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Theory of Functions of a Complex Variable

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Chapter IV - The Method of Power Series, Analytic Continuation

$$\arctan x = \frac{1}{2i} \log \frac{1+ix}{1-ix} .$$

This may be deduced from the series

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \dots$$

by substituting the values ix and -ix for z and then subtracting, the result being 2i times the familiar series for arctan x. Not surprisingly, this uninhibited use of formal calculation occasionally led to paradox.*

Nonetheless, it was not until the 19^{th} century that this naive approach to mathematical analysis was replaced by the critical attitude of today. Functions of a complex variable were then studied systematically for the first time. The subsequent progress of mathematics has been largely in this field of function theory and the study of function theory has come to be regarded as the first step for any student of mathematics after he has mastered the elements of calculus.

§1 Complex numbers

"Imaginaries" emerged in algebra as early as the Middle Ages when mathematicians sought a general solution of quadratic equations. The choice of the word "imaginary" is unfortunate,

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* Suppose, for example, that the function tan α can be generalized so as to take on any given complex value. Specifically, let us suppose α is a complex number such that tan $\alpha = i$. Then, for all complex values $\beta \neq \pm \alpha$ we have, according to the elementary trigonometric identity:

$$\tan(\alpha + \beta) = \frac{i+\tan \beta}{1-i \tan \beta} = i \frac{1-i \tan \beta}{1-i \tan \beta} = i .$$

But this is plainly absurd. We are therefore led to a contradiction if we assume the ordinary tangent addition laws to hold for such a quantity; hence, if we wish to retain these laws, such an angle cannot exist.

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but it indicates the distrust in which complex numbers were held. These unwarranted suspicions were finally dispelled at the end of the 18^{th} century when Gauss in his doctoral thesis[*] gave a simple geometrical representation to complex numbers. Imaginaries could then be handled by the intuitive devices of geometry and they soon lost their awkward artificiality. In recent years mathematicians have turned the other way, preferring to define complex numbers abstractly as symbols subject to certain algebraic operations.

1.1 Definition of complex number

To the set of real numbers we adjoin a new symbol i, the imaginary unit, with which we add and multiply as for ordinary real numbers with the additional provision that [1.10] $i^2 = 1$. The set of complex numbers consists of all possible finite products and sums of i with itself and with real numbers. Thus a complex number z is a polynomial in i with real coefficients

$$z = a_o + a_1 i + a_2 i^2 + \dots + a_n i^n \quad .$$

Applying the rule $i^2 = -1$ we obtain

$$z = [a_o - a_2 + a_4 - \dots] + [a_1 - a_3 + a_5 - \dots]i \ .$$

Any complex number can therefore be represented in the form

[1.11] $$z = a + bi \ , \ a,b \ \text{real}.$$

This mode of representation is unique, i.e., if $a + bi = c + di$ then $a = c$ and $b = d$. For suppose

$$a + bi = c + di$$

then $$(a - c) + (b - d)i = 0.$$

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* Helmstadt (1799).

But $0 + 0i$ is the only representation of 0 in the prescribed form. For if $\alpha + \beta i = 0$ then $(\alpha + \beta i)(\alpha - \beta i) = \alpha^2 + \beta^2 = 0$. Hence $\alpha = \beta = 0$. It follows that

$$a - c = b - d = 0 \quad \text{or} \quad a = c \quad \text{and} \quad b = d \ .$$

Sums and products of complex numbers are clearly given by the formulae

$$[1.12a] \qquad (a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

$$[1.12b] \qquad (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

Division is defined as follows:

If $z = c + di$ is any complex number $z \neq 0$, then z possesses a unique reciprocal, a complex number z^{-1} such that $z^{-1}z = 1$. It is easy to show that

$$z^{-1} = \frac{1}{z} = \frac{c-di}{c^2+d^2} \ .$$

Hence for any complex number $\zeta = a + bi$ we define the quotient,

$$[1.12c] \qquad \frac{\zeta}{z} = \frac{a+bi}{c+di} = (a+bi)\left(\frac{c-di}{c^2+d^2}\right) = \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2} \ .$$

