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A PRIMER OF REAL FUNCTIONS

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By

RALPH P. BOAS, JR.

Professor of Mathematics
Northwestern University

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THE CARUS MATHEMATICAL MONOGRAPHS are an expression of the desire of Mrs. Mary Hegeler Carus, and of her son, Dr. Edward H. Carus, to contribute to the dissemination of mathematical knowledge by making accessible at nominal cost a series of expository presentations of the best thoughts and keenest researches in pure and applied mathematics. The publication of the first four of these monographs was made possible by a notable gift to the Mathematical Association of America by Mrs. Carus as sole trustee of the Edward C. Hegeler Trust Fund. The sales from these have resulted in the Carus Monograph Fund, and the Mathematical Association has used this as a revolving book fund to publish the succeeding monographs.

The expositions of mathematical subjects which the monographs contain are set forth in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields, and especially to the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics, wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises. The scope of this series includes also historical and biographical monographs.

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TO MY EPSILONS

PREFACE

I. To the beginner. In this little book I have presented some of the concepts and methods of "real variables" and used them to obtain some interesting results. I have not sought great generality or great completeness. My idea is to go reasonably far in a few directions with a minimum amount of special terminology. I hope that in this way I have been able to preserve some of the sense of wonder that was associated with the subject in its early days but has now largely been lost. I hope also that someone who has read this book will be able to go on to one of the many more forbidding systematic treatises, of which there is no lack.

No previous knowledge of the subject is assumed of the reader, but he should have had at least a course in calculus. In general, each topic is developed slowly but rises to a moderately high peak; a reader who finds the slope too steep may skip to the beginning of the next section.

Since this is not a handbook, but more in the nature of a course of informal lectures, I have not been at all consistent about either the proportion of detailed proof to general discussion, or about strict logical arrangement of material.

All phrases like "it is clear," "plainly," "it is trivial" are intended as abbreviations for a statement something like "it should seem reasonable, the reader should be able to supply the proof, and he is invited to do so." On the other hand, "it can be shown . . ." is usually to suggest that the proof is too complicated to give here, or depends on notions that are not discussed here, and that it will probably be better for the reader not to try to supply the proof himself.

In stating definitions, I have frequently used "if" where I should really have used "if and only if." For example, "If a set is both bounded above and bounded below, it is called *bounded*." This definition is to be understood to carry an additional clause, "and if it is not both bounded above and bounded below, it is not called bounded."

There are a number of exercises, some of which merely supply illustrative material, and some of which are essential parts of the book. An exercise that merely states a proposition is to be interpreted as a demand for a proof of the proposition. Answers to all exercises are given at the end of the book.

Paragraphs in small type deal either with peripheral material or with more difficult questions.

I apologize in advance for whatever mistakes the alert reader may be able to detect. None were intentionally included; nevertheless, the detection and rectification of mistakes is a good exercise, and fosters a healthy skepticism about the printed word.

II. To the expert. Experts are not supposed to read this book at all; since this statement will doubtless be taken as an invitation for them to do so, I must explain what I have tried (and not tried) to do. I have set out to tell readers with no previous experience of the subject some of the results that I find particularly interesting. I have therefore tried to present the material that seemed essential for the results I had in mind, together with as much related material as seemed interesting and not too complicated. Since this is not a systematic treatise, I have deliberately tried not to introduce any concepts or notations, however significant or convenient, that I did not really need to use. I have omitted integration, reluctantly, because of the many technical details that are needed before one gets to the interesting results.

Since this is not a treatise it has not been written like one. The style is deliberately wordy. The axiom of choice is frequently used but never mentioned; this book is not the place to discuss philosophical questions, and, in any case, after Gödel's results, the assumption of the axiom of choice can do no mathematical harm that has not already been done. I therefore see no point in avoiding the axiom of choice whenever it seems natural to use it, even in cases where it is known to be avoidable, in a book that is not concerned with the precise logical structure of the subject.

III. Acknowledgements. I am indebted to my teachers, J. L. Walsh and D. V. Widder, for introducing me to this kind of mathematics; to M. L. Boas and to E. F. and R. C. Buck for criticizing early drafts of the book; and to H. M. Clark and H. M. Gehman for help with the proofreading.

Ralph P. Boas, Jr.

Northwestern University
March 1960

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CHAPTER 1

SETS

1. Sets. In order to read anything about our subject, the reader will have to learn the language that is used in it. I shall try to keep the number of technical terms as small as possible, but there is a certain minimum vocabulary that is essential. Much of it consists of ordinary words used in special senses; this practice has both advantages and disadvantages, but has in any case to be endured since it is now too late to change the language completely. Much of the standard language is taken from the theory of sets, a subject with which we are not concerned for its own sake. The theory of sets is, indeed, an independent branch of mathematics. It has its own basic undefined concepts, subject to various axioms; one of these undefined concepts is the notion of "set" itself.

