Foundations of Algebraic Analysis

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Preface

Prior to its founding in 1963, the Research Institute for Mathematical Sciences (to which we are gratefully indebted for support) was the focus of divers discussions concerning goals. One of the more modest goals was to set up an institution that would create a "Courant-Hilbert" for a new age.[1] Indeed, our intention here—even though this book is small in scale and only the opening chapter of our utopian "Treatise of Analysis"—is to write just such a "Courant-Hilbert" for the new generation. Each researcher in this field may have his own definition of "algebraic analysis," a term included in the title of this book. On the other hand, algebraic analysts may well share a common attitude toward the study of analysis: the essential use of algebraic methods such as cohomology theory. This characterization is, of course, too vague: one can observe such common trends whenever analysis has made serious reformations. Professor K. Oka, for example, once spoke of the "victory of abstract algebra" in regard to his theory of ideals of undetermined domains.[2] Furthermore, even Leibniz's main interest, in the early days of analysis, seems to have been in the algebraization of infinitesimal calculus. As used in the title of our book, however, "algebraic analysis" has a more special meaning, after Professor M. Sato: it is that analysis which holds onto substance and survives the shifts of fashion in the field of analysis, as Euler's mathematics, for example, has done. In this book, as the most fruitful result of our philosophy, we pay particular attention to the microlocal theory of linear partial differential equations, i.e. the new thinking on the local analysis on contangent bundles. We hope that the fundamental ideas that appear in this book will in the near future become the conventional wisdom

[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vols. 1 and 2 (Interscience, 1953 and 1962). These two volumes seem to reflect the strong influence of the Courant Institute; the countervailing influence must be strong as well.

[2] Quoted by Professor Y. Akizuki in *Sūgaku 12* (1960), 159. A general theory of ideals of undetermined domains has been reorganized by H. Cartan and Serre and is now called the theory of coherent sheaves (see Hitotumatu [1]).

among analysts and theoretical physicists, just as the Courant-Hilbert treatise did.

Despite our initial determination and sense of purpose, the task of writing was a heavy burden for us. It has been a time-consuming project, while our first priority has been to be at the front of the daily rapid progress in this field. Thus, we cannot deny the existence of minor areas that do not yet meet with our full satisfaction. Still, a proverb says, "Striving for the best is an enemy of the good." We are content, then, to publish our book in this form, hoping that the intelligent reader will benefit despite several defects, and expecting that this will become the first part of our "Treatise of Analysis." We would also like to emphasize that our comparison of this book with "Courant-Hilbert" is only a goal, and that we do not pretend to equate the maturity of this book with that of Courant and Hilbert's. Theirs is the crystallization of the great scholar Courant's extended effort. Therefore, we would appreciate hearing the critical reader's opinions on the content of this book, for the purpose of improvement.

Let us turn to the content of each chapter. In Chapter I, §1, a review of cohomology theory is given, with which we define the sheaf of hyperfunctions. Since students of analysis nowadays seem to be given little opportunity to learn cohomology theory, despite its importance, we have prepared a rather comprehensive treatment of sheaf cohomology theory as an introduction to notions and notations used in later chapters. One may skip this material if it is familiar. The main purpose of Chapter I, §2, is to present the mathematical formulation, via the Čech cohomology group, of the idea that "hyperfunctions are boundary values of holomorphic functions." The reader can then obtain the explicit description of a hyperfunction by combining this with the results in §3 of Chapter II.

In Chapter II, §1, the sheaf of microfunctions is constructed on a cotangent bundle, by which the stage for our main theme, microlocal analysis, is established. After some preparation of the theory of holomorphic functions of several complex variables, in §2, the properties of microfunctions will be studied in detail in §3. Furthermore, in §4, specific examples will be treated.