Every complex number z can be uniquely described in the form

$$z = a + bi \qquad a, b \ \text{real} \ .$$

The real numbers a and b are said respectively to be the real and imaginary part of z and are denoted by

$$[1.13] \qquad a = \text{Re } z \qquad b = \text{Im } z$$

after Weierstrass. The complex number

[1.14] $$\bar{z} = a - bi$$

is called the conjugate of z and has the property that both the sum $z + \bar{z}$ and the product $z \cdot \bar{z}$ are real. Evidently

[1.15] $$\text{Re } z = \frac{1}{2}(z + \bar{z}), \quad \text{Im } z = \frac{1}{2i}(z - \bar{z}) \quad .$$

With every complex number $z = a + bi$ we associate a real, non-negative number called the absolute value or modulus of z, written $|z|$, and defined as

[1.16] $$|z| = \sqrt{a^2 + b^2} \quad .$$

We have $|z| \geq 0$ and $|\bar{z}| = |z| = \sqrt{z\bar{z}}$. It is easy to verify that $|z_1 z_2| = |z_1|\,|z_2|$ and, provided $z_2 \neq 0$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$. Clearly $|z| = 0$ implies $z = 0$ and conversely.

It is possible to give a complete axiomatic definition of complex number without introducing the auxiliary symbol i.* A complex number is defined as a pair of real numbers (a,b) given in some definite order. Two complex numbers (a,b) and (c,d) are said to be equal (a,b) = (c,d) if and only if a = c and b = d. The imaginary unit is the ordered pair (0,1). The real numbers correspond to the ordered pairs (a,0). By developing the algebra of ordered pairs of real numbers in this fashion we may introduce complex numbers without recourse to the imaginary elements which once were considered so disturbing.

1.2 The complex number plane

The complex numbers may be represented geometrically by the points of an ordinary cartesian plane - each complex number z being represented by the point with abscissa Re z and ordinate Im z. The representation is clearly one-to-one; that is, every point of the plane is used to represent some complex number and no two complex numbers are represented by the same point.

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*Cf. E. Landau: Grundlagen der Analysis.

We shall therefore adopt a certain looseness of language and use the words "point" and "complex number" interchangeably. This description of complex numbers may be regarded as an extension of the representation of the real number system by the points on a line, since here the real numbers appear simply as the points on the x-axis.

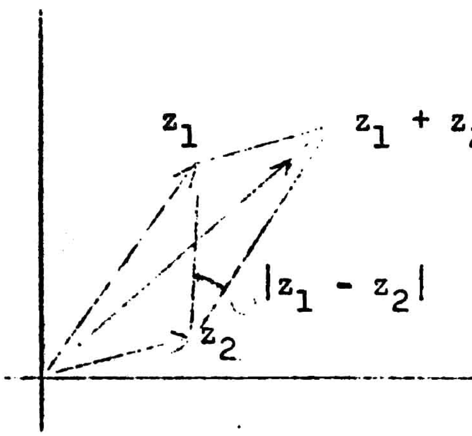

Fig. 1

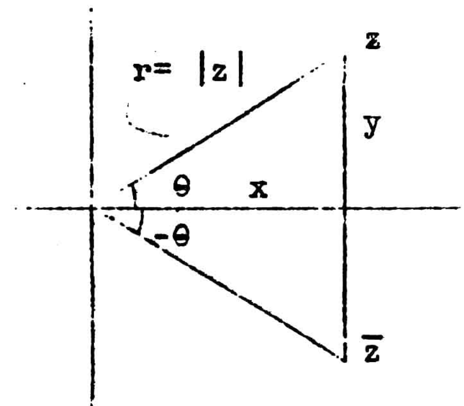

Fig. 2

Each point of the complex plane determines a vector (directed line segment) from the origin to the point. Since addition of two complex numbers is performed by addition of their x and y components, it is seen that addition of complex numbers corresponds to geometric vector addition in the complex plane, according to the parallelogram law. (Fig. 1).