From an intuitive point of view, however, we may think of a *set* as being a collection of objects of some kind, called its *elements*, or *members*, or *points*. We say that a set contains its elements, or that the elements belong to the set, or simply are in the set. The normal usage of set, as in "a set of dishes" or "a set of the works of Bourbaki," is fairly close to what we should have in mind, although the second phrase suggests some sort of arrangement of the elements which is irrelevant to the mathematical concept. Sets may, for example, be formed of ordinary geometrical points, or of functions, or indeed of other sets. We shall use the words *class, aggregate* and *collection* interchangeably with set, especially to make complicated situations clearer: thus we may speak of a collection of aggregates of sets rather than of a set of sets of sets.

If E is a set, a set H is called a *subset* of E if every element of H is also an element of E. For example, if E is the set whose elements are the numbers 1, 2, 3, there are eight subsets of E. Three of them contain one element each; three contain two elements each; one is the set E itself (a subset does not have to be, in any sense, "smaller" than the original set); the eighth subset of E is, by convention, the *empty set,* which is the set that has no elements at all. If H is a subset of E we write $H \subset E$ or $E \supset H$; sometimes we say that E contains H or that E covers H. If H is a subset of E but is not all of E, we call H a *proper* subset of E.

We write $x \in E$ to mean that x is an element of E. We often say that x is in E, or that x belongs to E, or that E contains x, meaning the same thing. Since the elements of sets are usually things of a different kind from the sets themselves, we should distinguish between the element x and the set whose only element is x. It is often convenient to denote the latter set by (x). The notations $x \in E$ and $(x) \subset E$ mean the same thing.

A *space* is a set that is being thought of as a universe from which sets can be extracted. If Ω is a space and $E \subset \Omega$, the *complement* of E (with respect to Ω) is the set consisting of all the elements of Ω that are not elements of E. The complement of E is denoted by $C(E)$. For example, if Ω consists of the letters of the alphabet and E of the consonants (including y as a consonant), $C(E)$ consists of the vowels. If, however,* E consists of the single letter

* For simplicity of notation we frequently use, as here, a letter that has just been used as the name of a set, that we are now through with, to denote a different set.

a, $C(E)$ consists of the letters b, c, $\cdots$, z. If E consists of the entire alphabet, $C(E)$ is empty. If E is empty, $C(E) = \Omega$.

Exercise 1.1. Show that $C(C(E)) = E$.

Occasionally it is necessary to consider the complements of a set with respect to different spaces; in such cases, special notations will be used.

If E and F are two sets, there are two other sets that can be formed by using them, and that occur so frequently that they have special names. One of these sets is the *union* of the two sets, written $E \cup F$ (sometimes called their sum, and written $E + F$); it consists of all elements that are in E or in F (or in both; an element that is in both is counted only once). The other is the *intersection* of the two sets, written $E \cap F$ (sometimes called their product, and written $E \cdot F$ or EF); it consists of all elements that are in both E and F. If $E \cap F$ is empty, E and F are called *disjoint;* that is, E and F are disjoint if they have no element in common.

Exercise 1.2. Let Ω consist of the 26 letters of the alphabet. Let E consist of all the consonants (including y), and F of all the letters that occur in the words *real functions* (the n is counted only once). Show that (a) $E \cup F = \Omega$; (b) $F \supset C(E)$; (c) $C(F) \subset E$; (d) $F \cap E$ and $C(E)$ are disjoint.

There are various logical difficulties inherent in the uncritical use of the terminology of the theory of sets, and they have given rise to a great deal of discussion. Fortunately, however, they arise only at a higher level of abstraction than we shall attain in the rest of this book, and in contexts that we should consider rather artificial, so that we may safely ignore them hereafter. Some forms of words which appear to define sets may actually not do so, somewhat as some combinations of letters which might well represent English words (e.g., "frong") do not actually do so. For example, although we can safely speak of sets whose elements are sets, we cannot safely talk about the set of all sets whatsoever. Supposing that we could, the set of all sets would necessarily have itself as one of its elements. This is a peculiar property, although there are other ostensible sets that have it, for example, the set of all objects definable in fewer than thirteen words (since this "set" is itself defined in fewer than thirteen words). We might well decide to exclude from consideration those sets that are elements of themselves. The remaining sets do not have themselves as elements; form the aggregate of all such acceptable sets, say A. Now is A one of the sets that we accept, or one of the sets that we exclude? If we accept A, it does not have itself as an element and so must be included in the aggregate of all sets with this property; that is, A belongs to A, and therefore we do not accept A. On the other hand, if we do not accept A, A is an element of itself; then since all elements of A are sets that are not elements of themselves, and so are acceptable, we must accept A. Thus if A is a set at all, we are involved in a logical contradiction. The only way out seems to be to declare that the words that seem to define A do not actually define a set.