In §1 and §2 of Chapter III, where we basically followed Sato, Kawai, and Kashiwara [1] (hereafter SKK [1]), fundamental operations on microfunctions are discussed. However, the approach taken in SKK [1] may not be suited to the novice; hence the method of description has been changed. There it was necessary to prove a certain lemma (Proposition 3.1.1) directly, which is technical and intricate and could be tiresome for the reader. Because of the introductory nature of this book, therefore, we decided to treat this lemma as an "axiom," so to speak, and to proceed to what follows from it. In §4 through §6, elliptic and hyperbolic differential equations are treated explicitly to show how effectively microfunction theory applies to the theory of linear partial differential equations. These three sections also serve as preparation for the theory of microdifferential equations considered in Chapter IV. Prior to these three sections, we discuss (in §3) the analyticity of Feynman integrals. This section has a somewhat different flavor than other sections; it is intended as an invitation to a new trend in mathematical physics: namely, the study of theoretical physics through methods of algebraic analysis. We also thought that it might be a good exercise to go through the operations on microfunctions. In §7, we prove the flabbiness of the microfunction sheaf; and, in §8, a hyperfunction containing holomorphic parameters is discussed. The last two sections are intended to take into account some important properties of microfunctions not covered by the previous sections.

In Chapter IV, we discuss the theory of microdifferential equations, the most effective application of microfunction theory. In §1, we define a microdifferential operator, and the fundamental properties are given. "Qauntized contact transformations" of microdifferential operators are treated in §2. A quantized contact transformation is an extremely important notion, one that revolutionized the theory of linear differential equations. The reader may be astonished to see how easily one can obtain profound results with the structures of solutions of linear (micro)-differential equations by combining microfunction theory with the theory of quantized contact transformations. This point should be considered as the quintessence of microlocal analysis. As in Chapter III, we proceed in Chapter IV in a manner accesible to the reader rather than in the most logical order, which may be less accessible. For example, in §1 we chose the plane-wave decomposition of the δ-function as a starting point for the introduction of microdifferential operators, and in §2 we restricted our discussion to those contact transformations which have generating functions. We decided not to present our more "algebro-analytic" treatments of the above topics until we write a treatise on microdifferential equations centered around the theory of holonomic systems. Likewise, so that the essence of the theory might be plain to the reader, we did not aim at full generality in §3.

As we close this preface, we would like to express our most sincere gratitude to our teacher Professor Mikio Sato, who indeed provided almost all the essential ideas this book contains. We hope that this book will succeed in imparting the emanation of Professor Sato's throbbing mathematics. It is quite fortunate that authors Kashiwara and Kawai, just at the point when they were choosing their specialities, were able to

attend Professor Hikosaburo Komatsu's introductory lectures in hyperfunction theory.[3] This book might be thought of as a report to Professor Komatsu ten years later. Furthermore, activity centered around Professor Sato and the authors' works has received warm encouragement and support from Professors Kōsaku Yosida and Yasuo Akizuki. Two graduate students at Kyoto University, Mr. Kimio Ueno and Mr. Akiyoshi Yonemura, have read our manuscript and have given beneficial advice. Mr. Yonemura and a graduate student at Sophia University, Mr. Masatoshi Noumi, helped us read the proofs; we would like to take this opportunity to offer our sincere thanks. During the preparation of this book, one or another of us was affiliated with the Research Institute for Mathematical Sciences, Kyoto University; the Department of Mathematics, Nagoya University; the Miller Institute for Basic Research in Science, University of California–Berkeley; the Mathematics Department, Harvard University; the Institute for Advanced Study, Princeton; the Department of Mathematics, Université Paris–Nord; and the Department of Mathematics, Massachusetts Institute of Technology. We thank these institutions and their members for their hospitality during our stay. Last, but not least, we would like to express our profound gratitude to Professor Seizō Itō, who not only gave us the opportunity to write this book, but also kept us from proceeding too slowly. We would again like to apologize to Professor Itō for our delay. Without his warm encouragement, in fact, it is doubtful that this book could ever have been published.

August of the coming-of-age year [1978] of hyperfunction theory[4]

The Authors

[3] *Sato's Hyperfunction Theory and Linear Partial Differential Equations with Constant Coefficients*, Seminar Notes 22 (University of Tokyo). At the time (1968), the above lecture note was at the highest level in the field, rather than at the introductory level.
[4] It was in 1958 that Professor Sato published his outline of hyperfunction theory.

Contents

Notations

(I) Sheaf Theory ($\mathscr{F}$ and $\mathscr{G}$ denote sheaves)

(II) Manifold Theory (M and N denote manifolds; however X and Y are sometimes used instead of M and N)

CHAPTER I

Hyperfunctions

§1. Sheaf Theory

Recall some of the basic concepts from sheaf theory.

