Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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T. tom Dieck (Ed.)

Algebraic Topology and Transformation Groups

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The Homotopy Type of a 4-Manifold with finite Fundamental Group

by Stefan Bauer*

ABSTRACT: ... is determined by its quadratic 2-type, if the 2-Sylow subgroup has 4-periodic cohomology.

The homotopy type of simply connected 4-manifolds is determined by the intersection form. This is a well-known result of J.H.C. Whitehead and J. Milnor. In the non-simply connected case the homotopy groups π_1 and π_2 and the first k-invariant $k \in H^3(\pi_1, \pi_2)$ give other homotopy invariants. The **quadratic 2-type** of an oriented closed 4-manifold is the isometry class of the quadruple $[\pi_1(M), \pi_2(M), k(M), \gamma(\tilde{M})]$, where $\gamma(\tilde{M})$ denotes the intersection form on $\pi_2(M) \cong H_2(\tilde{M})$. An isometry of two such quadruples is an isomorphism of π_1 and π_2 which induces an isometry on γ and respects the k-invariant.

Recently $[H-K]$ I. Hambleton and M. Kreck, studying the homeomorphism types of 4-manifolds, showed that for groups with periodic cohomology of period 4 the quadratic 2-type determines the homotopy type.

This result can be improved away from the prime 2.

Theorem: Suppose the 2-Sylow subgroup of G has 4-periodic cohomology. Then the homotopy type of an oriented 4-dimensional Poincaré complex with fundamental group G is determined by its quadratic 2-type.

I am indebted to Richard Swan for showing me proposition 6. Furthermore I am grateful to the department of mathematics at the University of Chicago for its hospitality during the last year.

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Let X be an oriented 4-dimensional Poincaré complex with finite fundamental group, $f : X \longrightarrow B$ its 2-stage Postnikov approximation, determined by π_1, π_2, and k, and let $\gamma(X)$ denote the intersection form on $H_2(\tilde{X})$. Then $\mathbf{S}_4^{PD}(B, \gamma(X))$ denotes the set of homotopy types of 4-dimensional Poincaré complexes Y, together with 3-equivalences $g : Y \longrightarrow B$, such that f and g induce an isometry of the quadratic 2-types.
The universal cover $\tilde{B}$ is an Eilenberg-MacLane space and hence, by [MacL], $H_4(\tilde{B}) \cong \Gamma(\pi_2(B))$, the $\mathbf{Z}\pi_1(B)$-module $\Gamma(\pi_2(B))$ being the module of symmetric 2-tensors, i.e. the kernel of the map $(1-\tau)$: $\pi_2(B) \otimes \pi_2(B) \longrightarrow \pi_2(B) \otimes \pi_2(B)$, $(1-\tau)(a \otimes b) = a \otimes b - b \otimes a$. The intersection form on $\tilde{X}$ corresponds to $\tilde{f}_*[\tilde{X}]$ of the fundamental class $[\tilde{X}] \in H_4(\tilde{X}; \mathbf{Z})$. Let $\hat{H}_*$ denote Tate homology.

Proposition 1: If X is a Poincaré space with finite fundamental group G, then there is a bijection $\hat{H}_0(G; \pi_3(X)) \longleftrightarrow \mathbf{S}_4^{PD}(B, \gamma(X))$.

The proof uses a lemma of [H-K]:

Lemma 2: Let (X, f) and (Y, g) be elements in $\mathbf{S}_4^{PD}(B, \gamma(X))$. Then the only obstruction for the existence of a homotopy equivalence $h : X \longrightarrow Y$ over B is the vanishing of$g_*[Y] - f_*[X] \in H_4(B)$.

Lemma 3: Given a diagram

$$\begin{array}{ccc} \mathbf{Z} & \overset{\alpha}{\hookrightarrow} & M \\ \cdot n \downarrow & & \\ \mathbf{Z} & & \end{array},$$

such that the torsion in the cokernel of α is annihilated by n, then the torsion subgroup in the pushout K is isomorphic to the torsion subgroup of $coker(\alpha)$.

