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Differential Systems Involving Impulses

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PREFACE

When a system described by an ordinary differential equation is subjected to perturbations, the perturbed system is again an ordinary differential equation in which the perturbation function is assumed to be continuous or integrable, and as such, the state of the system changes continuously with respect to time. However, in many physical problems (optimal control theory in particular), one can not expect perturbations to be well behaved. Biological systems such as heart beats, blood flows, pulse frequency modulated systems and models for biological neural nets exhibit an <u>impulsive</u> behaviour. Therefore, perturbations of impulsive type are more realistic. This gives rise to Measure Differential Equations. The derivative involved in these equations is the distributional derivative. The fact that their solutions are discontinuous (they are functions of bounded variation), renders most of the classical methods ineffective, thereby making their study interesting.

The systems involving impulsive behaviour are in abundance. We mention below some problems of this kind.

(i) <u>Growth Problem</u> : A fish breeding pond maintained scientifically is an example of this kind. Here the natural growth of fish population is disturbed by making catches at certain time intervals and by adding fresh breed. The natural growth of fish population is disturbed at some time intervals. This problem therefore involves impulses. We study such a model in some details in Chapter 1.

(ii) <u>Case and Blaquiere Problem</u> [2, 4] : The profit of a roadside inn on some prescribed interval of time $\tau \leq t \leq T$ is a function of the number of strangers who pass by on the road each day and of the number of times the inn is repainted during that period. The ability to attract new customers into the inn depends on its appearance which is supposed to be indexed by a number x_1. During time intervals between paint jobs, x_1 decays according to the law

$$x_1' = - k\, x_1 \;, \quad k = \text{positive constant.}$$

The total profit in the planning period $\tau \leq t \leq T$ is supposed to be

$$W(T) = A \int_{\tau}^{T} x_1(t)dt - \sum_{\alpha=1}^{N(T)} C_\alpha$$

where $N(T)$ is the number of times the inn is repainted, C_α, $\alpha = 1,\ldots, N(T)$, the cost of each paint job and $A > 0$ is a constant. The owner of the inn wishes to maximize his total profit or equivalently to minimize $- W(T)$. The problem has been treated in several details in [2,4].

(iii) Control Problem : In an optimal control problem given by a system

$$x' = f(t, x, u)$$

representing certain physical process, the central problem is to select the function $u(t)$ from a given set of controls so that the solution $x(t)$ of the system has a preassigned behaviour on a given time interval $[t_o,T]$ so as to minimize some cost functional. Suppose that the control function $u(t)$ has to be selected from the set of functions of bounded variation defined on $[t_o,T]$, then the solution $x(t)$ of the control system may possess discontinuities. Hence the given control problem has to be represented by a differential equation involving impulses.

(iv) Ito's Equation [15,50] : The dynamic system of control theory is representable by ordinary vector differential equation of the form

$$\frac{dx}{dt} = f(t, x), \quad t_o \leq t \leq T$$

where $x \in R^n$ and f is such that the system admits a unique solution $x(t, x_o)$, $t_o \leq t \leq T$ for the initial state $x(t_o)= x_o \in R^n$.

The Ito's stochastic differential equation is of the form

$$dx = f(t, x)dt + G(t, x)\, dw(t), \; t_o \leq t \leq T$$

where f is chosen as above, $G(t, x)$ is an $n \times m$ matrix valued function of (t, x), w is a separable Wiener process (Brownian motion) in Euclidean m-space. This equation is equivalent to a stochastic integral equation

$$x(t) = x(t_o) + \int_{t_o}^{t} f(s, x(s))ds + \int_{t_o}^{t} G(s,x(s))dw(s)$$

defined on $I = [t_o,T]$, $(T < \infty)$. Assume that f and G are measurable in (t,x) for $t \varepsilon I$ and $x \varepsilon R^n$ and satisfy certain growth conditions. It is to be noted that

$$\int_{t_o}^{t} G(s, x(s))dw(s) = \lim_{\gamma \to \infty} \sum_{j=o}^{\gamma - 1} G(t_j, x(t_j))[w(t_{j+1})-w(t_j)]$$

where $t_o \leq t_1 \leq \cdots \leq t_\gamma = t$ and that t_j become dense in $[t_o,t]$ as $\gamma \to \infty$.