Another paradoxical property of "the set of all sets" will turn up in § 3.

2. Sets of real numbers. Since we have to start somewhere, the reader will be supposed to be familiar with the real number system. Its algebraic properties—those connected with addition, subtraction, multiplication, and division, and with inequalities—will be taken completely for granted. However, there is one property of the real numbers that is less familiar to most people, even though it underlies concepts, such as limit and convergence, which are fundamental in calculus. This property can be stated in many equivalent forms, and the particular one that we select is a matter of taste. I shall take as fundamental the so-called *least upper bound property*. Before we can state what this property is, we need some more terminology. Let E be a nonempty set of real numbers. We say that E is *bounded above* if there is a number M such that every x in E satisfies the inequality $x \leq M$. For example, the set of all real numbers less than 2 is bounded above, and we can take $M = 2$, or $M = \pi$, or $M = 100$. On the other hand, the set of all positive integers is not bounded above. If E is bounded above, its *least upper bound* is B if B is the smallest M that can be used in the preceding definition. In our example, where E is the set of all real numbers less than 2, the least upper bound of E is 2. Another way of stating the definition of the least upper bound of E is to say that it is a number B such that every x in E satisfies $x \leq B$, while if $A < B$ there is at least one x in E satisfying $x > A$. The least upper bound of E may or may not belong to E. In the example just given, it does not. However, if we change the example so that E consists of all numbers not greater than 2, the least upper bound of E is still 2, and now it belongs to E.

So far, although we have talked about the least upper bound of a set, we have not known (except in our illustrative examples) whether there is any such thing. The least upper bound property, which we take as one of the axioms about real numbers, is just that *every nonempty set E that is bounded above does in fact have a least upper bound.* In other words, if we form the collection of all upper bounds of E, this collection has a smallest element (hence the name). We denote the least upper bound of E by $\sup E$ or $\sup_{x \in E} x$ (sup stands for supremum). When $\sup E$ belongs to E we sometimes write $\max E$ instead. Thus $\max E$ is the largest element of E if E has a largest element. The greatest lower bound, denoted by inf, is defined similarly. (Cf. exercise 2.2.)

An *interval* is a set consisting of all the real numbers between two other numbers, or of all the real numbers on one side or the other of a given number. More precisely, an interval consists of all real numbers x that satisfy an inequality of one of the forms $a < x < b$, $a \leq x < b$, $a < x \leq b$, $a \leq x \leq b$ (where $a < b$), $x > a$, $x \geq a$, $x < a$, or $x \leq a$. Using a square bracket to suggest $\leq$ or $\geq$ and a parenthesis to suggest $<$ or $>$, we shall often use the following notations for the corresponding intervals: (a, b), $[a, b)$, $(a, b]$, $[a, b]$, (a, ∞), $[a, \infty)$, $(-\infty, a)$, $(-\infty, a]$. Thus $(0, 1]$ means the set of all real numbers x such that $0 < x \leq 1$. (The use of the symbol ∞ in the notation for intervals is simply a matter of convenience, and is not to be taken as suggesting that there is a number ∞.)

Exercise 2.1. For each of the sets E described below, describe the set of all upper bounds, the set of all lower bounds, sup E, and inf E.

(a) E is the interval $(0, 1)$.
(b) E is the interval $[0, 1)$.
(c) E is the interval $[0, 1]$.
(d) E is the interval $(0, 1]$.
(e) E consists of the numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$.
(f) E is the set containing the single point 0.

Exercise 2.2. Give a detailed definition of inf E, formulate a greatest lower bound property, and prove that it is equivalent to the least upper bound property.

If E is not bounded above, we write $\sup E = +\infty$; if E is not bounded below, we write $\inf E = -\infty$. These are convenient abbreviations, but are not to be interpreted as implying that there are real numbers $+\infty$ and $-\infty$; there are not. We can, if we like, create such infinite numbers and adjoin them to the real number system, but for most purposes it is undesirable to do so. No matter how we introduce infinite numbers, we are bound to make arithmetic worse than it already is: there is one impossible operation to begin with (division by zero), but if we make this operation possible we introduce even more impossible operations.

Exercise 2.3. Explore the consequences of introducing numbers $+\infty$ and $-\infty$ such that $a/0 = +\infty$ if $a > 0$, $a/0 = -\infty$ if $a < 0$.