Lecture Notes in Mathematics 1470

E. Odell H. Rosenthal (Eds.)

Functional Analysis

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Further remarks and relevant addresses at the back of this book.

Lecture Notes in Mathematics 1470

Longhorn Notes

Preface

This is the sixth annual proceedings of our Functional Analysis Seminar at The University of Texas, and the second one to be published in the Springer-Verlag Lecture Notes. All the articles that appear are based on talks given in the seminar during the years 1987-89. We thank the participants for their effort in communicating mathematical ideas in both spoken and written form.

We again wish to express our deep appreciation to Margaret Combs for her expert craftsmanship and considerable patience in typesetting this issue. Thanks are also due to The University of Texas for supporting the publication of the Longhorn Notes.

Ted Odell
Haskell Rosenthal
August 1990

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On certain classes of Baire-1 functions with applications to Banach space theory

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0. Introduction.

Let X be a separable infinite dimensional Banach space and let K denote its dual ball, $Ba(X^*)$, with the weak* topology. K is compact metric and X may be naturally identified with a closed subspace of $C(K)$. X^{**} may also be identified with a closed subspace of $A_\infty(K)$, the Banach space of bounded affine functions on K in the sup norm. Our general objective is to deduce information about the isomorphic structure of X or its subspaces from the topological nature of the functions $F \in X^{**} \subseteq A_\infty(K)$. A classical example of this type of result is: X is reflexive if and only if $X^{**} \subset C(K)$.

A second example is the following theorem. ($B_1(K)$ is the class of bounded Baire-1 functions on K and $DBSC(K)$ is the subclass of differences of bounded semicontinuous functions on K. The precise definitions appear below in §1.) We write $Y \hookrightarrow X$ if Y is isomorphic to a subspace of X.

Theorem A. *Let X be a separable Banach space and let $K = Ba(X^*)$ with the weak* topology.*

a) **[35]** $\ell_1 \hookrightarrow X$ iff $X^{**} \setminus B_1(K) \neq \emptyset$.

b) **[7]** $c_0 \hookrightarrow X$ iff $[X^{**} \cap DBSC(K)] \setminus C(K) \neq \emptyset$.

Theorem A provides the motivation for this paper: What can be said about X if $X^{**} \cap [B_1(K) \setminus DBSC(K)] \neq \emptyset$? To study this problem we consider various subclasses of $B_1(K)$ for an arbitrary compact metric space K. J. Bourgain has also used this approach and some of our results and techniques overlap with those of **[8,9,10]**. In a different direction, generalizations of $B_1(K)$ to spaces where K is not compact metric with ensuing applications to Banach space theory have been developed in **[22]**.

In §1 we consider two subclasses of $B_1(K)$ denoted $B_{1/4}(K)$ and $B_{1/2}(K)$ satisfying

(0.1) $$C(K) \subseteq DBSC(K) \subseteq B_{1/4}(K) \subseteq B_{1/2}(K) \subseteq B_1(K)\ .$$

Research partially supported by NSF Grant DMS-8601752.

Our interest in these classes stems from Theorem B (which we prove in §3).

Theorem B. *Let K be a compact metric space and let (f_n) be a uniformly bounded sequence in $C(K)$ which converges pointwise to $F \in B_1(K)$.*

a) *If $F \notin B_{1/2}(K)$, then (f_n) has a subsequence whose spreading model is equivalent to the unit vector basis of ℓ_1.*

b) *If $F \in B_{1/4}(K) \setminus C(K)$, there exists (g_n), a convex block subsequence of (f_n), whose spreading model is equivalent to the summing basis for c_0.*

Theorem B may be regarded as a local version of Theorem A (see Corollary 3.10). In fact the proof is really a localization of the proof of Theorem A. In Theorem 3.7 we show that the converse to a) holds and thus we obtain a characterization of $B_1(K) \setminus B_{1/2}(K)$ in terms of ℓ_1 spreading models. We do not know if the condition in b) characterizes $B_{1/4}(K)$ (see Problem 8.1).

Given that our main objective is to deduce information about the subspaces of X from the nature of $F \in X^{**} \cap B_1(K)$, it is useful to introduce the following definition.

