CALCULUS

AND ANALYTIC GEOMETRY

AL SHENK

FOURTH EDITION

FOURTH EDITION

Calculus and Analytic Geometry

AL SHENK
University of California,
San Diego

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TO GENIE

There are two volumes of a *Student Handbook,* which parallel the topic coverage of the separately published volumes of the text. Each includes section summaries, learning hints, and detailed solutions to selected exercises.

A *Calculus Toolkit* is available in Apple or IBM, either as a separately sold item or in combination with the text (at a reduced price). Its main function is to illustrate the beauty of computer-generated graphics, as well as to provide ease and instruction in performing and understanding important mathematical calculations, from single- to multivariable calculus.

Two volumes of the combined book are available. *Volume I* covers single-variable calculus up to and including conic sections and polar coordinates, while *Volume II* (which begins with sequences and series) includes further topics in several variables, among them Green's, Divergence, and Stokes' theorems, as well as differential equations.

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PREFACE

Calculus and Analytic Geometry, Fourth Edition, covers analytic geometry and differential and integral calculus of one, two, and three variables and includes introductions to differential calculus of more than three variables, ordinary differential equations, infinite sequences and series, and vector analysis. It is designed primarily for one-and-a-half or two-year courses for mathematics, science, and engineering students, but can be used in other classes by limiting the topics and types of exercises covered.

This text emphasizes the geometry of calculus and problem solving. Numerous applications to physics, biology, chemistry, medicine and other fields are included. Exercises using computer-generated drawings enable students to work with curves and surfaces that they could not readily sketch, and graphical problems that require little or no algebra help teach the geometric meanings of results and procedures. Computer/calculator exercises illustrate limit processes and show how calculators and computers can be used with calculus. Step-by-step procedures are given for solving the most difficult types of problems, and brief historical notes describe how the basic ideas of calculus evolved.

The study problems before the exercise sets at the ends of sections are representative of the most important types of exercises and are designed to develop students' problem-solving skills and to help them make the transition from reading examples to solving exercises. The accompanying outlines of solutions show how the problems are solved and give the results of intermediate calculations and final answers, leaving some of the details to be worked out by the students.

The review exercises at the end of each chapter cover the main types of problems in the chapter. They can be used to supplement homework assignments, and because their answers are given in the back of the text, they can be also used by students to test their understanding. The sets of miscellaneous exercises in each chapter consist of problems that either do not fall in the categories covered in the sections or are more difficult than those in the sections.

Calculator exercises, indicated by the symbol, can be worked with any scientific calculator or computer and include traditional calculus exercises as well as problems on predicting limits, numerical differentiation, Newton's method in one variable, numerical integration, and the numerical solution of differential equations. Computer exercises, indicated by, require a computer or programmable calculator and include exercises on Newton's method in one variable, differential equations, and the calculation of partial sums of infinite series. Either one or both of these types of exercises can be omitted without loss of continuity.

CHANGES IN THE FOURTH EDITION

This edition has been extensively rewritten and reorganized in response to suggestions by users and the author's experience in his own classes. Notation and terminology have been standardized, and all drawings of xyz-space now have the traditional orientation of the x- and y-axes.

The precalculus Chapter 1 of the third edition has been condensed into new Sections 1.1 and 1.2. Section 1.4 on finite limits now contains step-by-step procedures for finding δ's in $\epsilon\delta$-exercises. Section 2.9 of the third edition has been combined with other parts of the discussion of inverse functions in a new Section 6.2. The tangent, cotangent, secant, and cosecant are now discussed with the sine and cosine in Sections 2.7 and 2.8. The section on numerical integration has been moved to Chapter 4 on the integral. The first seven sections of Chapter 5 on applications of integration have been streamlined, with several topics moved to a final Section 5.8. L'Hôpital's rule has been moved to Chapter 6 on transcendental functions; improper integrals are now covered in Chapter 7 on techniques of integration; and numerical methods of solving $dy/dx = f(x, y)$ are now in Chapter 8 with separable differential equations. The section on Taylor polynomials has been placed in Chapter 9 on infinite sequences and series; and the material on vectors, curves, and partial derivatives in Chapters 11 through 13 has been reorganized so that the two- and three-variable cases are presented together. A new section on Jacobians and changing variables in multiple integrals has been added to Chapter 15.