Proof of 3: Since the torsion subgroup of M maps injectively into K as well as into $coker(\alpha)$, we may assume it trivial. Then M is isomorphic to $N \oplus < x >$ with $\alpha(1) = mx$ for an integer m dividing n. The pushout then is isomorphic to $(N \oplus \mathbf{Z} \oplus \mathbf{Z})/ < (0, m, n) > \cong M \oplus \mathbf{Z}/m$. ♣

Proof of proposition 1: Let (X, f) and (Y, g) be elements in $\mathcal{S}_4^{PD}(B)$ such that f and g induce an isometry of the quadratic 2-types. Let $\gamma(X) = \gamma(Y) = \gamma$ denote the intersection form on $H_2(\tilde{X})$ and $H_2(\tilde{Y})$. By [W] one has $\pi_3(X) \cong \Gamma(\pi_2(X))/\langle\gamma\rangle \cong H_4(\tilde{B}, \tilde{X})$

and $\pi_3(X) \otimes_{ZG} \mathbf{Z} \cong H_4(B, X)$. In the pushout diagramm:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_4(\tilde{X}) \otimes_{ZG} \mathbf{Z} & \longrightarrow & H_4(\tilde{B}) \otimes_{ZG} \mathbf{Z} & \longrightarrow & H_4(\tilde{B}, \tilde{X}) \otimes_{ZG} \mathbf{Z} & \longrightarrow & 0 \\
 & & \phi \downarrow & & \downarrow & & \downarrow\cong & & \\
0 & \longrightarrow & H_4(X) & \longrightarrow & H_4(B) & \longrightarrow & H_4(B, X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_4(X, \tilde{X}) & \xrightarrow{\cong} & H_4(B, \tilde{B}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

the torsion subgroup of $H_4(B, X)$ is isomorphic to the torsion subgroup of $H_4(B)$ by lemma 3: The module $H_4(\tilde{B}, \tilde{X})$ is torsion free. Hence the torsion subgroup of $H_4(\tilde{B}, \tilde{X}) \otimes_{ZG} \mathbf{Z}$ is annihilated by the order n of the group G. Note that ϕ is just multiplication by n. In particular one has

$$Torsion(H_4(B)) \cong Torsion(H_4(B, X)) \cong \hat{H}_0(G; \pi_3(X))$$

Since X and Y have the same quadratic 2-type, $\tilde{f}_*[\tilde{X}] = \tilde{g}_*[\tilde{Y}]$, hence we have $f_*[X] - g_*[Y] \in Torsion(H_4 B)$. This gives an injection

$$\mathcal{S}_4^{PD}(B, \gamma) \hookrightarrow \hat{H}_0(G; \pi_3(X)).$$

What about surjectivity? Let $K \subset \tilde{X}$ denote a subspace, where one single orbit is deleted. Let $\alpha \in \pi_3(K)$ map via the surjection $\pi_3(K) \to \pi_3(X) \to \pi_3(X) \otimes_{ZG} \mathbf{Z}$ to a given element $\hat{\alpha} \in \hat{H}_0(G; \pi_3(X))$. Let β be the image of $1 \in \mathbf{Z}G \cong H_4(\tilde{X}, K) \cong \pi_4(\tilde{X}, K) \hookrightarrow \pi_3(K)$. Now let $k : S^3 \to K$ represent $\alpha + \beta$ and define $X_\alpha := (K \cup_k (G \times D^4))/G$. One has to show that X_α is an orientable Poincaré space. Orientability is clear, since $H_4(X_\alpha) \cong H_4(X_\alpha, K) \cong \mathbf{Z}$. Let $f : X_\alpha \to B$ extend $f|_{K/G}$. The intersection form on $\tilde{X}_\alpha$ is determined by

$$\tilde{f}_{\alpha *}[\tilde{X}_\alpha] = trf(f_{\alpha *}[X_\alpha]) \in H_4(\tilde{X}).$$

