Philosophic Foundations of Quantum Mechanics

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PREFACE

TWO GREAT theoretical constructions have shaped the face of modern physics: the theory of relativity and the theory of quanta. The first has been, on the whole, the discovery of one man, since the work of Albert Einstein has remained unparalleled by the contributions of others who, like Hendrik Anton Lorentz, came very close to the foundations of special relativity, or, like Hermann Minkowski, determined the geometrical form of the theory. It is different with the theory of quanta. This theory has been developed by the collaboration of a number of men each of whom has contributed an essential part, and each of whom, in his work, has made use of the results of others.

The necessity of such teamwork seems to be deeply rooted in the subject matter of quantum theory. In the first place, the development of this theory has been greatly dependent on the production of observational results and on the exactness of the numerical values obtained. Without the help of the army of experimenters who photographed spectral lines or watched the behavior of elementary particles by means of ingenious devices, it would have been impossible ever to carry through the theory of the quanta even after its foundations had been laid. In the second place, these foundations are very different in logical form from those of the theory of relativity. They have never had the form of one unifying principle, not even after the theory has been completed. They consist of a set of principles which, despite their mathematical elegance, do not possess the suggestive character of a principle which convinces us at first sight, as does the principle of relativity. And, finally, they depart much further from the principles of classical physics than the theory of relativity ever did in its criticism of space and time; their implications include, in addition to a transition from causal laws to probability laws, a revision of philosophical ideas about the existence of unobserved objects, even of the principles of logic, and reach down to the deepest fundamentals of the theory of knowledge.

In the development of the theoretical form of quantum physics, we can distinguish four phases. The first phase is associated with the names of Max Planck, Albert Einstein, and Nils Bohr. Planck's introduction of the quanta in 1900 was followed by Einstein's extension of the quantum concept toward that of a needle radiation (1905). The decisive step, however, was made in Bohr's application (1913) of the quantum idea to the analysis of the structure of the atom, which led to a new world of physical discoveries.

The second phase, which began in 1925, represents the work of a younger generation which had been trained in the physics of Planck, Einstein, and Bohr, and started where the older ones had stopped. It is a most astonishing fact that this phase, which led up to quantum mechanics, began without a clear insight into what was actually being done. Louis de Broglie introduced waves as companions of particles; Erwin Schrödinger, guided by mathematical analogies with wave optics, discovered the two fundamental differential equations of quantum mechanics; Max Born, Werner Heisenberg, Pascual Jordan, and, independently of this group, Paul A. M. Dirac constructed the matrix mechanics which seemed to defy any wave interpretation. This period represents an amazing triumph of mathematical technique which, masterly applied and guided by a physical instinct more than by logical principles, determined the path to the discovery of a theory which was able to embrace all observable data. All this was done in a very short time; by 1926 the mathematical shape of the new theory had become clear.

The third phase followed immediately; it consisted in the physical interpretation of the results obtained. Schrödinger showed the identity of wave mechanics and matrix mechanics. Born recognized the probability interpretation of the waves. Heisenberg saw that the mathematical mechanism of the theory involves an unsurmountable uncertainty of predictions and a disturbance of the object by the measurement. And here once more Bohr intervened in the work of the younger generation and showed that the description of nature given by the theory was to leave open a specific ambiguity which he formulated in his principle of complementarity.

The fourth phase continues up to the present day; it is filled with constant extensions of the results obtained toward further and further applications, including the application to new experimental results. These extensions are combined with mathematical refinements; in particular, the adaptation of the mathematical method to the postulates of relativity is in the foreground of the investigations. We shall not speak of these problems here, since our inquiry is concerned with the logical foundations of the theory.

It was with the phase of the physical interpretations that the novelty of the logical form of quantum mechanics was realized. Something had been achieved in this new theory which was contrary to traditional concepts of knowledge and reality. It was not easy, however, to *say* what had happened, i.e., to proceed to the *philosophical* interpretation of the theory. Based on the physical interpretations given, a philosophy for common use was developed by the physicists which spoke of the relation of subject and object, of pictures of reality which must remain vague and unsatisfactory, of operationalism which is satisfied when observational predictions are correctly made, and renounces interpretations as unnecessary ballast. Such concepts may appear useful for the purpose of carrying on the merely technical work of the physicist. But it seems to us that the physicist, whenever he tried to be conscious of what he did, could not help feeling a little uneasy with this philosophy. He then became aware that he was walking, so to speak, on the thin ice of a superficially frozen lake, and he realized that he might slip and break through at any moment.