VOLUMES I AND II

This text is available in one or two volumes. Volume I contains Chapters 1 through 10 of the combined book: single-variable calculus, separable differential equations, infinite sequences and series, and conic sections and polar coordinates. Volume II contains Chapters 9 through 17: infinite sequences and series, conic sections and polar coordinates, vectors, multivariable calculus, vector analysis, and further topics in ordinary differential equations.

MANUALS

The *Student Handbook,* which may be purchased by students, gives brief summaries of the theory and techniques in each section, along with pointers for problem solving and detailed solutions of exercises selected from those whose answers are given in the back of the text.

The *Instructor's Guide and Solutions Manual,* which is available to instructors on request, contains solutions of all exercises in the text, classifications of the exercises by level of difficulty, and the author's comments on, and suggestions for, using the material and exercises in each section.

COMPUTER SUPPLEMENTS

The *Student Handbook* and *Instructor's Guide* contain a set of sample BASIC and Pascal computer programs on (i) evaluating functions; (ii) predicting limits and derivatives; (iii) solving $F(x) = 0$ by the bisection method, by Newton's method, and by the secant method; (iv) solving $g(x) = x$ by fixed point iterations; (v) numerical integration by the midpoint rule, the trapezoid rule, and Simpson's rule; (vi) calculating partial sums of infinite series; (vii) solving $F(x, y) = 0$, $G(x, y) = 0$ by Newton's method in two variables; and (viii) numerical solution of first order differential equations by Euler's method, a trapezoid method, and a Runge-Kutta method. Sets of exercises on Newton's method in two variables and on numerical solutions of first order differential equations are also given in both manuals.

A *Calculus Toolkit* is available in IBM format, either as a separately sold item or, at a reduced price, in combination with the text. It illustrates computer-generated graphics and limit processes and shows students how the computer can be used with single- and multivariable calculus.

TRANSPARENCIES

More than 100 two- and four-color transparencies, taken from illustrations in the text, are available to instructors who adopt the book. The transparencies deal with (i) graphical differentiation, (ii) direction fields and solutions of separable differential equations, (iii) conic sections, (iv) curves and acceleration vectors, (v) quadric surfaces, (vi) graphs and level curves and the second derivative test for functions of two variables, (vii) Lagrange multipliers, (viii) double and triple integrals, (ix) vector fields, (x) surface integrals, and (xi) the theorems of Gauss and Stokes.

SHENKTEST

A testing system to be used with either Apple or IBM computers offers a large number of test items with integrated graphics and allows instructors to add, delete, or alter questions to fit their individual needs.

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STREAMLINED APPROACH

Recent conferences have called for improving calculus by streamlining rigid and overcrowded syllabi, by emphasizing concepts rather than problem solving based on pattern recognition, and by giving instructors more opportunity to convey their enthusiasm for mathematics to their students. This book is well suited for classes which follow these recommendations.

1. Because it is comprehensive, it gives instructors a great deal of flexibility in deciding what to cover and what to omit in order to decrease the amount of required material while emphasizing the topics that they enjoy and that best fit the interests and needs of their students. Also it can serve students as a reference for topics not covered.

2. To make it easier to set up streamlined courses, a chart is available through the Publisher with the discussions, study problems, and exercises which, in the author's opinion, could be omitted without sacrificing fundamentals. Instructors can reduce the amount of required material by telling their students the numbers of the topics on the list that are not required.

3. Because the book presents theory clearly, there is no need to give any proofs in class, except in those instances where students will find them interesting, comprehensible, and instructive. Instead, lecture time can be used for informal discussions of concepts and results.

4. Material on problem solving and accompanying study problems can be assigned for students to read before class so they can participate in working out examples during the lectures.