Definition 1.1.1. *A presheaf $\mathscr{F}$ over a topological space X associates with each open set U of X an abelian group $\mathscr{F}(U)$, such that there exists an abelian group homomorphism $\rho_{V,U}:\mathscr{F}(U) \to \mathscr{F}(V)$ for open sets $U \supset V$ with the following axioms:*

(1) $\rho_{U,U} = \mathrm{id}_U$ (*= the identity map on $\mathscr{F}(U)$*)

(2) *For $V_1 \subset V_2 \subset V_3$, open sets of X, we have*

$$\rho_{V_1,V_2} \circ \rho_{V_2,V_3} = \rho_{V_1,V_3}.$$

The homomorphism $\rho_{V,U}$ is called the restriction map, and for $s \in \mathscr{F}(U)$ $\rho_{V,U}(s)$ is often denoted by $s|_V$.

Definition 1.1.2. *Let $\mathscr{F}$ be a presheaf over X. The stalk of the presheaf $\mathscr{F}$ at $x \in X$ is defined as $\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$, where $\varinjlim$ denotes the inductive limit, where an equivalence relation "$\sim$" on $\bigcup_{x \in U} \mathscr{F}(U)$ is defined as follows: $s_1 \sim s_2$, for $s_1 \in F(U)$ and $s_2 \in F(V)$, if and only if there exists a sufficiently small open set $W \subset U \cap V$ such that $s_1|_W = s_2|_W$. Therefore a canonical map is induced: $\mathscr{F}(U) \to \mathscr{F}_x$ for $x \in U$. The image of $s \in \mathscr{F}(U)$ under the canonical map is denoted by s_x. Hence we have $(s_1)_x = (s_2)_x$ if and only if there exists an open set V such that $x \in V \subset U$ and such that $s_1|_V = s_2|_V$.*

Definition 1.1.3. *A presheaf $\mathscr{F}$ over X is said to be a sheaf if the following axioms are satisfied: it is given an open covering $\{U_i\}_{i \in I}$ of U in X, $U = \bigcup_{i \in I} U_i$.*

(a) *Let $s \in \mathscr{F}(U)$. If $s|_{U_i} = 0$ for each $i \in I$, then $s = 0$.*

(b) *Suppose that for each $i \in I$ there exists $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for $i, j \in I$. Then there exists $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for each $i \in I$.*

Definition 1.1.4. *Suppose that $\mathscr{F}$ and $\mathscr{G}$ are presheaves. Then $f:\mathscr{F} \to \mathscr{G}$ is said to be a morphism if for each open set U the morphism $f(U):\mathscr{F}(U) \to \mathscr{G}(U)$ is an abelian group homomorphism and if open sets U and V are given such that $U \supset V$, then the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f(U)} & \mathscr{G}(U) \\ {\scriptstyle \rho_{V,U}}\downarrow & & \downarrow{\scriptstyle \rho_{V,U}} \\ \mathscr{F}(V) & \xrightarrow{f(V)} & \mathscr{G}(V) \end{array}$$

Hence there is induced a homomorphism on each stalk, $f_x:\mathscr{F}_x \to \mathscr{G}_x$.

Definition 1.1.5. *Let $\mathscr{F}$ be a presheaf over X. A sheaf $\mathscr{F}'$ is said to be the sheaf associated to the presheaf $\mathscr{F}$ (or $\mathscr{F}'$ is the sheafication of $\mathscr{F}$, or $\mathscr{F}'$ is the induced sheaf from the presheaf $\mathscr{F}$) if for each open set U of X the presheaf $\mathscr{F}'(U)$ (which is actually a sheaf) associates all the maps: $U \xrightarrow{s} \bigcup_{x \in U} \mathscr{F}_x$ such for each $X \in U$ there exists a neighborhood U' of x and $s' \in \mathscr{F}(U')$ such that $s(x') = s'_{x'}$ is true for any x' in U'.*

For a given morphism from a presheaf $\mathscr{F}$ into a sheaf $\mathscr{G}$ there is induced a unique morphism from $\mathscr{F}'$ into $\mathscr{G}$. Note that $\mathscr{F}$ and $\mathscr{F}'$ are isomorphic on each stalk.