It was this feeling of uneasiness which led the author to attempt a philosophical analysis of the foundations of quantum mechanics. Fully aware that philosophy should not try to construct physical results, nor try to prevent physicists from finding such results, he nonetheless believed that a logical analysis of physics which did not use vague concepts and unfair excuses was possible. The philosophy of physics should be as neat and clear as physics itself; it should not take refuge in conceptions of speculative philosophy which must appear outmoded in the age of empiricism, nor use the operational form of empiricism as a way to evade problems of the logic of interpretations. Directed by this principle the author has tried in the present book to develop a philosophical interpretation of quantum physics which is free from metaphysics, and yet allows us to consider quantum mechanical results as statements about an atomic world as real as the ordinary physical world.

It scarcely will appear necessary to emphasize that this philosophical analysis is carried through in deepest admiration of the work of the physicists, and that it does not pretend to interfere with the method of physical inquiry. All that is intended in this book is clarification of concepts; nowhere in this presentation, therefore, is any contribution toward the solution of physical problems to be expected. Whereas physics consists in the analysis of the physical world, philosophy consists in the analysis of our knowledge of the physical world. The present book is meant to be philosophical in this sense.

The division of the book is so planned that the first part presents the general ideas on which quantum mechanics is based; this part, therefore, outlines our philosophical interpretation and summarizes its results. The presentation is such that it does not presuppose mathematical knowledge, nor an acquaintance with the methods of quantum physics. In the second part we present the outlines of the mathematical methods of quantum mechanics; this is so written that a knowledge of the calculus should enable the reader to understand the exposition. Since we possess today a number of excellent textbooks on quantum mechanics, such an exposition may appear unnecessary; we give it, however, in order to open a short cut toward the mathematical foundations of quantum mechanics for all those who do not have the time for thorough studies of the subject, or who would like to see in a short review the methods which they have applied in many individual problems. Our presentation, of course, makes no claim to be complete. The third part deals with the various interpretations of quantum mechanics; it is here that we make use of both the philosophical ideas of the first part and the mathematical formulations of the second. The properties of the different interpretations are discussed, and an interpretation in terms of a three-valued logic is constructed which appears as a satisfactory logical form of quantum mechanics.

I am greatly indebted to Dr. Valentin Bargmann of the Institute of Advanced Studies in Princeton for his advice in mathematical and physical questions; numerous improvements in the presentation, in Part II in particular, are due to his suggestions. I wish to thank Dr. Norman C. Dalkey of the University of California, Los Angeles, and Dr. Ernest H. Hutten, formerly at Los Angeles, now in the University of Chicago, for the opportunity of discussing

with them questions of a logical nature, and for their assistance in matters of style and terminology. Finally I wish to thank the staff of the University of California Press for the care and consideration with which they have edited my book and for their liberality in following my wishes concerning some deviations from established usage in punctuation.

A presentation of the views developed in this book, including an exposition of the system of three-valued logic introduced in § 32, was given by the author at the Unity of Science Meeting in the University of Chicago on September 5, 1941.

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CONTENTS

Part I: General Considerations

Part II: Outlines of the Mathematics of Quantum Mechanics

Part III: Interpretations

Part I

GENERAL CONSIDERATIONS

§ 1. Causal Laws and Probability Laws

The philosophical problems of quantum mechanics are centered around two main issues. The first concerns the transition from causal laws to probability laws; the second concerns the interpretation of unobserved objects. We begin with the discussion of the first issue, and shall enter into the analysis of the second in later sections.