It is natural to introduce polar coordinates (r,θ) in the complex plane. We then have (Fig. 2)

[1.21] $$r = \sqrt{x^2 + y^2} = |z|$$

$$x = r \cos \theta , \; y = r \sin \theta \; .$$

Using these relations we may write z in the <u>polar form</u>:

[1.22] $$z = r(\cos \theta + i \sin \theta) \; .$$

The angle θ is called the <u>amplitude</u> of z and denoted by am z. The polar coordinates completely define the complex number z. On the other hand for a given $z \neq 0$, am z is only determined to within an added multiple of 2π, while for $z = 0$, am z is undetermined.

The conjugate $\bar{z} = x - iy$ has a simple geometric interpretation: it is the reflection of the point z in the real axis. Clearly, we have am $\bar{z}$ = - am z, $|\bar{z}| = |z|$; hence

[1.23] $$\bar{z} = r(\cos\theta - i\sin\theta).$$

The product of two complex numbers is most simply expressed in polar form:

[1.24] $$z_1z_2 = r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\} = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

This verifies the relation $|z_1z_2| = r_1r_2 = |z_1|.|z_2|$. The rule for multiplication may now be stated as follows: <u>To multiply two complex numbers we multiply their absolute values and add their</u> amplitudes. Setting

$$e(\theta) = \cos\theta + i\sin\theta,$$

we use the above relation to find

[1.25] $$e(\theta_1).e(\theta_2) = e(\theta_1+\theta_2)$$

which is known as <u>De Moivre's formula.</u> This identity is similar to the addition theorem for the exponential function and, in fact, it will be established later that

$$e(\theta) = e^{i\theta}$$

At present we wish merely to state a few simple properties of $e(\theta)$: $e(\theta)$ is periodic with period 2π, i.e., $e(\theta \pm 2\pi) = e(\theta)$. Furthermore, $e(0) = 1$, $e(\frac{\pi}{2}) = i$, $e(\pi) = -1$, and if $z = e(\theta)$, then $\bar{z} = e(-\theta)$.

If z_1 and z_2 are any two complex numbers or points, then the distance between two points is given by $|z_1 - z_2|$, (see Fig.1). Thus $|z| = 1$ is the equation for points z on the circle of radius 1 about the origin, the so-called unit circle. More generally, the points z for which $|z - z_0| = R$ form a circle of radius R about the point z_0.

In conclusion we prove an important inequality which we shall use very frequently; it is known as the triangle inequality:

[1.26] $$|z_1| + |z_2| \geq |z_1 + z_2| \geq ||z_1| - |z_2||$$

Geometrically this inequality is quite obvious, since the vectors z_1, z_2 and $z_1 + z_2$ define a triangle. It simply states that the length of any side of a triangle is not greater than the sum of the lengths of the other two sides and not less than their difference in absolute value. To prove [1.26] analytically set $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$. Then for the first inequality in [1.26] we have

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} \geq \sqrt{(a_1+a_2)^2+(b_1+b_2)^2} \quad .$$

But by squaring both sides we see that this is equivalent to

$$a_1a_2 + b_1b_2 \leq \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

which is a direct consequence of the Schwarz Inequality*

* cf. Courant: Calculus V. I, p. 13.

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$$(a_1a_2 + b_1b_2)^2 \le (a_1^2 + b_1^2)(a_2^2 + b_2^2) .$$

To obtain the right side of the inequality we note that

$$|(z_1 + z_2) - z_2| \le |z_1 + z_2| + |z_2| .$$

Thus

$$| z_1 + z_2 | \ge |z_1| - |z_2| .$$

Since z_1 and z_2 occur symmetrically their roles may be interchanged in the argument. Consequently

$$|z_1 + z_2| \ge ||z_1| - |z_2|| .$$

Exercises

1) Prove by mathematical induction the following consequence of De Moivre's formula

$$[e(\theta)]^n = e(n\theta)$$

For positive integers. Finally show that this relation is still valid if n is any rational number whatever.