But we have $f_{\alpha *}[X_\alpha] = f_*[X] + \alpha$: In the following diagram $1 \in \mathbf{Z} \cong \pi_4(X, B)$ is mapped to $f_*[X] \in H_4(B)$.

$$\begin{array}{ccccccccc}
H_4(X) & \cong & H_4(X, K/G) & \longleftarrow & H_4(\tilde{X}, K) & \cong & \pi_4(\tilde{X}, K) & \longrightarrow & \pi_3(K) \\
f\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow= \\
H_4(B) & \cong & H_4(B, K/G) & \longleftarrow & H_4(\tilde{B}, K) & \cong & \pi_4(\tilde{B}, K) & \cong & \pi_3(K)
\end{array}$$

If the upper row is replaced by the corresponding row for X_α and the vertical maps by the ones induced by f_α, then $1 \in \mathbf{Z}G$ is mapped (counterclockwise) to $f_{\alpha *}[X_\alpha]$ on the one hand, on the other hand (clockwise) to $f_*[X] + \alpha$.

Since the torsion element α lies in the kernel of the transfer, one immediately gets $\tilde{f}_{\alpha *}[\tilde{X}_\alpha] = \tilde{f}_*[\tilde{X}]$. ♣

In the sequel all $\mathbf{Z}G$-modules have underlying a free abelian group.

The short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\gamma} \Gamma(\pi_2 X) \longrightarrow \pi_3(X) \to 0$$

gives rise to an exact sequence in Tate homology:

$$\hat{H}_0(G;\mathbf{Z}) \longrightarrow \hat{H}_0(G;\Gamma(\pi_2 X)) \longrightarrow \hat{H}_0(G;\pi_3(X)) \longrightarrow \hat{H}_{-1}(G;\mathbf{Z}) \xrightarrow{\gamma} \hat{H}_{-1}(G;\Gamma(\pi_2 X))$$

Here $\hat{H}_0(G;\mathbf{Z}) = 0$ and $\hat{H}_{-1}(G;\mathbf{Z}) \cong \mathbf{Z}/|G|$.The sequence above gives the connection to [H-K], theorem(1.1).

In order to analyze this sequence, I recall some facts from [H-K],§§2 and 3.

Facts:

1) $\Gamma(\mathbf{Z}G) = \bigoplus_i \mathbf{Z}[G/H_i] \oplus F$, where the summation is over all subgroups H_i of order 2 and F is a free $\mathbf{Z}G$-module.

2) $\Gamma(\mathbf{Z}G) \cong \Gamma(I) \oplus \mathbf{Z}G \cong \Gamma(I^*) \oplus \mathbf{Z}G$.Here I denotes the augmentation ideal, I^* its dual.

3) The modules $\Omega^3\mathbf{Z}$ and $S^3\mathbf{Z}$ are (stably!) defined by exact sequences
$$0 \to \Omega^3\mathbf{Z} \to F_2 \to F_1 \to F_0 \to \mathbf{Z} \to 0$$
and
$$0 \to \mathbf{Z} \to F_1 \to F_2 \to F_3 \to S^3\mathbf{Z} \to 0$$
with free modules F_i.
There is an exact sequence
$$0 \to \Omega^3\mathbf{Z} \longrightarrow \pi_2(X) \oplus r\mathbf{Z}G \longrightarrow S^3\mathbf{Z} \to 0$$

Lemma 4: If $0 \to A \to B \to C \to 0$ is a short exact sequence of $\mathbf{Z}$G-modules, which are free over $\mathbf{Z}$, then there are short exact sequences

$$0 \to \Gamma(A) \longrightarrow \Gamma(B) \longrightarrow D \to 0$$

and

$$0 \to A \otimes_{\mathbf{Z}} C \longrightarrow D \longrightarrow \Gamma(C) \to 0.$$

Proof: Given $\mathbf{Z}$-bases $\{a_i\}$, $\{c_j\}$ and $\{a_i, \tilde{c}_j\}$ of A, C and B, the map $h : a_i \otimes c_j \to a_i \otimes \tilde{c}_j + \tilde{c}_j \otimes a_i$ is well-defined and equivariant modulo $\Gamma(A)$. ♣

To prove the theorem, it suffices to show that $\hat{H}_0(G;\pi_3(X)) = 0$. This in turn can be done separately for each p-Sylow subgroup G_p of G.