Let $\mathcal{C}$ be a class of separable infinite-dimensional Banach spaces and let $F \in B_1(K)$. F is said to *govern* $\mathcal{C}$ if whenever $(f_n) \subseteq C(K)$ is a uniformly bounded sequence converging pointwise to F, then there exists a $Y \in \mathcal{C}$ which embeds into $[(f_n)]$, the closed linear span of (f_n). We also say that F *strictly governs* $\mathcal{C}$ if whenever $(f_n) \subseteq C(K)$ is a uniformly bounded sequence converging pointwise to F, there exists a convex block subsequence (g_n) of (f_n) and a $Y \in \mathcal{C}$ with $[(g_n)]$ isomorphic to Y.

Theorem A (b) can be more precisely formulated as: if $F \in DBSC(K) \setminus C(K)$, then F governs $\{c_0\}$. (In fact Corollary 3.5 below yields that $F \in B_1(K) \setminus C(K)$ strictly governs $\{c_0\}$ if and only if $F \in DBSC(K)$.) In §4 we prove that the same result holds if $F \in DSC(K) \setminus C(K)$. (A more general result, with a different proof, has been obtained by Elton [**13**].) We also note in §4 that there are functions that govern $\{c_0\}$ but are not in $DSC(K)$.

In §6 we give a characterization of $B_{1/4}(K)$ (Theorem 6.1) and use it to give an example of an $F \in B_{1/4}(K) \setminus C(K)$ which does not govern $\{c_0\}$. Thus Theorem B (b) is best possible.

In §7 we note that there exists a K and an $F \in B_{1/2}(K)$ which governs $\{\ell_1\}$. We also give an example of an $F \in B_{1/2}(K)$ which governs $\mathcal{C} = \{X : X \text{ is separable and } X^* \text{ is nonseparable}\}$ but does not govern $\{\ell_1\}$.

§1 contains the definitions of the classes $DBSC(K)$, $DSC(K)$, $B_{1/2}(K)$ and $B_{1/4}(K)$. At the end of §1 we briefly recall the notion of spreading model. In §2 we recall some ordinal indices which are used to study $B_1(K)$. A detailed study of such indices can be found in [**25**]. Our use of these indices and many of the results of this paper have been motivated by [**8,9,10**].

Proposition 2.3 precisely characterizes $B_{1/2}(K)$ in terms of our index.

In §5 we show that the inclusions in (0.1) are, in general, proper. We first deduce this from a Banach space perspective. Subsequently, we consider the case where K is countable. Proposition 5.3 specifies precisely how large K must be in order for each separate inclusion in (0.1) to be proper.

In §8 we summarize some problems raised throughout this paper and raise some new questions regarding $B_{1/4}(K)$.

We are hopeful that our approach will shed some light on the central problem: if X is infinite dimensional, does X contain an infinite dimensional reflexive subspace or an isomorph of c_0 or ℓ_1? A different attack has been mounted on this problem in the last few years by Ghoussoub and Maurey. The interested reader should also consult their papers (*e.g.*, [**18,19,20,21**]). Another fruitful approach has been via the theory of types ([**26**], [**24**], [**38**]). We wish to thank S. Dilworth and R. Neidinger for useful suggestions.

1. Definitions.

In this section we give the basic definitions of the Baire-1 subclasses in which we are interested. Let K be a compact metric space. $B_1(K)$ shall denote the class of bounded Baire-1 functions on K, *i.e.*, the pointwise limits of (uniformly bounded) pointwise converging sequences $(f_n) \subseteq C(K)$. $DBSC(K) = \{F : K \to \mathbb{R} \mid$ there exists $(f_n)_{n=0}^{\infty} \subseteq C(K)$ and $C < \infty$ such that $f_0 \equiv 0$, (f_n) converges pointwise to F and

$$\sum_{n=0}^{\infty} |f_{n+1}(k) - f_n(k)| \leq C \quad \text{for all} \quad k \in K\} \ . \tag{1.1}$$

If $F \in DBSC(K)$ we set $|F|_D = \inf\{C \mid$ there exists $(f_n)_{n=0}^{\infty} \subseteq C(K)$ converging pointwise to F satisfying (1.1) with $f_0 \equiv 0\}$. $DBSC(K)$ is thus precisely those F's which are the "difference of bounded semicontinuous functions on K." Indeed if (f_n) satisfies (1.1), then $F = F_1 - F_2$ where $F_1(k) = \sum_{n=0}^{\infty}(f_{n+1} - f_n)^+(k)$ and $F_2(k) = \sum_{n=0}^{\infty}(f_{n+1} - f_n)^-(k)$ are both (lower) semicontinuous. The converse is equally trivial.