Definition 1.1.6. *Let $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$ be sheaves over a topological space X. A sequence $\mathscr{F}' \xrightarrow{f'} \mathscr{F} \xrightarrow{f} \mathscr{F}''$ is said to be exact if $\mathscr{F}'_x \xrightarrow{f'_x} \mathscr{F}_x \xrightarrow{f_x} \mathscr{F}''_x$ is an exact sequence, i.e.* Ker f_x = Im f'_x, *at each $x \in X$.*

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves over X, and let $f:\mathscr{F} \to \mathscr{G}$ be a morphism. Then the presheaf assignment of U, an open subset, to $\mathrm{Ker}(\mathscr{F}(U) \xrightarrow{f(U)} \mathscr{G}(U))$ *is a sheaf, denoted by* Ker(f). *One also has the presheaf* $\mathrm{Coker}(\mathscr{F}(U) \xrightarrow{f(U)} \mathscr{G}(U)) = \mathscr{G}(U)/\mathrm{Im}\, f(U)$. *This presheaf is not a sheaf in general. The sheaf associated to this presheaf is denoted by* Coker(f). *Then, by definition, we have the exact sequence of sheaves*

$$0 \to \mathrm{Ker}(f) \to \mathscr{F} \xrightarrow{f} \mathscr{G} \to \mathrm{Coker}(f) \to 0.$$

In the case where Ker$(f) = 0$, *we often write $\mathscr{G}/\mathscr{F}$ instead of* Coker(f) *by identifying $\mathscr{F}$ with* Im f.

Definition 1.1.7. *Let X and Y be topological spaces, and let $f:X \to Y$ be a continuous map. For a sheaf $\mathscr{F}$ over X the presheaf assignment of an open subset U of Y to $\mathscr{F}(f^{-1}(U))$ is a sheaf over Y. This sheaf is called the direct image of $\mathscr{F}$ under the continuous map f, denoted by $f_*(F)$. For a sheaf $\mathscr{G}$ on Y there can be defined the presheaf $\varinjlim_{V \supset f(U)} \mathscr{G}(V)$ for an open set U of X. Generally this presheaf is not a sheaf. The associated sheaf is called the inverse image of $\mathscr{G}$ under f, denoted by $f^{-1}(\mathscr{G})$. Suppose that S is an arbitrary subset of X, and let $j_S:S \to X$ be the imbedding map. Then* *the inverse image $j_S^{-1}(\mathscr{F})$ of the sheaf $\mathscr{F}$ is called the restriction of $\mathscr{F}$ to S, and we often denote it by $\mathscr{F}|_S$.*

If $\mathscr{G}$ is a sheaf over Y, then there exists a natural morphism $\mathscr{G} \to f_*(f^{-1}(\mathscr{G}))$. Notice that $(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$ and that giving a morphism $\mathscr{G} \to f_*\mathscr{F}$ for a sheaf $\mathscr{F}$ over X is equivalent to giving a morphism $f^{-1}\mathscr{G} \to \mathscr{F}$.

Definition 1.1.8. *Let $\mathscr{F}$ be a sheaf over a topological space X, and let U be an open subset of X. The subset $\{x \in U \mid s_x \neq 0\}$ for $s \in \mathscr{F}(U)$ is called the support of s, denoted by* supp(*s*). *Note that* supp(*s*) *is closed in U.*

Definition 1.1.9. *Let $\mathscr{F}$ be a sheaf over a topological space X, and let S be a locally closed subset of X; i.e. it is the intersection of an open set and a closed set in X. Then define $\Gamma_S(X, \mathscr{F}) = \{s \in \mathscr{F}(U) \mid \mathrm{supp}(s) \subset S\}$, where U is an open set in X such that S is closed in U.*

The definition above is independent of the choice of U.

Proof. Let U_1 and U_2 be such open sets; then $U_1 \cap U_2$ contains S as a closed subset. Therefore one can assume that $S \subset U_1 \subset U_2 \subset X$ and that S is closed in U_1 and U_2. Define a map φ from $\{s \in \mathscr{F}(U_2) \mid \mathrm{supp}(s) \subset S\}$ to $\{s \in \mathscr{F}(U_1) \mid \mathrm{supp}(s) \subset S\}$ by $\varphi(s) = s|_{U_1}$. Then φ is bijective. Therefore $\Gamma_S(X, \mathscr{F})$ is independent of the choice of U.

