The reader may wish to begin with (1.11), where the main results of §1 are summarized.

We begin with a quick review of some basic notions in the Hochschild theory (as presented in [**M**, Chapter 10]).

Let A be a commutative ring, and let R be an A-algebra (associative, but not necessarily commutative), i.e. R is a ring together with a ring homomorphism $h: A \to R$ such that $h(A) \subset R^c$, the center of R. An R-R *bimodule* is by definition an A-module M equipped with compatible left and right R-module structures both of which induce (via h) the A-module structure; in other words there are given A-bilinear "scalar multiplication" maps $R \times M \to M$ (respectively $M \times R \to M$) satisfying the usual conditions for *left* (respectively *right*) R-modules; and "compatibility" means that (with self-explanatory notation) $(rm)r' = r(mr')$ for all $r, r' \in R$ and $m \in M$.

With R^{op} the opposite algebra of R (that is, the A-module R together with the multiplication $\mu: R \times R \to R$ given by $\mu(x, y) = yx$), and R^e the "enveloping algebra" $R \otimes_A R^{op}$, an R-R bimodule M is essentially the same thing as a left R^e-module, the scalar multiplications being related by

$$(r \otimes r')m = rmr' \qquad (r, r' \in R;\ m \in M);$$

and also the same as a right R^e-module, with scalar multiplication

$$m(r' \otimes r) = rmr'.$$

(Via the antiautomorphism of R^e taking $r \otimes r'$ to $r' \otimes r$, every left R^e-module becomes a right R^e-module and vice versa.)

(1.0) The "bimodule bar resolution" $\epsilon: \mathbf{B}.(h) \to R$:

$$\cdots \xrightarrow{\partial_3} B_2 \xrightarrow{\partial_2} B_1 \xrightarrow{\partial_1} B_0 = R^e \xrightarrow{\epsilon} R$$

is defined as follows [**M**, p.282]. For $n \geq 0$, $B_n = B_n(h)$ is the left R^e-module $R^e \otimes_A T_A^n(R/A)$ where "R/A" denotes the cokernel of h, and

$$T_A^n(R/A) = (R/A)\otimes(R/A)\otimes \cdots \otimes(R/A) \qquad (\text{n factors; } \otimes = \otimes_A).$$

With r^* the natural image of $r \in R$ in R/A, we denote the element

$$(r\otimes r')\otimes[r_1^*\otimes \cdots \otimes r_n^*] \in B_n$$

by

$$r[r_1 \mid r_2 \mid \ldots \mid r_n]r'.$$

(The notation suggests that we think of B_n as an R-R bimodule.) Here we may omit r if $r = 1$, and similarly for r'. In particular we set

$$r[\ \ ]r' = (r\otimes r')\otimes 1 \in R\otimes R\otimes A = R^e = B_0.$$

Then the R^e-linear maps $\epsilon: R^e \to R$ and $\partial_n: B_n \to B_{n-1}$ $(n \geq 1)$ are determined by

$$\epsilon(r[\ \ ]r') = rr',$$

$$\partial_n(r[r_1 \mid r_2 \mid \ldots \mid r_n]r') = rr_1[r_2 \mid \ldots \mid r_n]r'$$
$$+ \sum_{i=1}^{n-1} (-1)^i r[r_1 \mid \ldots \mid r_i r_{i+1} \mid \ldots \mid r_n]r'$$
$$+ (-1)^n r[r_1 \mid \ldots \mid r_{n-1}]r_n r'.$$

B.(h) is a *positive complex* of left R^e-modules (i.e. $\partial_n\partial_{n+1} = 0$ for $n \geq 1$, and we take $B_m = (0)$ for $m < 0$), and ϵ: **B.**(h) $\to$ R is a *resolution* of the left R^e-module R, R being considered as a left R^e-module (= R-R bimodule) in the obvious way. In fact, with the right R-module homomorphisms

$$s_{-1}: R \to R^e = B_0$$

$$s_n: B_n \to B_{n+1} \qquad (n \geq 0)$$

determined by

$$s_{-1}(r') = 1\otimes r' = [\ \ ]r'$$

$$s_n(r[r_1 \mid \ldots \mid r_n]r') = [r \mid r_1 \mid \ldots \mid r_n]r'$$

we have

$$\epsilon s_{-1} = \text{identity}$$

$$\partial_1 s_0 + s_{-1}\epsilon = \text{identity}$$

$$\partial_{n+1}s_n + s_{n-1}\partial_n = \text{identity} \qquad (n \geq 1);$$

in other words, the s_i constitute a *right* R-*module splitting* (= contracting homotopy) of the bimodule resolution $\epsilon\colon \mathbf{B}.(h) \to R$; and furthermore

$$s_n s_{n-1} = 0 \qquad (n \geq 0).$$

(Our terminology is as in [**M**, pp. 41, 87].)