The question of replacing causal laws by statistical laws made its appearance in the history of physics long before the times of the theory of quanta. Since the time of Boltzmann's great discovery which revealed the second principle of thermodynamics to be a statistical instead of a causal law, the opinion has been repeatedly uttered that a similar fate may meet all other physical laws. The idea of determinism, i.e., of strict causal laws governing the elementary phenomena of nature, was recognized as an extrapolation inferred from the causal regularities of the macrocosm. The validity of this extrapolation was questioned as soon as it turned out that macrocosmic regularity is equally compatible with irregularity in the microcosmic domain, since the law of great numbers will transform the probability character of the elementary phenomena into the practical certainty of statistical laws. Observations in the macrocosmic domain will never furnish any evidence for causality of atomic occurrences so long as only effects of great numbers of atomic particles are considered. This was the result of unprejudiced philosophical analysis of the physics of Boltzmann.[1]

With this result a decision of the question was postponed until it was possible to observe macrocosmic effects of individual atomic phenomena. Even with the use of observations of this kind, however, the question is not easily answered, but requires the development of a more profound logical analysis.

Whenever we speak of strictly causal laws we assume them to hold between idealized physical states; and we know that actual physical states never cor-

[1] It is scarcely possible to say who was the first to formulate this philosophical idea. We have no published utterances of Boltzmann indicating that he thought of the possibility of abandoning the principle of causality. In the decade preceding the formulation of quantum mechanics the idea was often discussed. F. Exner, in his book, *Vorlesungen über die physikalischen Grundlagen der Naturwissenschaften* (Vienna, 1919), is perhaps the first to have clearly stated the criticism which we gave above: "Let us not forget that the principle of causality and the need for causality has been suggested to us exclusively by experiences with macrocosmic phenomena and that a transference of the principle to microcosmic phenomena, viz. the assumption that every individual occurrence be strictly causally determined, has no longer any justification based on experience."—p. 691. With reference to Exner, E. Schrödinger has expressed similar ideas in his inaugural address in Zurich, 1922, published in *Naturwissenschaften*, 17: 9 (1929).

respond exactly to the conditions assumed for the laws. This discrepancy has often been disregarded as irrelevant, as being due to the imperfection of the experimenter and therefore negligible in a statement about causality as a property of nature. With such an attitude, however, the way to a solution of the problem of causality is barred. Statements about the physical world have meaning only so far as they are connected with verifiable results; and a statement about strict causality must be translatable into statements about observable relations if it is to have a utilizable meaning. Following this principle we can interpret the statement of causality in the following way.

If we characterize physical states in observational terms, i.e., in terms of observations as they are actually made, we know that we can construct probability relations between these states. For instance, if we know the inclination of the barrel of a gun, the powder charge, and the weight of the shell, we can predict the point of impact with a certain probability. Let A be the so-defined initial conditions and B a description of the point of impact; then we have a probability implication

$$A \underset{p}{\ni} B \tag{1}$$

which states that if A is given, B will happen with a determinate probability p. From this empirically verifiable relation we pass to an ideal relation by considering ideal states A' and B' and stating a logical implication

$$A' \supset B' \tag{2}$$

between them, which represents a law of mechanics. Since we know, however, that from the observational state A we can infer only with some probability the existence of the ideal state A', and that similarly we have only a probability relation between B and B', the logical implication (2) cannot be utilized. It derives its physical meaning only from the fact that in all cases of applications to observable phenomena it can be replaced by the probability implication (1). What then is the meaning of a statement saying that if we knew exactly the initial conditions we could predict with certainty the future states resulting from them? Such a statement can be meaningfully said only in the sense of a transition to a limit. Instead of characterizing the initial conditions of shooting only by the mentioned three parameters, the inclination of the barrel, the powder charge, and the weight of the shell, we can consider further parameters, such as the resistance of the air, the rotation of the earth, etc. As a consequence, the predicted value will change; but we know that with such a more precise characterization also the probability of the prediction increases. From experiences of this kind we have inferred that the probability p can be made to approach the value 1 as closely as we want by the introduction of further parameters into the analysis of physical states. It is in this form that we must state the principle of causality if it is to have physical meaning. The statement that nature is governed by strict causal laws means that we can predict the future with a determinate probability and that we can push this probability as close to certainty as we want by using a sufficiently elaborate analysis of the phenomena under consideration.