Ito's equation has been studied in several details in [15,50]. The stochastic integral equation is closely related to the measure integral equation studied in Chapter 2 subsequently.

In the classical analysis of the differential systems, solutions are generally continuous functions while in the case of impulsive systems, solutions are functions of bounded variations. Hence the methods of classical analysis are not sufficient to describe the impulsive behaviour of systems. It is therefore necessary to study existence, uniqueness and continuation of solution, stability, boundedness criteria in the case of linear and nonlinear differential systems involving impulses.

In this monograph an attempt is made to unify the results from several research papers published during the last fifteen years. We feel that a monograph of this kind would assist the readers to reach to the mainstream of this area of research for making new contributions to the subject and employing the techniques in physical models.

The monograph is divided into five chapters and deals with the problems of existence, uniqueness, stability, boundedness and asymptotic equivalence associated with measure differential equations. Chapter 1 is on preliminaries. Chapter 2 contains results on existence and uniqueness. A control problem is also considered. In Chapter 3, fixed-point theorems and generalized integral inequalities are employed to derive results on stability and asymptotic

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equivalence. To study the predominant effect of the impulses, it is necessary to consider systems wherein impulses occur not only in perturbation terms but are also present in the original system itself. Chapter 4 provides this study. Chapter 5 deals with the extension of Lyapunov's second method to the study of impulsive systems.

The notes given at the end of each chapter indicate the sources which have been consulted. Some research papers which are closely related but not included are given for guidance and completeness. The q.e.d. mark is used to indicate the end of the proof of a result. Complete bibliography is given at the end.

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C O N T E N T S

CHAPTER 1

PRELIMINARIES

The purpose of this chapter is to make the reader familiar with the prerequisites needed for the study of measure differential equations.

1.1 The Space BV(J).

Let $J = [t_o, \infty)$, $t_o \geq 0$ and R^n denote the Euclidean n-space. The norm of $x = (x_1, x_2, \ldots, x_n) \in R^n$ is defined as

$$|x| = \sum_{i=1}^{n} |x_i| ,$$

whereas that of an n by n matrix $M = (m_{ij})$ as

$$|M| = \sum_{i=1}^{n} \sum_{j=1}^{n} |m_{ij}| .$$

Let f be a function defined on the set of real numbers and taking values in R^n. Consider all possible partitions $\pi : a = t_o < t_1 < \ldots < t_N = b$ of an interval $[a,b]$ in R. The quantity

$$V(f,[a,b]) = \sup_{\pi} \{ \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| \}$$

where π runs over the set of all partitions of $[a,b]$, is defined as the total variation of f on $[a,b]$. f is said to be of bounded variation on $[a, \infty)$ if f has bounded variation on each interval $[a,t]$, $a \leq t < \infty$ and the set $V(f,[a,t])$ of total variations is bounded. In this case

$$V(f,[a,\infty)) = \sup_{t \geq a} V(f,[a,t]).$$

The space of all functions of bounded variation on J and taking values in R^n is denoted by $BV(J) = BV(J,R^n)$. The norm of $f \in BV(J)$ is defined by

$$\| f \|^* = V(f,J) + |f(t_o+)|.$$

Under this norm, $BV(J)$ is a Banach space.