As indicated above, any R-R bimodule M can be considered as a left R^e-module and as a right R^e-module. The Hochschild homology and cohomology A-modules of the R-R bimodule M are defined then by

$$H_n(R, M) = H_n(M \otimes_{R^e} \mathbf{B}.(h))$$

$$H^n(R, M) = H^n(\mathrm{Hom}_{R^e}(\mathbf{B}.(h), M))^{(1)}.$$

[The notation $H_n(R, M)$, $H^n(R, M)$ is customary, though it would be more precise to write $H_n(h, M)$, $H^n(h, M)$]. In particular

(1.0.1) $$H_0(R, M) = M \otimes_{R^e} R = M/\{rm - mr\}$$

where $\{rm - mr\}$ is the A-submodule of M consisting of all sums of elements of the form $rm - mr$ $(r \in R,\ m \in M)$; and

(1.0.2) $$H^0(R, M) = \mathrm{Hom}_{R^e}(R, M) = \{m \in M \mid rm = mr \text{ for all } r \in R\}.$$

If $r \in H^0(R, R) = R^c$, the *center* of R, then multiplication by $r \otimes 1$ is an R^e-endomorphism of the complex $\mathbf{B}.(h)$ (or of the R^e-module M); and hence $H_n(R, M)$ and $H^n(R, M)$ are left R^c-modules. Similarly multiplication by $1 \otimes r$ gives rise to right R^c-module structures. These left and right R^c-module structures actually *coincide* (i.e. $rz = zr$ for all $r \in R^c$ and $z \in H_n(R, M)$ or $H^n(R, M)$): for given $r \in R^c$, if $t_n\colon B_n \to B_{n+1}$ is the unique R^e-homomorphism satisfying

$$t_n(r'[r_1 \mid \dots \mid r_n]r'') = \sum_{i=0}^{n} (-1)^i r'[r_1 \mid \dots \mid r_i \mid r \mid r_{i+1} \mid \dots \mid r_n]r''$$

then we have (for $n \geq 0$, with $t_{-1} = 0$):

(1) As in [**M**, p.42] we use the following *sign convention*: the coboundary of an n-cochain $f \in \mathrm{Hom}_{R^e}(B_n, M)$ is the n+1-cochain $(-1)^{n+1} f \circ \partial_{n+1}$.

$$\partial_{n+1}t_n + t_{n-1}\partial_n = \text{multiplication by } r\otimes 1 - 1\otimes r,$$

so that multiplication by $r\otimes 1$ in $\mathbf{B}.(h)$ is homotopic to multiplication by $1\otimes r$. Thus we can just think of $H^n(R, M)$ and $H_n(R, M)$ as being R^c-modules.

* * *

A basic component of our definition of residues will be a natural R^c-linear map

(1.1) $$\rho_M^q\colon H^q(R, M)\otimes_{R^c} H_q(R, R) \to H_0(R, M) \qquad (q\geq 0)$$

defined as follows. For any $x \in B_q$, let

$$\bar{x} = 1\otimes x \in R\otimes_{R^e} B_q;$$

and for any R^e-linear map $f\colon B_q \to M$, let $\bar{f}$ be the A-linear map

$$\bar{f} = 1\otimes f\colon R\otimes_{R^e} B_q \to R\otimes_{R^e} M = M\otimes_{R^e} R = H_0(R, M).$$

If f is a q-cocycle representing $\xi \in H^q(R, M)$ and $\bar{x}$ is a q-cycle representing $\eta \in H_q(R, R)$, then $\bar{f}(\bar{x}) \in H_0(R, M)$ depends only on ξ and η, as we see at once from the relation $\overline{\delta g}(\bar{y}) = \pm\bar{g}(\bar{\partial}\bar{y})$ where δ (respectively $\bar{\partial}$) is the boundary map in the complex $\mathrm{Hom}_{R^e}(\mathbf{B}.(h), M)$ (respectively: in $R\otimes_{R^e}\mathbf{B}.(h)$); furthermore $\bar{f}(\bar{x})$ depends R^c-bilinearly on ξ and η; so we can set

$$\rho_M^q(\xi\otimes\eta) = \bar{f}(\bar{x}).$$

Remark (1.1.1). The map (1.1) "varies functorially" with M. In particular when R is *commutative* ($R^c = R$), then setting $\bar{M} = H_0(R, M)$ we can put (1.1) into a commutative diagram

$$\begin{array}{ccc}
H^q(R, M)\otimes_R H_q(R, R) & \longrightarrow & H_0(R, M) = \bar{M} \\
\downarrow & & \| \\
H^q(R, \bar{M})\otimes_R H_q(R, R) & \longrightarrow & H_0(R, \bar{M})
\end{array}$$

So when R is commutative, (1.1) is essentially determined by its restriction to the category of R-modules, any R-module M^* being considered as an R-R bimodule with $rm = mr$ for all $r \in R$, $m \in M^*$.

Example (1.2) $(q = 0)$. As above, $H^0(R, M) \subseteq M$ and $H_0(R, M)$ is a homomorphic image of M. Denoting by $\bar{m}$ the natural image of $m \in M$ in