Proposition 5: The map $\gamma_* : \hat{H}_{-1}(G_p;\mathbf{Z}) \longrightarrow \hat{H}_{-1}(G_p;\Gamma(\pi_2(X)))$ is injective, if either p is odd or $res^G_{G_p}\pi_2(X) \cong A \oplus B$ splits such that the rank of B over $\mathbf{Z}$ is odd. In general the kernel is at most of order 2.

Proof: For the sake of brevity, let π denote $\pi_2(X)$ and also let Γ denote the module $\Gamma(\pi)$. Now look at the following sequence of maps:

$$\psi : \mathbf{Z} \xrightarrow{\gamma} \Gamma \hookrightarrow \pi \otimes \pi \cong Hom(\pi^*, \pi) \xleftarrow{\alpha^* \cong} Hom(\pi, \pi) \xrightarrow{trace} \mathbf{Z}.$$

A generator of $\mathbf{Z}$ is mapped in $Hom(\pi^*, \pi)$ to the Poincaré map $\alpha : \pi^* \cong H^2(\tilde{X}) \xrightarrow{\cong} H_2(\tilde{X}) \cong \pi$, and then to the element $id \in Hom(\pi, \pi)$. So we have $\psi(1) = rank_{\mathbf{Z}}(\pi)$.

Fact 3) gives $rank_{\mathbf{Z}}(\pi) \equiv -2 \ mod \mid G \mid$, hence the induced selfmap ψ_* of $\mathbf{Z}/|G_p| \cong \hat{H}_{-1}(G_p; \mathbf{Z})$ is multiplication by -2. This proves, that the kernel is at most of order 2. In particular it is trivial, if p is odd.

In case p = 2 and $res^G_{G_p}\pi \cong A \oplus B$, such that the rank of the underlying group of B is odd, one can replace the map $Hom(\pi, \pi) \xrightarrow{trace} \mathbf{Z}$ by the map $Hom(\pi, \pi) \xrightarrow{p_* o i^*} Hom(B, B) \xrightarrow{trace} \mathbf{Z}$ in the defining sequence for ψ. A similar argument as above for p odd gives the claim. ♣

Remark: The module $res^G_{G_2}\pi_2(X)$ always splits, if $H_4(G;\mathbf{Z}) \cong Ext^1_{\mathbf{Z}G}(S^3\mathbf{Z}, \Omega^3\mathbf{Z})$ has no 2-torsion, in particular if G_2 has 4-periodic cohomology.

Proposition 6: Let A denote either $\Omega^n\mathbf{Z}$ or $S^n\mathbf{Z}$ and let τ be the selfmap of $A \otimes A$ which permutes the factors. Then $(-1)^n\tau$ induces the identity on $\hat{H}_0(G; A \otimes A)$.