In the case that $S = X$, we denote $\Gamma_S(X, \mathscr{F})$ with $\Gamma(X, \mathscr{F})$, whose elements are called the global sections of $\mathscr{F}$, i.e. $\Gamma(X, \mathscr{F}) = \mathscr{F}(X)$. Generally we also denote $\mathscr{F}(U)$ with $\Gamma(U, \mathscr{F})$ for an open set U in X, whose elements are called the sections of $\mathscr{F}$ over U.

Definition 1.1.10. *Let $\mathscr{F}$ be a sheaf over a topological space X, and let S be a locally closed subset of X. We denote the sheaf associated to a presheaf $\Gamma_{S \cap U}(U, \mathscr{F}) = \{s \in \mathscr{F}(U) \mid \mathrm{supp}(s) \subset S \cap U\}$, for an open set U of X, by $\Gamma_S(\mathscr{F})$.*

Definition 1.1.11. *A sheaf $\mathscr{F}$ over a topological space X is said to be flabby if for an arbitrary open set U the homomorphism $\rho_{U,X}: \mathscr{F}(X) \to \mathscr{F}(U)$ is an epimorphism. Therefore, for a flabby sheaf $\mathscr{F}$ any section of $\mathscr{F}$ over U can be extended to a section over X.*

Proposition 1.1.1. *Let $\mathscr{F}$ be a flabby sheaf over a topological space X, and let S be a locally closed subset of X. Then $\Gamma_S(\mathscr{F})$ is a flabby sheaf.*

Proof. Let U_1 be an open set such that S is closed in U_1. Then, for any open set U of X, the set $U_1 \cap U$ is open in X and contains $S \cap U$ as a closed set. Let s be an element of $\Gamma_S(\mathscr{F})(U) = \Gamma_{S \cap U}(U, \mathscr{F})$; then $s \in \mathscr{F}(U_1 \cap U)$ and $\mathrm{supp}(s) \subset S \cap U$. Therefore we have $s|_{(U_1 - S) \cap (U_1 \cap U)} = 0$. Then there exists a unique $s' \in \mathscr{F}((U_1 - S) \cup (U_1 \cap U))$ such that $s'|_{(U_1 - S)} = 0$ and $s'|_{U_1 \cap U} = s$. Since the sheaf $\mathscr{F}$ is flabby, s' can be extended to a section $\tilde{s} \in \mathscr{F}(U_1)$. Then we have $\tilde{s}|_{(U_1 - S)} = 0$. Hence $\mathrm{supp}(\tilde{s}) \subset S$, i.e. $\tilde{s} \in \Gamma_S(X, \mathscr{F}) = \Gamma_S(\mathscr{F})(X)$.

Proposition 1.1.2. *Let $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$ be sheaves over a topological space X, let U be an open set, and let S be a locally closed set in X.*

(1) *If* $0 \to \mathscr{F}' \xrightarrow{f'} \mathscr{F} \xrightarrow{f} \mathscr{F}''$ *is an exact sequence of sheaves, then*

(i) $0 \to \mathscr{F}'(U) \xrightarrow{f'(U)} \mathscr{F}(U) \xrightarrow{f(U)} \mathscr{F}''(U)$ *and*

(ii) $0 \to \Gamma_S(X, \mathscr{F}') \to \Gamma_S(X, \mathscr{F}) \to \Gamma_S(X, \mathscr{F}'')$ *are exact.*

(2) If $0 \to \mathscr{F}' \xrightarrow{f'} \mathscr{F} \xrightarrow{f} \mathscr{F}'' \to 0$ *is an exact sequence of sheaves, and if* $\mathscr{F}'$ *is a flabby sheaf, then*

(i) $0 \to \mathscr{F}'(U) \xrightarrow{f'(U)} \mathscr{F}(U) \xrightarrow{f(U)} \mathscr{F}''(U) \to 0$ *and*

(ii) $0 \to \Gamma_S(X, \mathscr{F}') \to \Gamma_S(X, \mathscr{F}) \to \Gamma_S(X, \mathscr{F}'') \to 0$ *are exact.*