It is easy to prove that $(DBSC(K), |\cdot|_D)$ is a Banach space by using the series criterion for completeness. The fact that $\|F\|_\infty \leq |F|_D$ but the two norms are in general not equivalent on $DBSC(K)$, leads naturally to the following two definitions.

$$B_{1/2}(K) = \big\{F \in B_1(K) \mid \text{ there exists a sequence } (F_n) \subseteq DBSC(K) \text{ converging uniformly to } F\big\} \text{ and}$$

$$B_{1/4}(K) = \big\{F \in B_1(K) \mid \text{ there exists } (F_n) \text{ converging uniformly to } F \text{ with } \sup_n |F_n|_D < \infty\big\} \ .$$

It can be shown that $DBSC(K)$ is a Banach algebra under pointwise multiplication, and hence $B_{1/2}(K)$ can be identified with $C(\Omega)$, where Ω is the "structure space" or "maximal ideal space" of Ω. Thus $B_{1/4}(K)$ also has a natural interpretation in the general context of commutative Banach algebras.

There is a natural norm on $B_{1/4}(K)$ given by

$$|F|_{1/4} = \inf\{C : \text{ there exists } (F_n) \text{ converging uniformly with } \sup_n |F_n|_D \leq C\}$$

Furthermore $(B_{1/4}(K), |\cdot|_{1/4})$ is a Banach space. One way to see this is to use the following elementary

Lemma 1.1. *Let (M, d_1) be a complete metric space and let d_2 be a metric on M with $d_1(x,y) \leq d_2(x,y)$ for all $x, y \in M$. If all d_2-closed balls in M are also d_1-closed, then (M, d_2) is complete.*

The hypotheses of the lemma apply to $M = \{F : |F|_{1/4} \leq 1\}$ and d_1, d_2 given, respectively, by $\|\cdot\|_\infty$ and $|\cdot|_{1/4}$.

Remark 1.2. While we shall confine our attention to $B_{1/2}$ and $B_{1/4}$, one could of course continue the game, defining

$$\begin{aligned} B_{1/8}(K) = \{F \in B_1(K) \mid & \text{ there exists } (F_n) \subseteq DBSC(K) \\ & \text{with } |F_n - F|_{1/4} \to 0\} \text{ and} \\ B_{1/16}(K) = \{F \in B_1(K) \mid & \text{ there exists } F_n \\ & \text{with } \sup_n |F_n|_D < \infty \text{ and } |F_n - F|_{1/4} \to 0\}\ . \end{aligned}$$

This could be continued obtaining

$$DBSC(K) \subseteq \cdots \subseteq B_{1/2^{2n}}(K) \subseteq B_{1/2^{2n-1}}(K) \subseteq \cdots \subseteq B_{1/2}(K)$$

with $B_{1/2^{2n}}(K)$ having a norm $|\cdot|_{1/2^{2n}}$ which, using Lemma 1.1, is easily seen to be complete.

There is another class of Baire-1 functions that shall interest us, the differences of (not necessarily bounded) semi-continuous functions on K.

$$\begin{aligned} DSC(K) = \{F : K \to \mathbb{R} \mid & \text{ there exists a uniformly bounded sequence} \\ & (f_n)_{n=0}^\infty \subseteq C(K) \text{ converging pointwise to } F \text{ with} \\ & \sum_{n=0}^\infty |f_{n+1}(k) - f(k)| < \infty \text{ for } k \in K\}\ . \end{aligned}$$

An interesting subclass of $DSC(K)$, is $PS(K)$, the pointwise limits of *pointwise stabilizing* (pointwise ultimately constant) sequences.

$$PS(K) = \{F \in B_1(K) \mid \text{ there exists a uniformly bounded sequence } (f_n) \subseteq C(K) \text{ with the property that for all } k \in K \text{ there exists } m \in \mathbb{N} \text{ such that } f_n(k) = F(k) \text{ for } n \geq m\} .$$