In the space $BC(J)$ of bounded, continuous functions on J, compact subsets are characterized by the well-known Ascoli-Arzela theorem. A similar, but rather weak result in $BV(J)$ is the following :

Theorem 1.1. Let $\mathcal{F}$ be an infinite family of functions $\in BV([a,b])$ such that all functions of the family together with their total variations are uniformly bounded. Then there exists a sequence $\{\phi_k\}$ in $\mathcal{F}$ which converges at every point of $[a,b]$ to some function $\phi \in BV([a,b])$; moreover,

$$V(\phi,[a,b]) \leq \liminf_{k \to \infty} V(\phi_k,[a,b]).$$

An important property of a function of bounded variation is that it is differentiable almost everywhere. Moreover, the set of its points of discontinuity is at most countable. Quite often, we shall be interested in the following type of functions of bounded variation :

Definition 1.1. A function $u : J \to R$ is said to be of type $\mathcal{S}$ if

(i) u is a right-continuous function which is of bounded variation on every compact subinterval of J;

(ii) the discontinuities $t_1 < t_2 < \ldots$ are isolated and are such that $t_1 > t_o$ and $t_k \to \infty$ as $k \to \infty$;

(iii) u is differentiable on each subinterval $[t_k, t_{k+1})$, $k = 1,2,\ldots$ of J, where the derivative at t_k is to be understood as the right hand derivative.

One of the useful properties of the function of bounded variation is that if $[a,b]$ is a closed interval in R and f be a function of finite variation on $[a,b]$ then $f = g + h$ where g is absolutely continuous on $[a,b]$ and $h' = 0$ a. e on $[a,b]$. Further this decomposition of f is unique except for additive constants.

1.2 Complex Measures.

Let Σ be a σ - algebra of subsets of a set X. A countable collection $\{E_i\}$ in Σ such that

$$E = \bigcup_{i=1}^{\infty} E_i \ , \quad E_i \cap E_j = \phi$$

for $i \neq j$, is called a partition of E. A complex-valued set function μ on Σ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i), \qquad E \in \Sigma$$

for every partition $\{E_i\}$ of E is called a complex measure on Σ. Real measures (usually called as signed measures) are defined similarly.

The total variation measure $|\mu|$ of a complex measure μ on Σ is a set function on Σ defined by

$$|\mu|(E) = \sup \{ \sum_{i=1}^{\infty} |\mu(E_i)| \ , E \in \Sigma \}$$

where the supremum runs over all partitions $\{E_i\}$ of E. It follows that $|\mu|$ is a positive measure (usually referred to as measure) on Σ, with the property that $|\mu|(X) < \infty$. If μ is a positive measure, then of course $|\mu| = \mu$.

Let μ be a real measure on Σ. Define

$$\mu^+ = (|\mu| + \mu)/2, \quad \mu^- = (|\mu| - \mu)/2.$$

μ^+ and μ^- are respectively called positive and negative variation measures of μ, and are positive measures on Σ. Further,

$$\mu = \mu^+ - \mu^- , \ |\mu| = \mu^+ + \mu^-.$$

The integral of a function f with respect to μ is defined as

$$\int_E f \, d\mu = \int_E f d\mu^+ - \int_E f d\mu^-.$$

f is said to be μ-integrable if it is both μ^+ - and μ^- - integrable.

Let μ be a complex measure on Σ. Then there is a measurable function g such that $|g(x)| = 1$ for all $x \in X$ and

$$\mu(E) = \int_E g d|\mu|, \quad E \in \Sigma.$$

Therefore, integration with respect to a complex measure μ may be defined by the formula

$$\int_E f d\mu = \int_E f g d|\mu|, \quad E \in \Sigma.$$

Definition 1.2. Let λ be a positive measure and μ any arbitrary (positive or complex) measure on a σ-algebra Σ. μ is said to be absolutely continuous with respect to λ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\mu(E)| < \varepsilon$ for all $E \in \Sigma$ with $\lambda(E) < \delta$.

If X is a topological space, there exists a smallest σ-algebra $\mathcal{B}$ in X such that every open set in X belongs to $\mathcal{B}$. Members of $\mathcal{B}$ are called Borel sets of X. A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a Borel measure on X.

