Lecture Notes in Mathematics 1606

Pei-Dong Liu Min Qian

Smooth Ergodic Theory of Random Dynamical Systems

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Introduction

This book aims to present a systematic treatment of a series of results concerning invariant measures, entropy and Lyapunov exponents of smooth random dynamical systems. We first try to give a short account about this subject and the brief history leading to it.

Smooth ergodic theory of deterministic dynamical systems, i.e. the qualitative study of iterates of a single differentiable transformation on a smooth manifold is nowadays a well-developed theory. Among the major concepts of this theory are the notions of invariant measures, entropy and Lyapunov (characteristic) exponents which culminated in a theorem well known under the name of Oseledec, and there have been numerous relevant results interesting in theory itself as well as in applications. One of the most important classes of the results is Pesin's work on ergodic theory of differentiable dynamical systems possessing a smooth invariant measure. Another is related to the ergodic theory of Anosov diffeomorphisms or Axiom A attractors developed mainly by Sinai, Bowen and Ruelle. A brief review of these two classes of works is now given in the next two paragraphs.

In his paper $[\text{Pes}]_1$, Pesin proved some general theorems concerning the existence and absolute continuity of invariant families of stable and unstable manifolds of a smooth dynamical system, corresponding to the non-zero Lyapunov exponents. This set up the machinary for transferring the linear theory of Lyapunov exponents into non-linear results in neighbourhoods of typical trajectories. Using these tools Pesin then derived a series of deep results in ergodic theory of diffeomorphisms preserving a smooth measure ($[\text{Pes}]_2$). Among these results is the remarkable Pesin's entropy formula which expresses the entropy of a smooth dynamical system in terms of its Lyapunov exponents. Part of the work above has been extended and applied to dynamical systems preserving only a Borel measure ([Kat], [Fat] and $[\text{Rue}]_2$).

We now turn to some results related to the ergodic theory of Axiom A attractors. Recall that for a given Axiom A attractor there exists a unique invariant measure, called Sinai-Bowen-Ruelle (or simply SBR) measure, that is characterized by each of the following properties ([Sin], $[\text{Bow}]_2$ and $[\text{Rue}]_3$):

(1) Pesin's entropy formula holds true for the associated system.

(2) Its conditional measures on unstable manifolds are absolutely continuous with respect to Lebesgue measures on these manifolds.

(3) Lebesgue almost every point in an open neighbourhood of the attractor is generic with respect to this measure.

Each one of these properties has been shown to be significant in its own right, but it is also remarkable that they are equivalent to one another. More crucially, Ledrappier and Young proved later in their well-known paper $[\text{Led}]_2$ that the properties (1) and (2) above remain equivalent for all C^2 diffeomorphisms (That (2) implies (1) was proved by Ledrappier and Strelcyn in $[\text{Led}]_3$). All results mentioned above are fundamental and stand at the heart of smooth ergodic theory of deterministic dynamical systems.

In recent years the counterpart in random dynamical systems has also been investigated. For an introduction to the scope of random dynamical systems, one can hardly find better description than that given by Walter $[\text{Wal}]_2$ in reviewing the pioneering book *Ergodic Theory of Random Transformations* by Kifer ($[\text{Kif}]_1$):

> *"Traditionally ergodic theory has been the qualitative study of iterates of an individual transformation, of a one-parameter flow of transformations (such as that obtained from the solutions of an autonomous ordinary differential equation), and more generally of a group of transformations of some state space. Usually ergodic theory denotes that part of the theory obtained by considering a measure on the state space which is invariant or quasi-invariant under the group of transformations. However in 1945 Ulam and von Neumann pointed out the need to consider a more general situation when one applies in turn different transformations chosen at random from some space of transformations. Considerations along these lines have applications in the theory of products of random matrices, random Schrödinger operators, stochastic flows on manifolds, and differentiable dynamical systems".*

In his book $[\text{Kif}]_1$, Kifer presented the first systematic treatment of ergodic theory of evolution processes generated by independent actions of transformations chosen at random from a certain class according to some probability distribution. Among the major contributions of this treatment are the introduction of the notions of invariant measures, entropy and Lyapunov exponents for such processes and a systematic exposition of some very useful properties of them. This pioneering book establishes a foundation for further study of this subject, especially for the purpose of the development of the present book.

In this book we are mainly concerned with ergodic theory of random dynamical systems generated by (discrete or continuous) stochastic flows of diffeomorphisms on a smooth manifold, which we sometimes call *smooth ergodic theory of random dynamical systems.* Our main purpose here is to exhibit a systematic generalization to the case of such flows of a major part of the fundamental deterministic results described above. Most generalizations presented in this book turn out to be non-trivial and some are in sharp contradistinction with the deterministic case. This is described in a more detailed way in the following paragraphs.