Remark 1.3. We discuss $PS(K)$ in Proposition 4.9. Both of these classes were considered in [**10**], and as noted there, if an indicator function $\mathbf{1}_A \in B_1(K)$, then $\mathbf{1}_A \in PS(K)$. Indeed A must be both F_σ and G_δ (cf. Proposition 2.1 below) and so we can write $A = \bigcup_n F_n = \bigcap_n G_n$ where $F_1 \subseteq F_2 \subseteq \cdots$ are closed sets and $G_1 \supseteq G_2 \supseteq \cdots$ are open sets. Then by the Tietze extension theorem, for each n choose $f_n \in Ba(C(K))$ with f_n identically 1 on F_n and identically 0 on $K \setminus G_n$. Thus for all $k \in K$, $(f_n(k))_n$ is ultimately $\mathbf{1}_A(k)$.

The *summing basis* (s_n) for (an isomorph of) c_0 is characterized by

$$\|\sum a_n s_n\| = \sup_k |\sum_{i=1}^k a_i| .$$

Let (x_n) be a seminormalized basic sequence. A basic sequence (e_n) is said to be a *spreading model* of (x_n) if for all $k \in \mathbb{N}$ and all $\varepsilon > 0$ there exist N so that if $N < n_1 < n_2 < \cdots < n_k$ and $(a_i)_1^k \subseteq \mathbb{R}$ with $\sup_i |a_i| \leq 1$, then

$$\Big| \|\sum_{i=1}^k a_i x_{n_i}\| - \|\sum_{i=1}^k a_i e_i\| \Big| < \varepsilon .$$

For further information on spreading models see [**4**].

We recall that if $(f_n) \subseteq Ba(C(K))$ converges pointwise to $F \in B_1(K) \setminus C(K)$ then there exists a $C = C(F)$ such that (f_n) has a basic subsequence (f_n') with basis constant C which C-dominates (s_n). Thus $C\|\sum a_n f_n'\| \geq \|\sum a_n s_n\|$, for all $(a_n) \subseteq \mathbb{R}$ (see *e.g.*, [**31**]). Furthermore (f_n') can be taken to have a spreading model [**4**]. The constant C depends only on $\sup\{\text{osc}(F,k) \mid k \in K\}$ (see §2 for the definition of $\text{osc}(F,k)$).

Finally we recall that a sequence (g_n) in a Banach space is a *convex block subsequence* of (f_n) if $g_n = \sum_{i=p_n+1}^{p_{n+1}} a_i f_i$ where (p_n) is an increasing sequence of integers, $(a_i) \subseteq \mathbb{R}^+$ and for each n, $\sum_{i=p_n+1}^{p_{n+1}} a_i = 1$.

2. Ordinal Indices for $B_1(K)$.

Let (K,d) be a compact metric space and let $F : K \to \mathbb{R}$ be a bounded function. The Baire characterization theorem [**3**] states that $F \in B_1(K)$ iff for all closed nonempty $L \subseteq K$, $F|_L$ has a point of continuity (relative to the compact space (L, d)). This leads naturally to an ordinal index for Baire-1 functions which we now describe.

For a closed set $L \subseteq K$ and $\ell \in L$ let the *oscillation of* $F\big|_L$ at ℓ be given by $\operatorname{osc}_L(F, \ell) = \lim_{\varepsilon \downarrow 0} \sup\{f(\ell_1) - f(\ell_2) \mid \ell_i \in L \text{ and } d(\ell_i, \ell) < \varepsilon \text{ for } i = 1, 2\}$. We define the *oscillation of* F *over* L by $\operatorname{osc}_L F = \sup\{F(\ell_1) - F(\ell_2) \mid \ell_1, \ell_2 \in L\}$.

For $\delta > 0$, let $K_0(F, \delta) = K$ and if $\alpha < \omega_1$ let

$$K_{\alpha+1}(F, \delta) = \left\{k \in K_\alpha(F, \delta) \mid \operatorname{osc}_{K_\alpha(F,\delta)}(F, k) \geq \delta\right\} .$$

For limit ordinals α, set

$$K_\alpha(F, \delta) = \bigcap_{\beta < \alpha} K_\beta(F, \delta) .$$

Note that $K_\alpha(F, \delta)$ is always closed and $K_\alpha(F, \delta) \supseteq K_\beta(F, \delta)$ if $\alpha < \beta$. The index $\beta(F, \delta)$ is given by

$$\beta(F, \delta) = \inf\{\alpha < \omega_1 \mid K_\alpha(F, \delta) = \emptyset\}$$

provided $K_\alpha(F, \delta) = \emptyset$ for some $\alpha < \omega_1$ and $\beta(F, \delta) = \omega_1$ otherwise. Since K is separable, the transfinite sequence $(K_\alpha(F, \delta))_{\alpha < \omega_1}$ must stabilize: there exists $\beta < \omega_1$ so that $K_\alpha(F, \delta) = K_\beta(F, \delta)$ for $\beta \geq \alpha$.