2) Prove the identity

$$|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2(|\alpha|^2 + |\beta|^2)$$

where α, β are arbitrary complex numbers; interpret this result geometrically.

3) Write the following expressions in standard form a + bi:

a) $\sqrt{1 + i}$, b) $\sqrt[3]{i}$, c) $\sqrt[3]{p + q\,i}$.

4) In 3c there are three solutions $a_j + b_j i, (j = 1,2,3)$

Find the cubic equation whose roots are a_1, a_2, a_3.

1.3 The Complex Number Sphere; Stereographic Projection.

For certain purposes it is simpler to represent complex numbers by the points on a sphere rather than those of a plane. To this end we use the unit sphere

$$S: \xi^2 + \eta^2 + \zeta^2 = 1$$

Where ξ, η, ζ are rectangular coordinates in space. For the complex number plane we choose the equator plane $\zeta = 0$ and take the real axis in the direction of ξ, the imaginary axis in the direction of η.

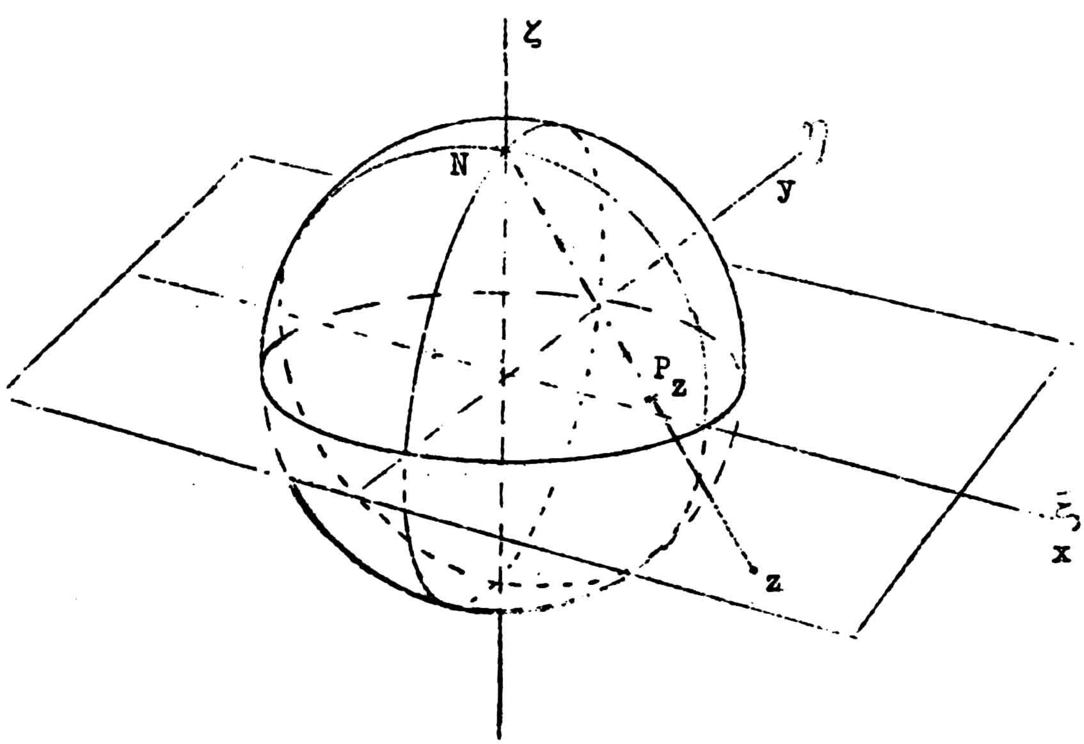


Fig. 3

The ray which joins each point $z = x + iy$ of the equator plane to the north pole, N: $\xi = \eta = 0$, $\zeta = 1$, intersects S in another point P_z which is then taken as the geometrical representation of z on the sphere. In this manner, the number plane is mapped on the unit sphere in a one-to-one way except that the north pole N does not correspond to any point of the z-plane. However, if P_z on S approaches N the distance $|z|$ of the corresponding point of the plane from the origin increases without bound. Accordingly we sometimes denote N by the symbol ∞ and call N the point at infinity of the complex number sphere S. It is advantageous to employ the notion of the point at infinity of the z-plane, an ideal point which is assigned to N in order to complete the one-to-one correspondence between the z-plane and the z-sphere. In like fashion, we speak of the value ∞ attained by a complex variable z although such a value cannot be included in the complex number system without violating the ordinary rules of algebra. The purpose in introducing the phrase "point at infinity" is to obviate the necessity of differentiating special cases in certain discussions concerned