Proof. (1.i) First we will show that $f'(U)$ is a monomorphism. Suppose that $f'(U)s' = 0$ for $s' \in \mathscr{F}'(U)$. Then $f'_x s'_x = 0$ for each x in U. Therefore $s'_x = 0$; i.e. there exists a neighborhood $V(x)$ of x such that $s'|_{V(x)} = 0$. By the definition of a sheaf, we know that $s' = 0$. Therefore $f'(U)$ is monomorphic. Next we will prove that $\operatorname{Im} f'(U) \subset \operatorname{Ker} f(U)$. Since $(f_x \circ f'_x)s'_x = 0$ for $s' \in \mathscr{F}(U)$, for each x one can find a neighborhood $V(x)$ of x such that $f(U)f'(U)s'|_{V(x)} = 0$. Therefore, since $\mathscr{F}''$ is a sheaf we have $f(U)f'(U)s' = 0$. It remains to be proved that $\operatorname{Im} f'(U) \supset \operatorname{Ker} f(U)$. Let $s \in \mathscr{F}(U)$ such that $f(U)s = 0$. Then, for each $x \in U$, $f_x s_x = 0$ holds. By the exactness there exists $s'_x \in \mathscr{F}'_x$ such that $f'_x s'_x = s_x$. This implies that $f'(V(x))s'(x) = s|_{V(x)}$ for some $s'(x) \in \mathscr{F}(V(x))$ in some neighborhood $V(x)$ of x such that $V(x) \subset U$. Since $f'(V(x))$ is a monomorphism, $s'(x)$ is unique. Therefore we have $s'(x)|_{V(x)\cap V(y)} = s'(y)|_{V(x)\cap V(y)}$. By the sheaf axiom, we have $s' \in \mathscr{F}'(U)$ and $s'|_{V(x)} = s'(x)$. Then $f'(U)s' = s$.

Next we will give a proof of (1.ii). Let U be an open set in X such that S is closed in U. It is to be shown that $\operatorname{supp}(s') \subset S$ for the s', as in the above, provided that $\operatorname{supp}(s) \subset S$ for an $s \in \mathscr{F}(U)$. Note that $f'(U-S)s'|_{(U-S)} = s|_{(U-S)} = 0$ and that $f'(U-S)$ is a monomorphism. Therefore $s'|_{(U-S)} = 0$, i.e. $\operatorname{supp}(s') \subset S$.

It suffices to show that $f(U)$ is an epimorphism to prove (2.i). Let $s'' \in \mathscr{F}''(U)$, and let $\mathscr{M} = \{(s, V) \mid V$ is an open subset of U, $s \in \mathscr{F}(V)$ and $f(V)s = s''|_V\}$. Then define an order relation, denoted with $\succ$, in $\mathscr{M}$ as follows: let (s_1, V_1) and (s_2, V_2) be elements of $\mathscr{M}$. The expression $(s_1, V_1) \succ (s_2, V_2)$ holds if and only if $V_1 \supset V_2$ and $s_1|_{V_2} = s_2$. Then $\mathscr{M}$ is a non-empty, inductively ordered set. Therefore there exists a maximal element in $\mathscr{M}$ by Zorn's lemma. Let (s, V) be a maximal element. $V = U$ is left to be proved. Suppose $V \neq U$, and let $x \in U - V$. Then there exists a neighborhood $V(x)$ and $s(x) \in \mathscr{F}(V(x))$ such that $(s(x), V(x)) \in \mathscr{M}$. Then notice that $f(V \cap V(x))(s - s(x))|_{V\cap V(x)} = 0$. One can then find $s' \in \mathscr{F}'(V \cap V(x))$ such that $f'(V \cap V(x))s' = (s - s(x))|_{V\cap V(x)}$ by (1.i). The flabbiness of $\mathscr{F}'$ implies that there exists $\tilde{s}' \in \mathscr{F}'(V(x))$ such that $\tilde{s}'|_{V \cap V(x)} = s'$. Define $\tilde{s} \in \mathscr{F}(V \cup V(x))$ as $\tilde{s}|_V = s$ and $\tilde{s}|_{V(x)} = s(x) + f'(V(x))\tilde{s}'$. Then $(\tilde{s}, V \cup V(x)) \in \mathscr{M}$, which contradicts the maximality of (s, V) in $\mathscr{M}$. Therefore $V = U$; that is, $f(U)$ is an epimorphism. (2.ii) can be proved similarly. Let $\mathscr{M}' = \{(s, U) | s \in \Gamma_{S \cap U}(U, \mathscr{F}), U$ is an open set such that $f(U)s = s''|_U\}$. Then take a maximal element of $\mathscr{M}'$ to be (s, V) satisfying that $(s, V) \succ (0, X - \operatorname{supp}(s''))$.