With this formulation the principle of causality is stripped of its disguise as a principle *a priori*, in which it has been presented within many a philosophical system. If causality is stated as a limit of probability implications, it is clear that this principle can be maintained only in the sense of an empirical hypothesis. There is, logically, no need for saying that the probability of predictions can be made to approach certainty by the introduction of more and more parameters. In this form the possibility of a limit of predictability was recognized even before quantum mechanics led to the assertion of such a limit.[2]

The objection has been raised that we can know only a finite number of parameters, and that therefore we must leave open the possibility of discovering, at a later time, new parameters which lead to better predictions. Although, of course, we have no means of excluding with certainty such a possibility, we must answer that there may be strong inductive evidence against such an assumption, and that such evidence will be regarded as given if continued attempts at finding new parameters have failed. Physical laws, like the law of conservation of energy, have been based on evidence derived from repeated failures of attempts to prove the contrary. If the existence of causal laws is denied, this assertion will always be grounded only in inductive evidence. The critics of the belief in causality will not commit the mistake of their adversaries, and will not try to adduce a supposed evidence *a priori* for their contentions.

The quantum mechanical criticism of causality must therefore be considered as the logical continuation of a line of development which began with the introduction of statistical laws into physics within the kinetic theory of gases, and was continued in the empiricist analysis of the concept of causality. The specific form, however, in which this criticism finally was presented through Heisenberg's principle of indeterminacy was different from the form of the criticism so far explained.

In the preceding analysis we have assumed that it is possible to measure the independent parameters of physical occurrences as exactly as we wish; or more precisely, to measure the *simultaneous values* of these parameters as exactly as we wish. The breakdown of causality then consists in the fact that these values do not strictly determine the values of dependent entities, including the values of the same parameters at later times. Our analysis therefore contains an assumption of the measurement of simultaneous values of independent parameters. It is this assumption which Heisenberg has shown to be wrong.

The laws of classical physics are throughout *temporally directed laws*, i.e., laws stating dependences of entities at different times and which thus establish causal lines extending in the direction of time. If simultaneous values of differ-

[2] Cf. the author's "Die Kausalstruktur der Welt," *Ber. d. Bayer. Akad., Math. Kl.* (Munich, 1925), p. 138; and his paper, "Die Kausalbehauptung und die Möglichkeit ihrer empirischen Nachprüfung," which was written in 1923 and published in *Erkenntnis* 3 (1932), p. 32.

ent entities are regarded as dependent on one another, this dependence is always construed as derivable from temporally directed laws. Thus the correspondence of various indicators of a physical state is reduced to the influence of the same physical cause acting on the instruments. If, for instance, barometers in different rooms of a house always show the same indication, we explain this correspondence as due to the effect of the same mass of air on the instruments, i.e., as due to the effect of a common cause. It is possible, however, to assume the existence of *cross-section laws*, i.e., laws which directly connect simultaneous values of physical entities without being reducible to the effects of common causes. It is such a cross-section law which Heisenberg has stated in his relation of indeterminacy.

This cross-section law has the form of a *limitation of measurability*. It states that the simultaneous values of the independent parameters cannot be measured as exactly as we wish. We can measure only one half of all the parameters to a desired degree of exactness; the other half then must remain inexactly known. There exists a coupling of simultaneously measurable values such that greater exactness in the determination of one half of the totality involves less exactness in the determination of the other half, and vice versa. This law does not make half of the parameters functions of the others; if one half is known, the other half remains entirely unknown unless it is measured. We know, however, that this measurement is restricted to a certain exactness.

This cross-section law leads to a specific version of the criticism of causality. If the values of the independent parameters are inexactly known, we cannot expect to be able to make strict predictions of future observations. We then can establish only statistical laws for these observations. The idea that there are causal laws "behind" these statistical laws, which determine exactly the results of future observations, is then destined to remain an unverifiable statement; its verification is excluded by a physical law, the cross-section law mentioned. According to the verifiability theory of meaning, which has been generally accepted for the interpretation of physics, the statement that there are causal laws therefore must be considered as physically meaningless. It is an empty assertion which cannot be converted into relations between observational data.