with limits. It will be recalled that an analogous device is used in projective geometry where, in order to avoid special cases in discussions involving the intersection of lines, we introduce not merely an ideal point but an entire ideal line consisting of points at infinity. However, it should not be forgotten that the concept of point at infinity is a natural and convenient invention but not a logical necessity.

Stereographic projection.

This mapping of the sphere onto the plane is the familiar stereographic projection of the cartographers who, in view of its special properties, find it indispensable for navigational maps. Before we consider these properties let us formulate the analytical description of the mapping:

By elementary geometry we obtain relations between z and the coordinates of P_z, namely

[1.31] $$x = \frac{\xi}{1-\zeta} \quad \text{and} \quad y = \frac{\eta}{1-\zeta}.$$

From [2.31] and $\xi^2 + \eta^2 + \zeta^2 = 1$ we obtain the inverse transformation:

[1.32] $$\xi = \frac{2x}{x^2+y^2+1}; \quad \eta = \frac{2y}{x^2+y^2+1}; \quad \zeta = \frac{x^2+y^2-1}{x^2+y^2+1}$$

or, alternatively,

[1.33] $$\xi = \frac{z+\bar{z}}{|z|^2+1}, \quad \eta = \frac{1}{i}\,\frac{z-\bar{z}}{|z|^2+1}, \quad \zeta = \frac{|z|^2-1}{|z|^2+1}.$$

Stereographic projection is characterized by the following property:

Every circle in the z-plane corresponds to a circle on the z-sphere and every straight line to a circle passing through the north pole. Conversely, every circle on the z-sphere is projected into a straight line or a circle (in the z-plane), according to whether it passes through the north pole or not.

If a straight line is regarded as a special kind of circle, namely, a "circle" through the point at infinity, we may express the theorem simply: Stereographic projection preserves circles.

Proof: Let

[1.34] $$a(x^2 + y^2) + bx + cy + d = 0$$

be the equation of any circle in the z-plane. To find the image of this circle, we substitute the transformation formulae [1.31], giving

$$a\,\frac{\xi^2+\eta^2}{(1-\zeta)^2} + b\,\frac{\xi}{1-\zeta} + c\,\frac{\eta}{1-\zeta} + d = 0.$$