The Baire characterization theorem yields that $\beta(F, \delta) < \omega_1$ for all $\delta > 0$ iff $F \in B_1(K)$. In fact we have the following proposition. In its statement $\mathcal{A}$ denotes the algebra of *ambiguous* subsets of K. Thus $A \in \mathcal{A}$ iff A is both F_σ and G_δ. Also we write $[F \leq a]$ for the set $\{k \in K \mid F(k) \leq a\}$.

Proposition 2.1. *Let $F : K \to \mathbb{R}$ be a bounded function on the compact metric space K. The following are equivalent.*

1) *$F \in B_1(K)$.*
2) *$\beta(F, \delta) < \omega_1$ for all $\delta > 0$.*
3) *For a and b real, $[F \leq a]$ and $[F \geq b]$ are both G_δ subsets of K.*
4) *For U an open subset of $\mathbb{R}$, $F^{-1}(U)$ is an F_σ subset of K.*
5) *For $a < b$, $[F \leq a]$ and $[F \geq b]$ may be separated by disjoint sets in $\mathcal{A}$. Equivalently, there exists $A \in \mathcal{A}$ with $[F \leq a] \subseteq A$ and $A \cap [F \geq b] = \emptyset$.*
6) *F is the uniform limit of a sequence of $\mathcal{A}$-simple functions ($\mathcal{A}$-measurable functions with finite range).*
7) *F is the uniform limit of a sequence $(g_n) \subseteq DSC(K)$.*
8) *F is the uniform limit of a sequence $(g_n) \subseteq PS(K)$.*

The proof is standard and can be compiled from [**23**]. We are more interested in an analogous characterization of $B_{1/2}(K)$. Before stating that proposition we need a few more definitions.

$\mathcal{D}$ shall denote the algebra of all finite unions of differences of closed subsets of K. $\mathcal{D}$ is easily seen to be a subalgebra of $\mathcal{A}$.

One of the statements in our next proposition involves another ordinal index for Baire-1 functions, $\alpha(F;a,b)$, which as we shall see is closely related to our index. For $a<b$, let $K_0(F;a,b)=K$ and for any ordinal α, let

$$\begin{aligned} K_{\alpha+1}(F;a,b) = \{k \in K_\alpha(F;a,b) \mid \ & \text{for all } \ \varepsilon>0 \text{ and } i=1,2, \\ & \text{there exist } k_i \in K_\alpha(F;a,b) \ \text{ with } \ d(k_i,k)\le\varepsilon\ , \\ & F(k_1)\ge b \ \text{ and } \ F(k_2)\le a\}\ . \end{aligned}$$

Equivalently, $K_{\alpha+1} = \overline{K_\alpha \cap [F\le a]} \cap \overline{K_\alpha \cap [F\ge b]}$. At limit ordinals α we set

$$K_\alpha(F;a,b) = \bigcap_{\beta<\alpha} K_\beta(F;a,b)\ .$$

As before these sets are closed and decreasing. We let $\alpha(F;a,b)=\inf\{\gamma<\omega_1 \mid K_\gamma(F;a,b)=\emptyset\}$ if $K_\gamma(F;a,b)=\emptyset$ for some $\gamma<\omega_1$ and let $\alpha(F;a,b)=\omega_1$ otherwise.