This book has the following structure. Chapter 0 consists of some necessary preliminaries. In this chapter we first present some basic concepts and theorems of measure theory. Proofs are only included when they cannot be found in standard references. Secondly, we give a quick review of the theory of measurable partitions of Lebesgue spaces and conditional entropies of such partitions.

Contents of this part come from Rohlin's fundamental papers $[\text{Roh}]_{1,2}$. The major part of this chapter is devoted to developing a general theory of conditional entropies of measure-preserving transformations on Lebesgue spaces. The concept of conditional entropies of measure-preserving transformations was first introduced by Kifer (see Chapter II of $[\text{Kif}]_1$), but his treatment was only justified for finite measurable partitions of a probability space. Here we deal with the concept in the case of general measurable partitions (maybe uncountable) of Lebesgue spaces and prove some associated properties mainly along the line of $[\text{Roh}]_2$, though the paper of Rohlin only deals with the usual entropies of measure-preserving transformations. Results presented in this chapter serve as a basis of the later chapters.

The concepts in Chapter I are mainly adopted from Kifer's book $[\text{Kif}]_1$. But for an adequate treatment of entropy formula (in Chapters II, IV, VI and VII) an extension of the notion of entropy to general measurable partitions is indispensible. So we have to formulate and prove the related theorems in this setting, which could be accomplished if the reader is familiar with the preliminaries in Chapter 0. In Section I.1 we first introduce the random dynamical system $\mathcal{X}^+(M,v)$ (see Section I.1 for its precise meaning). Then we discuss some properties of invariant measures of $\mathcal{X}^+(M,v)$. When associated with an invariant measure μ, $\mathcal{X}^+(M,v)$ will be referred to as $\mathcal{X}^+(M,v,\mu)$. Section I.2 consists of the concept of the (measure-theoretic) entropy $h_\mu(\mathcal{X}^+(M,v))$ of $\mathcal{X}^+(M,v,\mu)$ and of some useful properties of it deduced from its relationship with conditional entropies of (deterministic) measure-preserving transformations. In Section I.3 we introduce the notion of Lyapunov exponents of $\mathcal{X}^+(M,v,\mu)$ by adapting Oseledec multiplicative ergodic theorem to this random case.

In Chapter II we carry out the estimation of the entropy of $\mathcal{X}^+(M,v,\mu)$ from above through its Lyapunov exponents. We prove that for any given $\mathcal{X}^+(M,v,\mu)$ the following inequality holds true:

$$h_\mu(\mathcal{X}^+(M,v)) \leq \int \sum_i \lambda^{(i)}(x)^+ m_i(x) d\mu,$$

where $\lambda^{(1)}(x) < \lambda^{(2)}(x) < \cdots < \lambda^{(r(x))}(x)$ are the Lyapunov exponents of $\mathcal{X}^+(M,v,\mu)$ at point $x \in M$ and $m_i(x)$ is the multiplicity of $\lambda^{(i)}(x)$. This is an extension to the present random case of the well-known Ruelle's (or Margulis-Ruelle) inequality in deterministic dynamical systems. As in the deterministic case the above inequality is sometimes also called Ruelle's (or Margulis-Ruelle) inequality. In the random case this type of inequality was first considered by Kifer in Chapter V of $[\text{Kif}]_1$ (see Theorem 1.4 there), but the proof of his theorem contains a nontrivial mistake and this led the authors of the present book to an essentially different approach to this problem (see Chapter II for details). Our presentation here comes from $[\text{Liu}]_1$. As compared with the deterministic case, it involves substantially new techniques (especially the introduction of relation numbers and the related estimates).

After the first version of this book was completed, the authors received a preprint [Bah] by J. Banmüller and T. Bogenschütz which gives an alternative treatment of Ruelle's inequality. Their argument shows that the mistake mentioned above is inessential and can be corrected with some careful modifications, and their argument is also carried out within a more general framework of "stationary" random dynamical systems. It turns out, then, that the correction of the mistake in the original Chapter II is at the expense of an extraneous hypothesis (see Remark 2.1 of Chapter II). However, the treatment in that chapter (for example, the argument about the C^2-norms and relation numbers) is besides its own right very useful for the later chapters. For this consideration and in order not to change drastically the original (carefully organized) sketch of the book, we retain here the original Chapter II and also introduce Bahnmüller and Bogenschütz's argument (with some modifications) in the Appendix (it involves some results in Chapter VI).