Definition 1.3. A complex Borel measure μ defined on the σ-algebra $\mathcal{B}$ of all Borel sets in a locally compact Hausdorff space X is said to be regular if for each $E \in \mathcal{B}$ and $\varepsilon > 0$, there exist a set $F \in \mathcal{B}$ whose closure is contained in E and a set $G \in \mathcal{B}$ whose interior contains E such that $|\mu(C)| < \varepsilon$ for every $C \in \mathcal{B}$ with $C \subset G - F$.

Let f be a right continuous function on an open interval $I = (a,b)$. Extend the domain of f to $[a,b]$ by defining $f(a)=f(b)= 0$. The set function μ defined by $\mu([a,d]) = f(d) - f(a)$ and $\mu((c,d]) = f(d) - f(c)$, for $a < c < d \leq b$ has a regular, countably additive extension to the σ-algebra of all Borel sets in $[a,b]$. The restriction of this extension to the σ-algebra of Borel subsets of I is called the Borel - Stieltjes measure in I determined by f. Now let $\mathcal{B}^*$ consist of all sets of the form $E \cup N$ where $E \in \mathcal{B}$ and N is a subset of a set $M \in \mathcal{B}$ with $|\mu|(M) = 0$. Then $\mathcal{B}^*$ is a σ-algebra and if the domain of μ is extended to $\mathcal{B}^*$ by defining $\mu(E \cup N) = \mu(E)$, the extended function is countably additive on $\mathcal{B}^*$.

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The function μ with domain $\mathcal{B}^*$ is called the Lebesgue-Stieltjes measure determined by f, and the integral $\int_I g\, d\mu$ is written as

$$\int_a^b g(t)df(t).$$

Since $|\mu|(I') = V(f,I')$ if I' is any interval in I, the integral $\int_I g\, d|\mu|$ is written as

$$\int_a^b g(t)dVf(t),$$

where Vf is the total variation function of f. If $f(t) = t$, μ is the Lebesgue measure and the integral $\int_I g\, d\mu$ is written as

$$\int_a^b g(t)dt.$$

When the set E is regarded as a variable, $\int_E g\, d\mu$ is called the indefinite integral of g with respect to μ. It is of bounded variation and absolutely continuous with respect to μ.

<u>Lemma 1.1</u> Let (X, Σ, μ) be a measure space and f a complex-valued μ-integrable function and

$$\lambda(E) = \int_E f\, d\mu \quad , \quad E \in \Sigma.$$

Then a function g on X to a Banach space Y is λ-integrable if and only if fg is μ-integrable and in this case the relation

$$\int_E g\, d\lambda = \int_E f g\, d\mu \quad , \quad E \in \Sigma$$

holds.

1.3 Distribution and Distributional Derivative.

Let X be a topological space. The support of a complex function f on X is the closure of the set

$$\{ x \in X ; f(x) \neq 0 \}.$$

For an open subset Ω of R^n, we denote by $C_c^\infty(\Omega)$, the collection of all infinitely partially differentiable functions defined on Ω and having a compact support. A classical example of a function

$\in C_c^\infty(R^n)$ is

$$\phi(x) = \begin{cases} \exp[(\|x\|^2 - a^2)^{-1}], & \text{for } \|x\| = (\sum_{i=1}^{n} |x_i|^2)^{1/2} < a,\ a > 0 \\ 0 & , \text{for } \|x\| \geq a. \end{cases}$$

$C_c^\infty(\Omega)$ is a normed linear space under addition, scalar multiplication and norm defined by

$$(\phi_1 + \phi_2)(x) = \phi_1(x) + \phi_2(x)$$

$$(\alpha\,\phi)(x) = \alpha\,\phi(x)$$

$$\|\phi\| = \sup_{x \in \Omega} |\phi(x)|.$$

A continuous linear functional on $C_c^\infty(\Omega)$ is called a distribution on Ω. The space of distributions on Ω, being the dual space of $C_c^\infty(\Omega)$, is denoted by $C_c^\infty(\Omega)'$.