There is only one way left in which a physically meaningful statement about causality can be made. If statements of causal relations between the exact values of certain entities cannot be verified, we can try to introduce them at least in the form of *conventions* or *definitions;* that is, we may try to establish arbitrarily causal relations between the strict values. This means that we can attempt to assign definite values to the unmeasured, or not exactly measured, entities in such a way that the observed results appear as the causal consequences of the values introduced by our assumption. If this were possible, the causal relations introduced could not be used for an improvement of predictions; they could be used only after observations had been made in the sense of a causal construction *post hoc.* Even if we wish to follow such a procedure, however, we must answer the question of whether such a *causal supplementation of observable data by interpolation of unobserved values* can be consistently done. Although the interpolation is based on conventions, the answer to the latter question is not a matter of convention, but depends on the structure of the physical world. Heisenberg's principle of indeterminacy, therefore, leads to a revision of the statement of causality; if this statement is to be physically meaningful, it must be made as an assertion about a possible causal supplementation of the observational world.

With these considerations the plan of the following inquiry is made clear. We shall first explain Heisenberg's principle, showing its nature as a cross-section law, and discuss the reasons why it must be regarded as being well founded on empirical evidence. We then shall turn to the question of the interpolation of unobserved values by definitions. We shall show that the question stated above is to be answered negatively; that the relations of quantum mechanics are so constructed that they do not admit of a causal supplementation by interpolation. With these results the principle of causality is shown to be in no sense compatible with quantum physics; causal determinism holds neither in the form of a verifiable statement, nor in the form of a convention directing a possible interpolation of unobserved values between verifiable data.

§ 2. The Probability Distributions

Let us analyze more closely the structure of causal laws by means of an example taken from classical mechanics and then turn to the modification of this structure produced by the introduction of probability considerations.

In classical physics the physical state of a free mass particle which has no rotation, or whose rotation can be neglected, is determined if we know the *position* q, the *velocity* v, and the *mass* m of the particle. The values q and v, of course, must be corresponding values, i.e., they must be observed at the same time. Instead of the velocity v, the *momentum* $p = m \cdot v$ can be used. The future states of the mass particle, if it is not submitted to any forces, is then determined; the velocity, and with it, the momentum, will remain constant, and the position q can be calculated for every time t. If external forces intervene, we can also determine the future states of the particle if these forces are mathematically known.

If we consider the fact that p and q cannot be exactly determined, we must replace strict statements about p and q by probability statements. We then introduce *probability distributions*

$$d(q) \quad \text{and} \quad d(p) \tag{1}$$

which coordinate to every value q and to every value p a probability that this value will occur. The symbol $d(\ )$ is used here in the general meaning of *distri-*

bution of; the expressions $d(q)$ and $d(p)$ denote, therefore, different mathematical functions. As usual, the probability given by the function is coordinated, not to a sharp value q or p, but to a small interval dq or dp such that only the expressions

$$d(q)dq \quad \text{and} \quad d(p)dp \tag{2}$$

represent probabilities, whereas the functions (1) are probability *densities.* This can also be stated in the form that the integrals

$$\int_{q_1}^{q_2} d(q)dq \quad \text{and} \quad \int_{p_1}^{p_2} d(p)dp \tag{3}$$

represent the probabilities of finding a value of q between q_1 and q_2, or a value of p between p_1 and p_2.

Probability distributions can be determined only for sets of measurements, not for an individual measurement. When we speak of the exactness of a measurement we therefore mean, more precisely, the exactness of a type of measurement made in a certain type of physical system. In this sense we can say that every measurement ends with the determination of probability functions d. Usually d is a Gauss function, i.e., a bell-shaped curve following an exponential law (cf. figure 1); the steeper this curve, the more precise is the measurement. In classical physics we make the assumption that each of these curves can be made as steep as we want, if only we take sufficient care in the elaboration of the measurement. In quantum mechanics this assumption is discarded for the following reasons.

Whereas, in classical physics, we consider the two curves $d(q)$ and $d(p)$ as independent of each other, quantum mechanics introduces the rule that they are not. This is the cross-section law mentioned in § 1. The idea is expressed through a mathematical principle which determines both curves $d(q)$ and $d(p)$, at a given time t, as derivable from a mathematical function $\psi(q)$; the derivation is so given that a certain logical connection between the shapes of the curves $d(q)$ and $d(p)$ follows. This contraction of the two probability distributions into one function ψ is one of the basic principles of quantum mechanics. It turns out that the connection between the distributions established by the principle has such a structure that if one of the curves is very steep, the other must be rather flat. Physically speaking, this means that measurements of p and q cannot be made independently and that an arrangement which permits a precise determination of q must make any determination of p unprecise, and vice versa.