INTRODUCTION

Calculus is a tool for solving problems that cannot be solved by other means. If a car moves on a straight road and the forces acting on it are constant, its motion can be determined using only arithmetic and algebra, but if the road is curved and the forces vary, calculus is required. Well-known formulas from geometry will suffice to find the area of a circle or the volume of a cone, but calculus is needed to find the area of a region bounded by a parabola or the volume of a nonspherical lens. Calculus is the foundation for much of higher mathematics and is fundamental in all branches of the physical sciences, including the study of planetary motion, aerodynamics and fluid flow, heat transfer, sound, electricity and magnetism, biological growth and decay, and chemical reactions. It also has numerous applications in medicine, economics, and the social sciences.

The basic operations of calculus, *differentiation* and *integration,* are not difficult to understand because they are based on the familiar concepts of velocity and area. With this book, you will learn how these concepts lead to the basic results and techniques of calculus and you will learn how to apply calculus to practical problems.

HOW TO USE THIS BOOK

Review all of the precalculus material in Sections 1.1 and 1.2, including the brief catalog of curves in Section 1.2, even if it is not assigned to your class.

The most frequently employed techniques from algebra for solving narrative problems are illustrated in the exercises in Section 1.3 and in the review and miscellaneous exercises at the end of Chapter 1. Study these techniques before you begin work on the calculus narrative problems in Sections 3.6, 3.7, and 3.8.

You will probably find that your greatest resource for understanding and using calculus is your geometric intuition. Learn the geometric meanings of results and procedures and develop the ability to use the underlying geometry to guide your reasoning when you work exercises. Draw lots of pictures.

Although you can learn to work routine problems mechanically, you need to understand the theory to use calculus in new types of problems. Study the definitions, theorems, and proofs. They may be difficult to understand at first, but their meaning and purpose will become clear as you apply them and as your overall understanding of the subject increases.

The theory of calculus as presented in modern courses is so concise and formal that it may lead you to ask, "Where did this come from?" Calculus was not invented all at once, but has evolved over the past 300 years. The historical notes at the ends of Chapters 1, 2, 4, and 6 are included to give you a better understanding of its origins and development.

You will master calculus by learning to work exercises. First study each section, reading carefully the definitions and results presented in it and the solutions of examples. Next, work through the study problems, with the outlines of solutions as your guide, supplying all the details of the calculations yourself. Then you can apply the procedures you have learned to the exercises. The answers to exercises with colored exercise numbers and the answers and outlines of solutions of those marked with degree signs are given in the back of the book for you to check your results. You might also want to consult the *Student Handbook,* which is available through your bookstore. It gives summaries of the theory and techniques, points for problem solving, and detailed solutions of selected exercises in each section. Use the review exercises to test yourself when you have finished studying an entire chapter.

The sets of miscellaneous exercises at the ends of the chapters contain nonroutine problems that will extend your understanding of the material in the chapter and show more ways in which it can be applied.

If you have access to a calculator or computer, work the calculator and computer exercises, which are indicated with the symbols  and , respectively. They will show you how calculators and computers can be used in problem solving and will help you understand the various types of limit processes that are fundamental to calculus. Instructions for using some of the common types of calculators and sample computer programs are given in the *Student Handbook*.

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CHAPTER 1

Functions and Limits

SECTION 1.1 COORDINATE LINES AND PLANES

The first two sections of this chapter provide a brief review of the analytic geometry that is required to apply calculus to geometric problems and to give geometric interpretations of concepts and results in calculus. We will see that most of the definitions and theorems in this book have geometric meanings which make them easier to understand.

The first step in analytic geometry is to associate the ***real numbers**** (the positive numbers, the negative numbers, and 0) with points on a ***coordinate line,*** such as the ***x-axis*** as in Figure 1.1, where the letter ''x'' indicates that x is being used to refer to the points on the line and to the numbers that they represent. (Because we deal almost exclusively with real numbers in introductory calculus, we say ''number'' when we mean ''real number,'' and because of the association of each real number with a point on a coordinate line, we use the terms ''the number x'' and ''the point x'' interchangeably.)