If μ is a complex Borel measure on Ω, then

$$T_\mu(\phi) = \int_\Omega \phi\, d\mu \quad , \quad \phi \in C_c^\infty(\Omega)$$

defines a distribution on Ω. In fact, by Riesz Representation Theorem, the set of all complex Borel measures on Ω is, by $\mu \longleftrightarrow T$, in one - one correspondence with the set of all distributions on Ω.

Let a complex function f, defined almost everywhere on Ω, be locally integrable on Ω with respect to the Lebesgue measure, in the sense that $\int_K |f(x)|dx < \infty$ for each compact subset K of Ω. Then

$$T_f(\phi) = \int_\Omega f(x)\,\phi(x)dx, \quad \phi \in C_c^\infty(\Omega) \tag{1.1}$$

defines a distribution on Ω. Two distributions T_f and T_g are equal as functionals (that is $T_f(\phi) = T_g(\phi)$ for every $\phi \in C_c^\infty(\Omega)$) if and only if $f(x) = g(x)$ a.e. on Ω. Hence, the set of all locally integrable functions on Ω is, by $f \longleftrightarrow T_f$, in one-one correspondence with a subset of $C_c^\infty(\Omega)'$ in such a way that (f and g being considered equivalent if and only if $f(x) = g(x)$ a.e.),

$$T_f + T_g = T_{f+g} \; , \; \alpha T_f = T_{\alpha f}.$$

The derivative of a distribution T with respect to x_i, denoted by $D_i T$ (or $\partial T / \partial x_i$), is defined by

$$D_i T = - T(\frac{\partial \phi}{\partial x_i}), \quad \phi \in C_c^{\infty}(\Omega)$$

and is also a distribution on Ω . A distribution is infinitely differentiable in the sense of above definition.

Since a locally integrable function f on an open interval I of the real line can be identified with the distribution T_f on I, $D T_f$ ($\equiv d T_f / dt$) is denoted by $D T$ and is called the distributional derivative of f to distinguish it from the ordinary derivative f'($\equiv df / dt$). If f is absolutely continuous, then Df is the ordinary derivative f' (which is defined a.e.), f' being identified with the distribution $T_{f'}$. If f is of bounded variation, then Df is the Lebesgue-Stieltjes measure df, df being identified with the distribution T_{df}. Thus, for the Heaviside function $H(t)$ defined by

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases} ,$$

we have

$$D H \equiv D T_H \equiv T_{\delta} \equiv \delta ,$$

where δ is the Dirac measure. Indeed, for any $\phi \in C_c^{\infty}(R)$, we have

$$D T_H(\phi) = - T_H(\frac{d\phi}{dt}) = - \int_{-\infty}^{\infty} H(t) \phi'(t)dt$$

$$= - \int_{0}^{\infty} \phi'(t)dt = \phi(0) ,$$

since ϕ has compact support. Note that the ordinary derivative of H is the zero function a.e. on R.

Now consider the measure differential equation

$$Dx = F(t,x) + G(t,x)Du \tag{1.2}$$

where F and G are defined on $J \times R^n$ with values in R^n and u

is a right-continuous function $\in BV(J,R)$. Let S be an open connected set in R^n and I an interval with left end point $T_o \geq t_o$.

<u>Definition 1.4.</u> A function $x(.) = x(.; T_o, x_o)$ is said to be a solution of (1.2) through (T_o, x_o) on I if $x(.)$ is a right-continuous function $\in BV(I,S)$, $x(T_o) = x_o$ and the distributional derivative of $x(.)$ on (T_o, τ) for any arbitrary $\tau \in I$ satisfies (1.2).