Proof: Let $F. \to \mathbf{Z}$ be a free resolution of $\mathbf{Z}$ and let $\tilde{F}.$ be the truncated complex with $\tilde{F}_i = F_i$ for $i \leq n-1$, $\tilde{F}_n = \Omega^n$ and $\tilde{F} = 0$ else. There is an obvious projection $f : F. \to \tilde{F}.$, such that $f_n = \partial_n$. The tensor product $F.\otimes F. = F.^2$ again is a free resolution of $\mathbf{Z}$ and $\tilde{F}.^2$ is a truncatedfree resolution of $\mathbf{Z}$ with $\tilde{F}^2_{2n} = \Omega\mathbf{Z} \otimes \Omega\mathbf{Z}$. The chain map $f \otimes f$ induces an isomorphism of $H_*(F.^2 \otimes_{\mathbf{Z}G} \mathbf{Z})$ and $H_*(\tilde{F}.^2 \otimes_{\mathbf{Z}G} \mathbf{Z})$ in the dimensions $* \leq 2n$. The selfmap t of $F^2.$, as usual defined by $t(x \otimes y) = (-1)^{deg(x)deg(y)} x \otimes y$, is a chain automorphism, inducing the identity on the augmentation, hence on all derived functors, in particular on $H_*(F.^2 \otimes_{\mathbf{Z}G} \mathbf{Z}) = H_*(G; \mathbf{Z})$. In the same way an involution t can be defined on $\tilde{F}.^2$ and $f \otimes f$ commutes with t. Obviously $t_{2n} = (-1)^n\tau$. Hence $(-1)^n\tau$ induces the identity on $H_{2n}(\tilde{F}^2_{2n} \otimes_{\mathbf{Z}G} \mathbf{Z}) = \hat{H}_0(G; \mathbf{Z})$.
The proof for $S^n\mathbf{Z}$ is dual. ♣

Proof of the theorem: By proposition 1, it suffices to show that $\hat{H}_0(G; \pi_3(X))$ vanishes. By proposition 4 and the remark following it, this group is isomorphic to $\hat{H}_0(G; \Gamma(\pi_2(X)))$. In order to show that this group vanishes it suffices, by lemma 3, to show that $\hat{H}_0(G; A)$ vanishes for $A \in \{\Gamma(\Omega^3\mathbf{Z}), \Gamma(S^3\mathbf{Z}), \Omega^3\mathbf{Z} \otimes S^3\mathbf{Z}\}$ But $\hat{H}_0(G; \Omega^3\mathbf{Z} \otimes S^3\mathbf{Z}) \cong \hat{H}_0(G; \mathbf{Z}) = 0$. Given a module B (with underlying free abelian group), there is a short exact sequence

$$0 \to \Gamma(B) \longrightarrow B \otimes B \longrightarrow \Lambda^2(B) \to 0.$$

The map τ, which flips the both factors, induces, if applied to $B \in \{\Omega^3\mathbf{Z}, S^3\mathbf{Z}\}$ the following diagram:

$$\begin{array}{ccccccc} \to & \hat{H}_1(G;\Lambda(B)) & \longrightarrow & \hat{H}_0(G;\Gamma(B)) & \longrightarrow & \hat{H}_0(G;B\otimes B) & \to \\ & \downarrow (-id) & & \downarrow id & & \downarrow (-id) & \\ \to & \hat{H}_1(G;\Lambda(B)) & \longrightarrow & \hat{H}_0(G;\Gamma(B)) & \longrightarrow & \hat{H}_0(G;B\otimes B) & \to \end{array}$$

The right vertical map is $(-id)$ by proposition 5. This diagram shows that any element in $\hat{H}_0(G;\Gamma(B))$ is annihilated by 4.In particular this group vanishes, if G is a p-group for an odd prime p. That $\hat{H}_0(G_2;\Gamma(B))$ vanishes, if G_2 has 4-periodic cohomology, follows at once from the facts 1 - 3, since in this case $\Omega^3\mathbf{Z} = I^* \oplus n\mathbf{Z}G$ and $S^3\mathbf{Z} = I \oplus n\mathbf{Z}G$ ♣

Final Remark: An elementary but lengthy computation shows $\Gamma(S^3\mathbf{Z}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ and $\Gamma(\Omega^3\mathbf{Z}) = 0$ for $G = \mathbf{Z}/2 \oplus \mathbf{Z}/2$. In particular the group $\hat{H}_0(\mathbf{Z}/2 \oplus \mathbf{Z}/2; \Gamma(\Omega^3\mathbf{Z} \otimes S^3\mathbf{Z}))$ is nontrivial. Hence the argument above won't work in general.