Chapter III deals with the theory of stable invariant manifolds of $\mathcal{X}^+(M, v, \mu)$. We present there an extension to the random case of Pesin's results concerning the existence and absolute continuity of invariant families of stable manifolds ([Pes]$_1$). Although some new technical approaches are employed, our treatment goes mainly along Pesin's line with some ideas being adopted from [Kat], [Fat] and [Bri]. Besides their own rights, results in this chapter serve as powerful tools for the treatment of entropy formula given in later chapters.

In Chapter IV we extend Pesin's entropy formula to the case of $\mathcal{X}^+(M, v, \mu)$, i.e. we prove that

$$h_\mu(\mathcal{X}^+(M, v)) = \int \sum_i \lambda^{(i)}(x)^+ m_i(x) d\mu$$

when μ is absolutely continuous with respect to the Lebesgue measure on M. This formula takes the same form as in the deterministic case, but now the meaning of the invariant measure μ is quite different since it is no longer necessarily invariant for individual sample diffeomorphisms; we also have to point out that the implication of this result exhibits a sharp contradistinction with that in the deterministic case (see the arguments in Section IV.1 and those at the end of Chapter V). This result was first proved by Ledrappier and Young ([Led]$_1$) in the setting of the two-sided random dynamical system $\mathcal{X}(M, v, \mu)$ (see Chapter VI for its meaning), and a more readable treatment of it was later given in [Liu]$_2$ within the present one-sided setting $\mathcal{X}^+(M, v, \mu)$. In this chapter we follow the latter paper. As compared with the deterministic case ([Pes]$_2$ and [Led]$_3$), the proof of the result given here involves the new ideas of employing the theory of conditional entropies and of applying stable manifolds instead of unstable manifolds. Aside from these points, the proof follows essentially the same line as in the deterministic case, although the technical details are much more complicated.

In Chapter V we apply our results obtained in the previous chapters to the case of stochastic flows of diffeomorphisms. Such flows arise essentially as solution flows of stochastic differential equations and all the assumptions made in the previous chapters can be automatically verified in this case. Thus we reach and finish with an important application to the theory of stochastic processes.

Chapter VI is devoted to an extension of the main result (Theorem A) of Ledrappier and Young's remarkable paper $[\text{Led}]_2$ to the case of random diffeomorphisms. Roughly speaking, in the deterministic case one has Theorem A in $[\text{Led}]_2$ which asserts that Pesin's entropy formula holds true if and only if the associated invariant measure has SBR (Sinai-Bowen-Ruelle) property, i.e. it has absolutely continuous conditional measures on unstable manifolds; for the case of random diffeomorphisms we prove in this chapter that Pesin's entropy formula holds true if and only if the associated family of sample measures, i.e. the natural invariant family of measures associated with individual realizations of the random process has SBR property. This result looks to be a natural generalization of the deterministic result to the random case, but it has a non-trivial consequence (Corollary VI.1.2) which looks unnatural and which seems hopeless to obtain if one follows a similar way as in the deterministic case (i.e. by using the absolute continuity of unstable foliations). This generalization was actually known first to Ledrappier and Young themselves, though not clearly stated. Here we present the first detailed treatment of this result. Although the technical details are rather different, our treatment follows the line in the deterministic case provided by $[\text{Led}]_{2,3}$. The sources of this chapter are $[\text{Led}]_{1,2,3}$ and $[\text{Liu}]_3$.

In Chapter VII we study the case when a hyperbolic attractor is subjected to certain random perturbations. Based on our elaboration given in the previous chapters, a random version of the deterministic results mentioned above for Axiom A attractors is derived here. The idea of this chapter comes from [You] and $[\text{Liu}]_5$.

Random dynamical systems, though only at an early stage of development by now, have been widely used and taken care of, especially in applications. In this book, our intention is to touch upon only a part of this subject which we can treat with mathematical rigor. For this reason, we naturally restrict ourselves to the finite dimensional case. Infinite dimensional dynamical systems with random effect should be more interesting from a physical point of view. Scientists from both probability theory and partial differential equations have already paid jointly sufficient attention to this new and important field (a conference was organized by P. L. Chow and Skorohod in 1994). We hope their efforts will lead to a substantially new mathematical theory which, we believe, could be considered as the core of the so-called "Nonlinear Science".