Remark. Conversely, if $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)(\to 0)$ is exact for any open set U, then it is plain that $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''(\to 0)$ is exact.

Corollary 1. *Let* $0 \to \mathscr{F}' \xrightarrow{f'} \mathscr{F} \xrightarrow{f} \mathscr{F}'' \to 0$ *be an exact sequence of sheaves.*

(1) *If* $\mathscr{F}'$ *and* $\mathscr{F}$ *are flabby sheaves, then* $\mathscr{F}''$ *is a flabby sheaf.*

(2) *If* $\mathscr{F}'$ *and* $\mathscr{F}''$ *are flabby sheaves, then* $\mathscr{F}$ *is a flabby sheaf.*

Proof. Since $\mathscr{F}'$ is a flabby sheaf, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F}'(X) & \xrightarrow{f'(X)} & \mathscr{F}(X) & \xrightarrow{f(X)} & \mathscr{F}''(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho'_{U,X}} & & \downarrow{\scriptstyle \rho_{U,X}} & & \downarrow{\scriptstyle \rho''_{U,X}} & & \\
0 & \longrightarrow & \mathscr{F}'(U) & \xrightarrow{f'(U)} & \mathscr{F}(U) & \xrightarrow{f(U)} & \mathscr{F}''(U) & \longrightarrow & 0
\end{array}$$

with exact rows, where U is any open set in X. To prove (1), first notice that $\rho_{U,X}$ is an epimorphism since $\mathscr{F}$ is flabby. On the other hand, $f(U)$ is an epimorphism. Therefore, for any $s'' \in \mathscr{F}''(U)$, there exists an $s \in \mathscr{F}(X)$ such that $s'' = f(U)\rho_{U,X}(s) = \rho''_{U,X}(f(X)s)$. This implies that $\rho''_{U,X}$ is an epimorphism; i.e. $\mathscr{F}''$ is a flabby sheaf. Next we will prove (2). Let $s \in \mathscr{F}(U)$. Since $\rho''_{U,X}$ and $f(X)$ are both epimorphisms, one can find $\tilde{s} \in \mathscr{F}(X)$ such that $\rho''_{U,X} f(X)\tilde{s} = f(U)s$. By commutativity we have $\rho''_{U,X} f(X)\tilde{s} = f(U)\rho_{U,X}\tilde{s}$. Therefore $f(U)(s - \rho_{U,X}\tilde{s}) = 0$. Then note that $\rho'_{U,X}$ is an epimorphism. Hence there exists $\tilde{s}' \in \mathscr{F}'(X)$ such that $s - \rho_{U,X}\tilde{s} = f'(U)\rho'_{U,X}\tilde{s}' = \rho_{U,X} f'(X)\tilde{s}'$. That is, $s = \rho_{U,X}(\tilde{s} + f'(X)\tilde{s}')$, showing that $\rho_{U,X}$ is an epimorphism, Therefore $\mathscr{F}$ is a flabby sheaf.

Corollary 2. *Suppose that* $0 \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots \to \mathscr{F}^r \to \mathscr{G} \to 0$ *is an exact sequence of sheaves and that each* $\mathscr{F}^i$, $i = 0, 1, \ldots, r$, *is a flabby sheaf. Then* $\mathscr{G}$ *is a flabby sheaf, and the following sequences are exact:*

$$0 \to \Gamma_S(X, \mathscr{F}^0) \to \Gamma_S(X, \mathscr{F}^1) \to \cdots \to \Gamma_S(X, \mathscr{F}^r) \to \Gamma_S(X, \mathscr{G}) \to 0$$

and

$$0 \to \mathscr{F}^0(U) \to \mathscr{F}^1(U) \to \cdots \to \mathscr{F}^r(U) \to \mathscr{G}(U) \to 0,$$

where S is a locally closed subset of X, and where U is an open subset of X.