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Rational Cohomology of Configuration Spaces of Surfaces

C.-F. Bödigheimer and F.R. Cohen

1. Introduction. The k-th configuration space $C^k(M)$ of a manifold M is the space of all unordered k-tuples of distinct points in M. In previous work [BCT] we have determined the rank of $H_*(C^k(M);\mathbb{F})$ for various fields $\mathbb{F}$. However, for even dimensional M the method worked for $\mathbb{F}=\mathbb{F}_2$ only. The following is a report on calculations of $H^*(C^k(M);\mathbb{Q})$ for M a deleted, orientable surface. This case is of considerable interest because of its applications to mapping class groups, see [BCP]. Similar results for (m-1)-connected, deleted 2m-manifolds will appear in [BCM].

2. Statement of results. The symmetric group Σ_k acts freely on the space $\tilde{C}^k(M)$ of all ordered k-tuples $(z_1,\ldots,z_k)$, $z_i \in M$, such that $z_i \neq z_j$ for $i \neq j$. The orbit space is $C^k(M)$. As in [BCT] we will determine the rational vector space $H^*(C^k(M);\mathbb{Q})$ as part of the cohomology of a much larger space. Namely, if X is any space with basepoint x_o, we consider the space

$$(1) \qquad C(M;X) = \left(\coprod_{k\geq 1} \tilde{C}^k(M) \underset{\Sigma_k}{\times} X^k \right) \Big/ \approx$$

where $(z_1,\ldots,z_{ki};x_1,\ldots,x_k) \approx (z_1,\ldots,z_{n-1};x_1,\ldots,x_{k-1})$ if $x_k = x_o$. The space C is filtered by subspaces

$$(2) \qquad F_kC(M;X) = \left(\coprod_{j=1}^{k} \tilde{C}^j(M) \underset{\Sigma_j}{\times} X^j \right) \Big/ \approx$$

and the quotients $F_kC/F_{k-1}C$ are denoted by $D_k(M;X)$.

Let $\bar{M}_g$ denote a closed, orientable surface of genus g, and M_g is $\bar{M}_g$ minus a point. We study $C(M_g;S^{2n})$ for $n\geq 1$. H^* will always stand for rational cohomology, and P[ ] resp. E[ ] for polynomial resp. exterior algebras over $\mathbb{Q}$.

Theorem A. *There is an isomorphism of vector spaces*

(3) $$H^*C(M_g;S^{2n}) \cong P[v,u_1,\dots,u_{2g}]\otimes H_*(E[w,z_1,\dots,z_{2g}],d)$$

with $|v|=2n$, $|u_i|=4n+2$, $|w|=4n+1$, $|z_i|=2n+1$, *and the differential* d *is given by* $d(w) = 2(z_1z_2+\dots+z_{2g-1}z_{2g})$.

Giving the generators weights, $\mathrm{wght}(v) = \mathrm{wght}(z_i) = 1$ and $\mathrm{wght}(u_i) = \mathrm{wght}(w) = 2$, makes H^*C into a filtered vector space. We denote this weight filtration by F_kH^*C. The length filtration F_kC of C defines a second filtration H^*F_kC of H^*C.

Theorem B. *As vector spaces*

(4) $$H^*F_kC(M_g;S^{2n}) = F_kH^*C(M_g;S^{2n}).$$

It follows that $H^*D_k(M_g;S^{2n})$ is isomorphic to the vector subspace of $H^*(g,n) = P[v,u_i]\otimes H_*(E[w,z_i],d)$ spanned by all monomials of weight exactly k. To obtain the cohomology of $C^k(M_g)$ itself, we consider the vector bundle

(5) $$\eta^k: \tilde{C}^k(M_g)\times_{\Sigma_k}\mathbb{R}^k \to C^k(M_g)_+$$

which has the following properties. First, the Thom space of m times η^k is homomorphic to $D_k(M_g;S^m)$. Secondly, it has finite even order, see [CCKN]. Hence

(6) $$D_k(M_g;S^{2nk}) = \Sigma^{2n_k\cdot k}\,C^k(M_g)_+$$

for $2n_k = \mathrm{ord}(\eta^k)$. Thus we have

Theorem C. *As a vector space, $H^*C^k(M_g)$ is isomorphic to the vector subspace generated by all monomials of weight k in $H^*(g,n_k)$, desuspended $2n_k k$ times.*