We would like to express our sincere thanks to Prof. Ludwig Arnold since conversations with him were very useful for the preparation of Chapter VII. Our gratitude also goes to Profs. Qian Min-Ping and Gong Guang-Lu for helpful discussions. During the elaboration of this book the first author is supported by the National Natural Science Foundation of China and also by the Peking University Science Foundation for Young Scientists. Finally, it is acknowledged that part of the work on this book was done while the first author was in the Institute of Mathematics, Academia Sinica as a postdoctor and he expresses here his gratitude for its hospitality.

Table of Contents

Chapter 0 Preliminaries

In this chapter we first present some basic concepts and facts from measure theory. Then we give a quick review of the theory of measurable partitions of Lebesgue spaces and conditional entropies of such partitions. A detailed treatment of this theory is presented in Rohlin's fundamental papers $[\mathrm{Roh}]_{1,2}$. The rest of this chapter is devoted to developing, following the scheme of $[\mathrm{Roh}]_2$, a general theory of conditional entropies of measure-preserving transformations on Lebesgue spaces.

§1 Measure Theory

Let $(X, \mathcal{B}, \mu)$ be a measure space. Two sets $B_1, B_2 \in \mathcal{B}$ are said to be equivalent modulo zero, written $B_1 = B_2 \pmod 0$, if the symmetric difference $B_1 \triangle B_2$ has measure zero. When we write $\mathcal{A}_1 = \mathcal{A}_2 \pmod 0$ for two subsets $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{B}$ we mean that for each $A_1 \in \mathcal{A}_1$ there exists $A_2 \in \mathcal{A}_2$ such that $A_1 = A_2 \pmod 0$ and vice versa. Let $\mathcal{A}$ be a subset of $\mathcal{B}$, we say that $\mathcal{A}$ generates $\mathcal{B} \pmod 0$ if $\mathcal{B} = \mathcal{B}_0 \pmod 0$, where $\mathcal{B}_0$ is the σ-algebra generated by $\mathcal{A}$. The following is the well-known approximation theorem (see, e.g. [Rud]):

Theorem 1.1. *If $(X, \mathcal{B}, \mu)$ is a probability space, a subalgebra $\mathcal{A} \subset \mathcal{B}$ generates $\mathcal{B}$ (mod 0) if and only if, for every $B \in \mathcal{B}$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(A \triangle B) \leq \varepsilon$.*

Before going further, let us first review some definitions and simple facts about function spaces on a measure space. Let $(X, \mathcal{B}, \mu)$ be a measure space and let $1 \leq p < +\infty$. We denote by $L^p(X, \mathcal{B}, \mu)$ the quotient of the set of functions $f : X \to \mathbf{C}$ such that $|f|^p$ is integrable, under the equivalence relation that identifies functions which coincide a.e. We endow $L^p(X, \mathcal{B}, \mu)$ with the norm $\|\cdot\|_p$ defined by

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}$$

which makes $L^p(X, \mathcal{B}, \mu)$ a Banach space. For $p = 2$, the norm $\|\cdot\|_2$ comes from an inner product

$$< f, g >= \int_X f g d\mu$$

with respect to which $L^2(X, \mathcal{B}, \mu)$ is a Hilbert space.

Given a probability space $(X, \mathcal{B}, \mu)$, if there exists a countable subset of $\mathcal{B}$ which generates $\mathcal{B}$ (mod 0), then we say it is *separable*, several equivalent descriptions of this kind of separability are given by the following

Theorem 1.2. *For a probability space* $(X, \mathcal{B}, \mu)$, *the following properties are equivalent:*

1) $(X, \mathcal{B}, \mu)$ *is separable;*
2) $L^1(X, \mathcal{B}, \mu)$ *is separable;*
3) $L^p(X, \mathcal{B}, \mu)$ *is separable for every* $1 \leq p < +\infty$;
4) *There exists a countable subalgebra* $\mathcal{A} \subset \mathcal{B}$ *which generates* $\mathcal{B}$ *(mod 0).*

A complete separable metric space is known as a *Polish space*. This kind of spaces provide an important class of separable measure spaces by the following theorem (see, for instance, $[\mathrm{Man}]_1$):

Theorem 1.3. *If* X *is a Polish space,* $\mathcal{B}$ *is the Borel* σ*-algebra of* X *and* μ *is a probability measure on* $\mathcal{B}$, *then* $(X, \mathcal{B}, \mu)$ *is separable.*

We formulate below a theorem concerning the regularity of finite Borel measures on Polish spaces (see [Coh]).

Theorem 1.4. *Let* $(X, \mathcal{B}, \mu)$ *be a probability space, where* X *is a Polish space and* $\mathcal{B}$ *is the Borel* σ*-algebra of* X. *Then for every Borel set* $B \in \mathcal{B}$ *and* $\varepsilon > 0$ *there exists a compact set* $K \subset B$ *such that* $\mu(B \backslash K) < \varepsilon$.