The function $\psi(q)$ has the character of a wave; it is even a complex wave, i.e., a wave determined by complex numbers ψ. Historically speaking, the introduction of this wave by L. de Broglie and Schrödinger goes back to the struggle between the wave interpretation and the corpuscle interpretation in the theory of light. The ψ-function is the last offspring of generations of wave concepts stemming from Huygens's wave theory of light; but Huygens would scarcely recognize his ideas in the form which they have assumed today in Born's probability interpretation of the ψ-function. Let us put aside for the present the discussion of the physical nature of this wave; we shall be concerned with this important question in later sections of our inquiry. In the present section we shall consider the ψ-waves merely as a mathematical instrument used to determine probability distributions; i.e., we shall restrict our presentation to show the way in which the probability distributions $d(q)$ and $d(p)$ can be derived from $\psi(q)$.

The derivation which we are going to explain coordinates to a curve $\psi(q)$ at a given time the curves $d(q)$ and $d(p)$; this is the reason that t does not enter into the following equations. If, at a later time, $\psi(q)$ should have a different shape, different functions $d(q)$ and $d(p)$ would ensue. Thus, in general, we have functions $\psi(q,t)$, $d(q,t)$, and $d(p,t)$. We omit the t for the sake of convenience.

The derivation will be formulated in two rules, the first determining $d(q)$, and the second determining $d(p)$. We shall state these rules here only for the simple case of free particles. The extension to more complicated mechanical systems will be given later (§ 17). We present first the rule for the determination of $d(q)$.

Rule of the squared ψ-function: The probability of observing a value q is determined by the square of the ψ-function according to the relation

$$d(q) = |\psi(q)|^2 \tag{4}$$

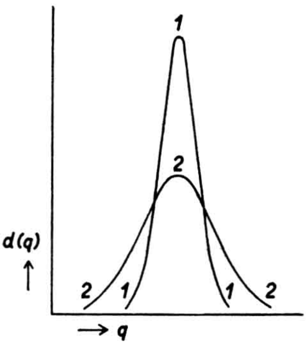

Fig. 1. Curve 1–1–1 represents a precise measurement, curve 2–2–2 a less precise measurement of q. Both curves are Gauss distributions, or *normal curves*.

The explanation of the rule for the determination of $d(p)$ requires some introductory mathematical remarks. According to Fourier a wave of any shape can be considered as the superposition of many individual waves having the form of sine curves. This is well known from sound waves, where the individual waves are called *fundamental tone* and *overtones*, or *harmonics*. In optics the individual waves are called *monochromatic waves*, and their totality is called the *spectrum*. The individual wave is characterized by its frequency ν, or its wave length λ, these two characteristics being connected by the relation $\nu \cdot \lambda = w$, where w is the velocity of the waves. In addition, every individual wave has an amplitude σ which does not depend on q, but is a constant for the whole individual wave. The general mathematical form of the Fourier expansion is explained in § 9; for the purposes of the present part it is not necessary to introduce the mathematical way of writing.

The Fourier superposition can be applied to the wave ψ, although this wave is considered by us, at present, not as a physical entity, but merely as a mathematical instrument. In case the wave ψ consists of periodic oscillations extended over a certain time, such as in the case of sound waves produced by musical instruments, the spectrum furnished by the Fourier expansion is *discrete*. Thus the individual waves of musical instruments have the wave lengths $\lambda, \frac{\lambda}{2}, \frac{\lambda}{3}, \frac{\lambda}{4}, \cdots$ where λ is the wave length of the fundamental tone and the other values represent the harmonics. In case the wave ψ consists of only one simple impact moving along the q-axis, i.e., in case the function ψ is not periodic, the Fourier expansion furnishes a continuous spectrum, i.e., the frequencies of the individual waves constitute, not a discrete, but a continuous set. As before, each of these individual waves possesses an amplitude σ, which can be written $\sigma(\lambda)$, since it depends on the wave length λ but is independent of q.