Remark 2.2. The index $\alpha(F;a,b)$ is only very slightly different from the index $L(F,a,b)$ considered by Bourgain [**8**]. $L(F;a,b) = \inf\{\eta<\omega_1 \mid$ there exists a transfinite increasing sequence of open sets $(G_\alpha)_{\alpha\le\eta}$ with $G_0=\emptyset$, $G_\eta=K$, $G_{\alpha+1}\setminus G_\alpha$ is disjoint from either $[F\le a]$ or $[F\ge b]$ for all $\alpha<\eta$ and $G_\gamma=\bigcup_{\alpha<\gamma} G_\alpha$ if $\gamma\le\eta$ is a limit ordinal$\}$. In fact one can show that if $\alpha(F;a,b)=\eta+n$ where η is a limit ordinal and $n\in\mathbb{N}$, then $L(F,a,b)\in\{\eta+2n,\ \eta+2n-1\}$. In Proposition 2.3 we shall show that $\alpha(F;a,b)<\omega$ for all $a<b$ iff $\beta(F,\delta)<\omega$ for all $\delta>0$. We note that a more general result has subsequently been obtained in [**25**]. Indeed if we define $\beta(F)=\sup\{\beta(F;\delta)\mid\delta>0\}$ and $\alpha(F)=\sup\{\alpha(F;a,b)\mid a<b \text{ rational}\}$ then Kechris and Louveau have shown that $\beta(F)\le\omega^\xi$ iff $\alpha(F)\le\omega^\xi$.

Also we note that the following result follows from [**8**]. Let X be a separable Banach space not containing ℓ_1. Let $K=Ba(X^*)$ in its weak* topology. Then

$$\sup\{\beta(x^{**}|_K) : x^{**}\in X^{**}\} < \omega_1\ .$$

Proposition 2.3. *Let $F: K\to\mathbb{R}$ be a bounded function on the compact metric space K. The following are equivalent*

1) $F\in B_{1/2}(K)$.

2) F is the uniform limit of $\mathcal{D}$-simple functions on K.

3) For $a < b$, $[F \le a]$ and $[F \ge b]$ may be separated by disjoint sets in $\mathcal{D}$.

4) $\beta(F) \le \omega$.

5) $\alpha(F; a, b) < \omega$ for all $a < b$.

Proof.

4) $\Rightarrow$ 5). This follows from the elementary observation that for all ordinals α and reals $a < b$, $K_\alpha(F; a, b) \subseteq K_\alpha(F, b - a)$, and the fact that 4) holds if and only if $\beta(F, \delta) < \omega$ for all $\delta > 0$.

5) $\Rightarrow$ 3). Let $K_i = K_i(F; a, b)$. Thus $K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n = \emptyset$ where $n = \alpha(F; a, b)$. Let

$$D = \bigcup_{i=1}^{n} \overline{(F \le a \cap K_{i-1})} \setminus \overline{([F \ge b] \cap K_{i-1})} \in \mathcal{D} .$$

Since $K_i = \overline{([F \le a] \cap K_{i-1})} \cap \overline{([F \ge b] \cap K_{i-1})}$,

$$\begin{aligned} D &= \bigcup_{i=1}^{n} \left(\overline{[F \le a] \cap K_{i-1}} \setminus K_i \right) \\ &\supseteq \bigcup_{i=1}^{n} \Big[([F \le a] \cap K_{i-1}) \setminus K_i \Big] \\ &= \bigcup_{i=1}^{n} ([F \le a] \cap (K_{i-1} \setminus K_i)) = [F \le a] . \end{aligned}$$

Since K_{i-1} is closed,

$$\begin{aligned} D &\subseteq \bigcup_{i=1}^{n} \left(K_{i-1} \setminus \overline{[F \ge b] \cap K_{i-1}} \right) \\ &\subseteq \bigcup_{i=1}^{n} \Big[K_{i-1} \setminus ([F \ge b] \cap K_{i-1}) \Big] \\ &= \bigcup_{i=1}^{n} (K_{i-1} \setminus [F \ge b]) = K \setminus [F \ge b] . \end{aligned}$$

3) $\Rightarrow$ 2). This is a standard exercise in real analysis.

2) $\Rightarrow$ 1). Since every $\mathcal{D}$-simple function can be expressed in the form $\sum_{i=1}^{k} a_i \mathbf{1}_{L_i}$ where the L_i's are closed sets and $DBSC(K)$ is a linear space it suffices to recall that $\mathbf{1}_L \in DBSC(K)$ whenever L is closed. In fact $\mathbf{1}_L$ is upper semicontinuous.