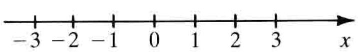

FIGURE 1.1

The number x_2 is *greater* than the number x_1 (equivalently, x_1 is *less* than x_2) if $x_2 - x_1$ is positive. In this case, x_2 is to the right of x_1 on a coordinate line as in Figure 1.1, and we write $x_2 > x_1$ or $x_1 < x_2$. The condition $x_2 \geq x_1$ means that x_2 is either greater than or equal to x_1 and is equivalent to the condition $x_1 \leq x_2$, which states that x_1 is less than or equal to x_2.

*The basic properties of the real numbers are reviewed in Appendix 1.

The sets (collections) of numbers most frequently encountered in calculus are ***intervals.*** The set of numbers x that satisfy the inequalities $a \le x \le b$ for constants a and b with $a \le b$ is a ***finite*** (or ***bounded***) ***closed interval*** and is denoted by the symbol $[a, b]$ (Figure 1.2). The numbers x satisfying $a < x < b$ with $a < b$ form a ***finite open interval*** that is denoted (a, b) (Figure 1.3). As is illustrated in Figures 1.2 and 1.3, we sketch intervals with heavy lines and show that an endpoint is included in an interval by putting a dot at that point and use a circle if the endpoint is not in the interval. The interval $[a, b]$ is called closed because it includes its endpoints, and (a, b) is called open because it does not.

The closed interval $[a, b]$

FIGURE 1.2

The open interval (a, b)

FIGURE 1.3

The interval $[a, b)$, consisting of the numbers x with $a \le x < b$, and the interval $(a, b]$, formed by the numbers x with $a < x \le b$, shown in Figure 1.4, are called ***half open*** because they each contain one of their endpoints and not the other.

The half-open interval $[a, b)$

The half-open interval $(a, b]$

FIGURE 1.4

There are also several types of ***infinite*** (or ***unbounded***) ***intervals.*** The interval $[a, \infty)$, consisting of all x with $x \ge a$, and the interval $(-\infty, b]$, consisting of all x with $x \le b$, are ***closed half lines,*** while the interval (a, ∞), formed by the numbers x with $x > a$, and the interval $(-\infty, b)$, formed by the numbers x with $x < b$, are ***open half lines*** (Figures 1.5 and 1.6). The symbols ∞ (''infinity'') and $-\infty$ (''minus infinity'') in this notation do not represent numbers. They are used only to show whether the half lines extend to the left or right of their endpoints. The set of all real numbers is also an interval and is denoted $(-\infty, \infty)$. It is considered to be both open and closed because it has no endpoints to include or exclude.

The closed half line $[a, \infty)$

The closed half line $(-\infty, b]$

FIGURE 1.5

The open half line (a, ∞)

The open half line $(-\infty, b)$

FIGURE 1.6

The ***interior*** of a nonopen interval is the open interval obtained by removing its endpoints, and the interior of an open interval is the interval itself. The ***closure*** of a nonclosed interval is the interval obtained by adding its endpoints, and the closure of a closed interval is itself. Thus, the interiors of the intervals $[1, 5)$, $(-\infty, -1]$, and $(0, 1)$ are $(1, 5)$, $(-\infty, -1)$, and $(0, 1)$, and their closures are $[1, 5]$, $(-\infty, -1]$, and $[0, 1]$.

The ***absolute value*** of a real number x is denoted $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0. \end{cases}$$

Thus, $|3| = 3$ and $|-3| = -(-3) = 3$. This definition may also be expressed as $|x| = \sqrt{x^2}$ since $\sqrt{x^2}$ denotes the nonnegative square root of x^2.

EXAMPLE 1 Solve the equation $|2x - 3| = 4$ for x.

Solution. The equation $|2x - 3| = 4$ is equivalent to the statement

$$2x - 3 = 4 \qquad \text{or} \qquad 2x - 3 = -4.$$

Adding 3 to both sides of the two equations gives

$$2x = 7 \qquad \text{or} \qquad 2x = -1.$$

We divide by 2 to obtain the solutions $x = \frac{7}{2}, -\frac{1}{2}$. ■

The ***triangle inequality***

$$|a + b| \le |a| + |b|$$