It is the amplitudes $\sigma(\lambda)$ which are connected with the momentum. We shall not try to explain here the trend of thought which led to this connection and which is associated with the names of Planck, Einstein, and L. de Broglie. Such an exposition may be postponed to a later section (§ 13). Let us suppress therefore any question of *why* this connection holds true, and let us rely, instead, upon the authority of the physicist who says that this is the case. Suffice it to say, therefore, that every wave of the length λ is coordinated to a momentum of the amount

$$p = \frac{h}{\lambda} \tag{5}$$

where h is Planck's constant. The probability of finding a momentum p then is connected with the amplitude σ belonging to the coordinated wave λ. This is expressed in the following rule.[1]

Rule of spectral decomposition: The probability of observing a value p is determined by the square of the amplitude $\sigma(\lambda)$ occurring within the spectral decomposition of $\psi(q)$, in the form

$$d(p) = \frac{1}{h^3} |\sigma(\lambda)|^2 \tag{6}$$

The factor $\frac{1}{h^3}$ results from the relation between p and λ expressed in (5).[2]

The two rules show clearly the connection which the ψ-function establishes between the two distributions $d(q)$ and $d(p)$, so far as it reduces these two distributions to one root. We shall later show that this kind of connection is not

[1] The name "principle of spectral decomposition" has been introduced by L. de Broglie, *Introduction à l'Etude de la Mécanique ondulatoire* (Paris, 1930), p. 151. In his later book, *La Mécanique ondulatoire* (Paris, 1939), p. 47, he uses also the name "principle of Born," since this principle was introduced by Born. For the rule of the squared ψ-function he uses the name "principle of interference" and in his later book the name "principle of localization".

[2] Mathematically speaking, this factor corresponds to a density function r as introduced in (22), § 9. The third power in h originates from the fact that we assume the waves to be three-dimensional.

restricted to the simple case of one mass particle, and that the same logical pattern is established by quantum mechanics for the analysis of all physical situations. For every physical situation there exists a ψ-function, and the probability distributions of the entities involved are determined by two rules of the kind described. This is one of the basic principles of quantum mechanics. We shall now construct the implications of this principle, returning once more to the simple case of the mass particle.

§ 3. The Principle of Indeterminacy

It can be shown that the derivation of the two distributions $d(q)$ and $d(p)$ from a function ψ leads immediately to the principle of indeterminacy. Let us consider a particle moving in a straight line, and let us assume that the func-

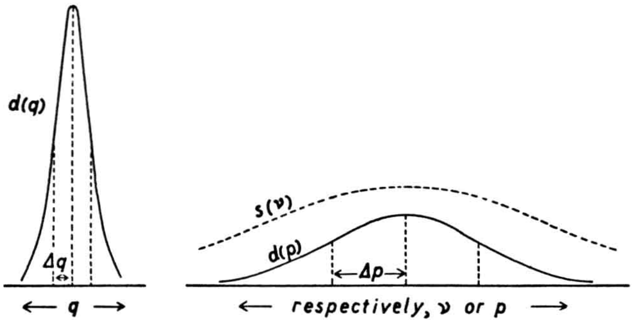

Fig. 2 Fig. 3

Fig. 2. Distribution of the position q, in the form of a Gauss curve.

Fig. 3. The dotted line indicates the direct Fourier expansion of the curve of fig. 2. The solid line is constructed through the Fourier expansion of a ψ-function from which the curve $d(q)$ of fig. 2 is derivable, and represents the distribution $d(p)$ of the momentum, coordinated to $d(q)$.

tion ψ is practically equal to zero except for a certain interval along the line. The function $|\psi(q)|^2$, i.e., the function $d(q)$, then will have the same property; let us assume that it is a Gauss curve such as is shown in figure 2. The shape of the curve means that we do not know the location of the particle exactly; with practical certainty it is within the interval where the curve is noticeably different from zero, but for a given place within this interval we know only with a determinate probability that the particle is there. Our diagram, of course, represents the situation only for a given time t; for a later time, when the particle has moved to the right, we shall have a similar curve, but it will be shifted to the right.[1]

[1] The curve will also gradually change its form. This, however, is irrelevant for the present discussion.