1) $\Rightarrow$ 4). Let F be the uniform limit of $(F_n) \subseteq DBSC(K)$. For $\delta > 0$ and n sufficiently large, $\beta(F, 2\delta) \le \beta(F_n, \delta)$ and thus is suffices to prove that for $G \in DBSC(K)$, $\beta(G, \delta) < \omega$ for $\delta > 0$. This is immediate from the following

Lemma 2.4. *If $m \in \mathbb{N}$, $\delta > 0$ and $G : K \to \mathbb{R}$ is such that $K_m(G,\delta) \neq \emptyset$, then $|G|_D \geq m\delta/4$.*

Proof. Let $(g_n) \subseteq C(K)$ converge pointwise to G. It suffices to show that there exist integers $n_1 < n_2 < \cdots < n_{m+1}$ and $k \in K$ such that $|g_{n_{i+1}}(k) - g_{n_i}(k)| > \delta/4$ for $1 \leq i \leq m$.

Let $n_1 = 1$, $k_0 \in K_m(G,\delta)$ and let U_0 be a neighborhood of k_0 for which $\text{osc}_{U_0}\, g_{n_1} < \delta/8$. Choose k_0^1 and k_0^2 in $U_0 \cap K_{m-1}(G,\delta)$ with $G(k_0^1) - G(k_0^2) > 3\delta/4$. Then choose $n_2 > n_1$ such that $g_{n_2}(k_0^1) - g_{n_2}(k_0^2) > 3\delta/4$. Thus there is a nonempty neighborhood $U_1 \subset U_0$ of either k_0^1 or k_0^2 such that for $k \in U_1$, $|g_{n_2}(k) - g_{n_1}(k)| > \delta/4$.

Similarly we can find a neighborhood $U_2 \subseteq U_1$ of a point in $K_{m-1}(G,\delta)$ and $n_3 > n_2$ so that for $k \in U_2$, $|g_{n_3}(k) - g_{n_2}(k)| > \delta/4$, etc. ∎

Remarks 2.5. 1. Of course by using a bit more care one can show that $|G|_D \geq m\delta/2$ whenever $K_m(G,\delta) \neq \emptyset$.

2. Following [**25**] we say that for $F \in B_1(K)$, $F \in B_1^\xi(K)$ iff $\beta(F) \leq \omega^\xi$. Thus $B_{1/2}(K) \equiv B_1^1(K)$ by Proposition 2.3, a result also observed in [**25**].

3. We do not yet have an index characterization of $B_{1/4}(K)$, however we have a necessary condition (which may be sufficient). To describe this we first must generalize our index above. Let $F : K \to \mathbb{R}$ and let $(\delta_i)_{i=1}^\infty$ be positive numbers. Set $K_0(F,(\delta_i)) = K$ and for $0 \leq i$ set

$$K_{i+1}(F,(\delta_j)) = \left\{k \in K_i(F,(\delta_j)) \mid \text{osc}_{K_i(F,(\delta_j))}(F,k) \geq \delta_{i+1}\right\} .$$

Proposition 2.6. *Let $F \in B_{1/4}(K)$. Then there exists an $M < \infty$ so that if $K_n(F,(\delta_i)) \neq \emptyset$, then $\sum_{i=1}^n \delta_i \leq M$.*

Proof. Let F be the uniform limit of (G_n) with $|G_n|_D \leq C < \infty$ for all n. Suppose that $K_n(F,(\delta_i)) \neq \emptyset$ for some sequence $(\delta_i)_{i=1}^\infty \subseteq \mathbb{R}^+$. Since $K_n(F,(\delta_i)) \subseteq K_n(G_m,(\delta_i/2))$ for large m, the latter set is non-empty as well. The proof of Lemma 2.4 yields

$$\text{(2.1)} \qquad \begin{cases} \text{If } G : K \to \mathbb{R} \text{ and } (\delta_i)_{i=1}^\infty \subseteq \mathbb{R}^+ \text{ is such that } K_n(G,(\delta_i)) \neq \emptyset, \\ \text{then } |G|_D \geq 4^{-1} \sum\limits_{i=1}^n \delta_i. \end{cases}$$

Thus by (2.1) we have, for large m, $C \geq |G_m|_D \geq 4^{-1} \sum_{i=1}^n \delta_i$ and so $\sum_{i=1}^n \delta_i \leq 4C$. ∎

We shall explore in greater detail in §3 and §8 some questions related to the problem of an index characterization of Baire-1/4. The following proposition gives a sufficient index criterion for a function to be Baire-1/4. It also shows (via Proposition 2.3) that if $F \in B_{1/2}(K) \backslash B_{1/4}(K)$, then $\beta(F) = \omega$.