Regarding the homology of $E = E[w, z_1, \ldots, z_{ig}]$ we have

Theorem D. *The homology $H_*(E,d)$ is as follows:*

(7) $\quad \operatorname{rank} H_{i(2n+1)} = \binom{2g}{i} - \binom{2g}{i-2}$ for $i = 0,1,\ldots g$, *and all (non-zero) elements have weight* i;

(8) $\quad \operatorname{rank} H_{i(2n+1)+4n+1} = \binom{2g}{i} - \binom{2g}{i+2}$ *for* $i = g,\ldots,2g$, *and all (non-zero) elements have weight* i+2;

(9) $\quad \operatorname{rank} H_j = 0$ *in all other degrees* j.

Note the apparent duality $\operatorname{rank} H_j = \operatorname{rank} H_{N-j}$ for $N = 2g(2n+1)+4n+1$.

We will give the proof of Theorem A in the next section. The proof of Theorem B is the same as for [BCT, Thm.B]. By what we said above Theorem C folows from Theorem B. And Theorem D will be derived in the last section.

3. Mapping spaces and fibrations. Let D denote an embedded disc in M_g. There is a commutative diagram

(10)
$$\begin{array}{ccc} C(D;S^{2n}) & \longrightarrow & \Omega^2 S^{2n+2} \\ \downarrow & & \downarrow \\ C(M_g;S^{2n}) & \longrightarrow & \mathrm{map}_o(\bar{M}_g;S^{2n+2}) \\ \downarrow & & \downarrow \\ C(M_g,D;S^{2n}) & \longrightarrow & (\Omega S^{2n+2})^{2g} \end{array}$$

where map_o stands for based maps. The right column is induced by restricting to the 1-section, and is a fibration. The left column is a quasifibration. Since S^{2n} is connected, all three horizontal maps are equivalences, see [M], [B] for details.

The E_2-term of the Serre spectral sequence of these (quasi)fibrations is as follows. From the base we have 2g-fold tensor product of

(11) $$H^*\Omega S^{2n+2} = H^*(S^{2n+1} \times \Omega S^{4n+3}) = E[z_i] \otimes P[u_i] \quad (i = 1, \ldots 2g),$$

where $|z_i| = 2n+1$ and $|u_i| = 4n+2$. From the fibre we have

(12) $$H^*\Omega^2 S^{2n+2} = H^*(\Omega S^{2n+1} \times \Omega^2 S^{4n+3}) = H^*(\Omega S^{2n+1} \times S^{4n+1})$$
$$= P[v] \otimes E[w] ,$$

where $|v| = 2n$ and $|w| = 4n+1$. The following determines all differentials in this spectral sequence.

<u>Lemma. The differentials are as follows:</u>

(13) $$d_{2n+1}(v) = 0$$

(14) $$d_{4n+2}(w) = 2z_1z_2 + 2z_2z_3 + \ldots + 2z_{2g-1}z_{2g}$$

Proof: Assertion (13) follows from the stable splitting of $C(M_g;S^{2n})$, on [B]. (14) results from symmetries of M_g and of the fibrations (10) which leave d invariant. ∎

The lemma implies $E_{4n+3} = E_\infty = H^*C(M_g;S^{2n})$. Furthermore, E_{4n+3} is a tensor product of the polynomial algebra $P[v,u_1, \ldots u_{2g}]$ and the homology module $H_*(E,d)$ of the exterior algebra $E = E[w,z_1, \ldots, z_{2g}]$ with differential d. This proves Theorem A.

<u>4. Homology of E.</u> Let us write $x_i = z_{2i-1}$ and $y_i = z_{2i}$ for $i = 1,\ldots g$. The form $d(w) = 2z_1z_2 + 2z_2z_3 + \ldots + 2z_{2g},z_{2g}$ is equivalent to the standard symplectic form $x_1y_1 + x_2y_2 + \ldots + x_gy_g$. The vector space