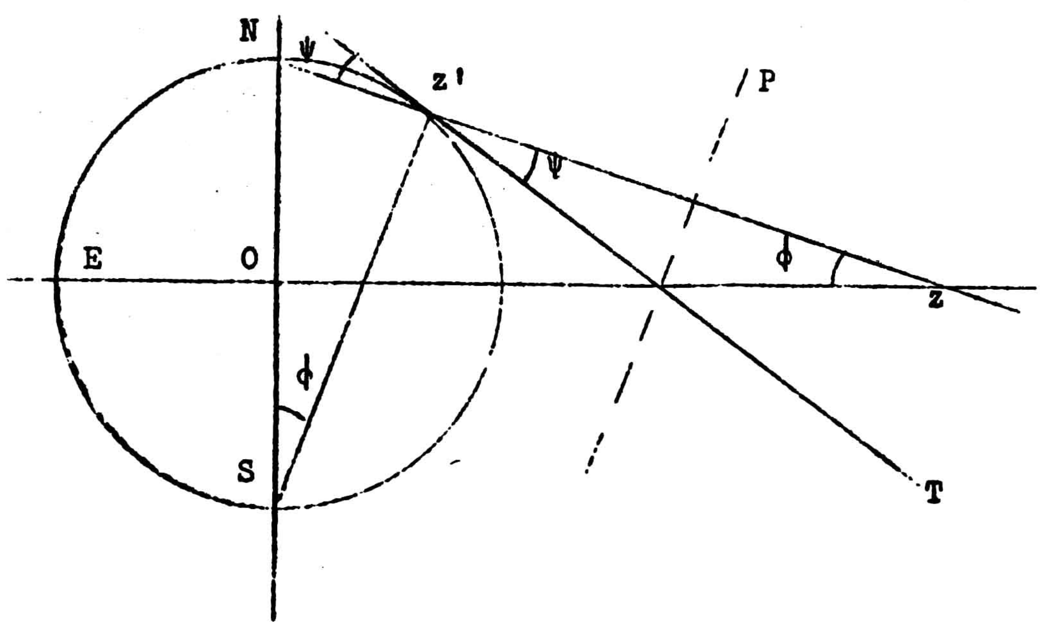
Remembering that $\xi^2 + \eta^2 = 1 - \zeta^2$, we reduce this to

[1.35] $$a(1+\zeta) + b\xi + c\eta + d(1-\zeta) = 0,$$

the equation of a plane, which, together with that of the unit sphere, determines a circle. For the straight line case let a be zero in [1.34]. The corresponding plane [1.35] becomes $b\xi + c\eta + d(1-\zeta) = 0$, which contains the north pole $\xi = \eta = 0, \zeta = 1$. The converse of the theorem is proved similarly.

The mapping defined by stereographic projection is <u>conformal</u> or <u>angle-preserving.</u> By this we mean that the images on the sphere of any two intersecting curves have the same angle of intersection as the original curves. The proof of conformality can be demonstrated analytically. Here we give a simple geometrical argument.

Let z be the point of intersection of the two curves, z' the corresponding projection point. It is sufficient to show that the angle between the tangent plane T at z' with the projection line Nz' z is the same as that made with the equatorial plane E. Donote the angle between E and zz' by ϕ, T and zz' by ψ. Now in the diagram



$\phi = \measuredangle NOS$ since they are both complementary to the same angle. But $\measuredangle NOS = \psi$ since both subtend the same arc. The theorem follows from the fact that T and E are mirror images in the plane P which bisects the angle between them.

Exercises

5) Show that the segment joining two points P_1 and P_2 on S is perpendicular to the z-plane if and only if their respective images z_1 and z_2 are inverse with respect to the unit circle $|z| = 1$, i.e. if $|z_1|\ |z_2| = 1$ and z_1/z_2 is real and positive.

6) Show that the endpoints P_1 and P_2 of a diameter of S are mapped onto two points z_1 and z_2 with $|z_1 z_2| = 1$ and z_1/z_2 real and negative.

7) Characterize the image on the sphere under stereographic projection of

a) A family of parallel lines

b) A pencil of lines

c) A set of concentric circles

Also characterize the image in the z-plane of a set of great circles through a fixed point on the z-sphere.

8) Give a geometrical proof of conformality by investigating the image of the pencil of circles passing through the north pole and a fixed point P on S.

1.4 Point Sets

With the geometrical interpretation of complex numbers in mind we shall consider a number of useful concepts of point set theory.

A set S of points of the complex plane is said to be bounded if S can be enclosed in a circle about the origin, i.e., if there exists a positive number R such that the inequality $|z| < R$ holds for all points of S.

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