Proposition 2.7. *Let $F \in B_1(K)$. If $\beta(F) < \omega$, then $F \in B_{1/4}(K)$.*

Proof. Without loss of generality let $F : K \to [0,1]$ with $\beta(F) \leq n$. Thus $\alpha(F; a, b) \leq n$ for all $a < b$. It follows from the proof of 5) $\Rightarrow$ 3) in Proposition 2.3 that for all $0 < a < b < 1$ there exists a $D \in \mathcal{D}$ with $|\mathbf{1}_D|_D \leq 2n$, $[F \leq a] \subseteq D$ and $[F \geq b] \cap D = \emptyset$. Thus for all $m < \infty$ there exist sets $D_1 \supseteq D_2 \supseteq \cdots \supseteq D_m$ in $\mathcal{D}$ with $[F \geq i/m] \subseteq D_i$, $[F \leq (i-1)/m] \cap D_i = \emptyset$ and $|\mathbf{1}_{D_i}|_D \leq 2n$ for $i \leq m$. In particular if $G = \sum_{i=1}^m m^{-1}\mathbf{1}_{D_i}$, then $\|F - G\|_\infty \leq m^{-1}$ and $|G|_D \leq 2n$. ∎

The following proposition is related to work of A. Sersouri [**39**]. It is of interest to us because it shows that a separable Banach space X can have functions of large index in X^{**} and yet be quite nice. In fact it shows there are Baire-1 functions of arbitrarily large index which strictly govern the class of quasireflexive (order 1) Banach spaces. Our proof was motivated by discussions with A. Pełczyński.

Proposition 2.8. *For all $\gamma < \omega_1$ there exists a quasireflexive (of order 1) Banach space Q_γ such that $Q_\gamma^{**} = Q_\gamma \oplus \langle F_\gamma \rangle$ where $\beta(F_\gamma) > \gamma$.*

(The index $\beta(F_\gamma)$ is computed with respect to $Ba(Q_\gamma^*)$.)

Remark 2.9. In §6 we shall show the existence of a quasireflexive space whose new functional (in the second dual) is Baire-1/4.

Proof of Proposition 2.8. We use interpolation, namely the method of [**12**]. (This has also been used in [**19**] in a slightly different manner to produce a quasireflexive space from a weak* convergent sequence.)

To begin let $\gamma < \omega_1$ be any ordinal and choose a compact metric space K containing an ambiguous set A_γ with $\alpha(\mathbf{1}_{A_\gamma}; \frac{1}{4}, \frac{3}{4}) > \gamma$. (For example $\mathbf{1}_{A_\alpha}$ could be taken to be one of the functions F_δ described in §5 with $\delta > \omega^\gamma +$.) Choose a sequence $(f_n) \subseteq Ba(C(K))$ converging pointwise to $\mathbf{1}_{A_\gamma}$ such that $(\mathbf{1}_{A_\gamma}, f_1, f_2, \ldots)$ is basic in $C(K)^{**}$. Let W be the closed convex hull of $\{\pm f_n\}_{n=1}^\infty$ in $C(K)$. Let Q_γ be the Banach space obtained from $W \subseteq Ba(C(K))$ by [DFJP]-interpolation. Thus for all $n \in \mathbb{N}$, $\|\cdot\|_n$ is the gauge of $U_n = 2^n W + 2^{-n} Ba(C(K))$, and $Q_\gamma = \{x \in C(K) : \|\!| x |\!\| \equiv (\sum_n \|x\|_n^2)^{1/2} < \infty\}$. Following the notation of [**12**], we let $C = Ba(Q_\gamma) = \{x \in C(K) : \|\!| x |\!\| \leq 1\}$ and let $j : Q_\gamma \to C(K)$ be the natural semiembedding.

We first observe that Q_γ is quasireflexive of order 1. Indeed it is easy to check that $\widetilde{W}$, the weak* closure of W in $C(K)^{**}$ is just

$$\widetilde{W} = \left\{ \sum_{i=1}^\infty a_i f_i + a_\infty \mathbf{1}_{A_\gamma} : |a_\infty| + \sum_{i=1}^\infty |a_i| \leq 1 \right\}.$$