For example, the solution of

$$Dx = 1 + DH , \quad x(o) = 0$$

where H is the Heaviside function, is

$$x(t) = \begin{cases} t - 1 & \text{if } t < 0 \\ t & \text{if } t \geq 0. \end{cases}$$

1.4 Growth Problem

Growth of bacteria in medicine, decay of radioactive elements in physics, increase in population and pollution etc. are some of the problems studied by employing differential equations of evolution type.

Let $x(t)$ denote the quantity of an entity under consideration at time t. Assuming that the rate of growth of $x(t)$ at any time t is proportional to $x(t)$, we arrive at a simple differential equation

$$(1) \qquad \frac{dx(t)}{dt} = \alpha\, x(t), \quad x(t_o) = x_o , \qquad (\alpha \text{ is real})$$

where α is the constant of proportionality. The growth or decay of the entity would depend on the sign of α. The unique solution of this problem is then given by

$$(2) \qquad x(t) = e^{\alpha(t - t_o)} x_o , \qquad t \geq t_o .$$

Let us study a somewhat new situation where this model needs to be modified.

Consider a fish breeding pond where in fish are grown scientifically. Some variety of fish breed is released in a pond and allowed to grow. After some fixed time intervals $t_1, t_2 \ldots$ partially grown fish are removed from the pond and simultaneously new breed of fish is released in it. The growth of fish population is impulsive. The impulses are given at times $t_1, t_2 \ldots$. This problem can be solved satisfactorily if the model (1) is replaced by

$$(3) \qquad Dx(t) = \alpha\, x(t)\, Du\ , \quad x(t_o) = x_o\ ;$$

where D is the distributional derivative, u is a right continuous function of bounded variation.

Assume that u is of the form $u(t) = t + \sum_{k=1}^{\infty} a_k H_k(t)$

$$\text{where} \quad H_k(t) = \begin{cases} 0 & \text{if} \quad t < t_k \\ 1 & \text{if} \quad t \geq t_k \end{cases}$$

where a_k are real numbers. Generally a right continuous function of bounded variation contains an absolutely continuous part and a singular part. Note that the discontinuities of u are isolated. Further

$$Du = 1 + \sum_{k=1}^{\infty} a_k H_k\ (t_k)$$

where $H_k(t_k)$ is the Dirac measure concentrated at t_k. It has been shown subsequently that the unique solution of this equation is given by

$$(4) \qquad x(t) = \frac{e^{\alpha(t - t_o)}}{\prod_{i=1}^{k-1} (1 - a_i)}\, x_o\ , \quad a_i \neq 1, \quad t_{k-1} \leq t < t_k\ .$$

Clearly, if $a_i = 0$ for $i = 1,2,\ldots$ in (4) then the equation (3) reduces to (1) and the solution (4) reduces to (2).

By considering several varieties of fish growing in one pond, the model of growth given in (3) can be generalized to a system of equations.

Before we proceed further, we wish to bring to the notice of readers the following important point : The terms distribution and distributional derivative used in the monograph are in a special sense and not in the general sense, as for example widely used in the literature on partial differential equations. Here, by a distribution, we shall always mean a continuous linear functional generated by a locally integrable function, given by formula (1.1). With this understanding, it follows that the measure differential equation (1.2) is identified with the ordinary differential equation $x' = F(t,x) + G(t,x)\, u'$ when u is an absolutely continuous function.

1.5 Notes.

Theorem 1.1 is known as Helly's Selection Principle, its proof is given in Graves [11, Chap. XII, Theorem 33]. Lemma 1.1 is taken from Dunford and Schwartz [8, Cor.6, p.180]. The approach to the Theory of distributions used in this monograph is due to Halperin [13], see also Yosida [51] for more details. For an alternative approach to the study of differential systems with discontinuous solutions see Stallard [46], Halany and Wexler [12], Fleishman and Mahar [9], Ligeza[24,25], Stuart [48] and Mottoni and Texi [27].

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