Let $(X, \mathcal{B})$ be a measurable space and let $\mu : \mathcal{B} \to [0, +\infty]$ and $\nu : \mathcal{B} \to [0, +\infty]$ be measures. We say that μ is *absolutely continuous* with respect to ν, and we write $\mu << \nu$, if $B \in \mathcal{B}$ and $\nu(B) = 0$ imply $\mu(B) = 0$. The following Radon-Nikodym theorem characterizes this kind of absolute continuity.

Theorem 1.5. *If* $(X, \mathcal{B}, \nu)$ *is* σ*-finite, then* $\mu << \nu$ *if and only if there exists* $f : X \to \mathbf{R}^+$, *integrable with respect to* ν *on all sets* $B \in \mathcal{B}$ *such that* $\nu(B) < +\infty$, *satisfying the following condition for every* $B \in \mathcal{B}$:

$$\mu(B) = \int_B f d\nu.$$

The function f *is unique a.e. (with respect to* ν*), and is denoted by* $d\mu/d\nu$. *A function* $g : X \to \mathbf{C}$ *is in* $L^1(X, \mathcal{B}, \mu)$ *if and only if* gf *is in* $L^1(X, \mathcal{B}, \nu)$, *and in this case we have*

$$\int_X g d\mu = \int_X g f d\nu.$$

If $\mu(X) < +\infty$ *then* $f \in L^1(X, \mathcal{B}, \nu)$.

The function f is called the *Radon-Nikodym derivative* of μ with respect to ν.

Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{B}$. For every $f \in L^1(X, \mathcal{B}, \mu)$, the Radon-Nikodym theorem allows us to prove easily that there exists a unique function, written $E(f|\mathcal{A})$, in $L^1(X, \mathcal{A}, \mu)$ such that $\int_A E(f|\mathcal{A}) d\mu = \int_A f d\mu$ for every $A \in \mathcal{A}$. This function $E(f|\mathcal{A})$ is called the *conditional expectation* of f with respect to $\mathcal{A}$. We now define the *conditional expectation operator* $E(\cdot|\mathcal{A}) : L^1(X,\mathcal{B},\mu) \to L^1(X,\mathcal{A},\mu)$, $f \mapsto E(f|\mathcal{A})$.

Theorem 1.6. *Let $(X,\mathcal{B},\mu)$ be a probability space and let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{B}$. The restriction of the conditional expectation operator $E(\cdot|\mathcal{A})$ to $L^2(X,\mathcal{B},\mu)$ is the orthogonal projection of $L^2(X,\mathcal{B},\mu)$ onto $L^2(X,\mathcal{A},\mu)$.*

Proof. For $f \in L^1(X,\mathcal{B},\mu)$ we know that $E(f|\mathcal{A})$ is the only $\mathcal{A}$-measurable function such that $\int_A E(f|\mathcal{A})d\mu = \int_A f d\mu$ for every $A \in \mathcal{A}$. Let P denote the orthogonal projection of $L^2(X,\mathcal{B},\mu)$ onto the closed subspace $L^2(X,\mathcal{A},\mu)$. If $f \in L^2(X,\mathcal{B},\mu)$, then Pf is $\mathcal{A}$-measurable and if $A \in \mathcal{A}$

$$\int_A f d\mu =< f, \chi_A >=< f, P\chi_A >=< Pf, \chi_A >= \int_A Pf d\mu.$$

Therefore $Pf = E(f|\mathcal{A})$. □

The Radon-Nikodym theorem also allows us to introduce the general definition of Jacobian of absolutely continuous maps between measure spaces.

Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{A},\nu)$ be two σ-finite measure spaces, and let $T : X \to Y$ be a map. We say that T is *absolutely continuous* if the following three conditions hold: (i) T is injective; (ii) if $B \in \mathcal{B}$ then $TB \in \mathcal{A}$; (iii) $B \in \mathcal{B}$ and $\mu(B) = 0$ imply $\nu(TB) = 0$.

Assume that T is absolutely continuous. We now define on $\mathcal{B}$ a new measure μ_T by the formula $\mu_T(B) = \nu(TB)$. The measure μ_T is absolutely continuous with respect to the measure μ. Thus by the Radon-Nikodym theorem one can introduce the measurable function $J(T) = d\mu_T/d\mu$ defined on X, it is called the *Jacobian* of the map T.

It is easy to see that, when the absolutely continuous map T is bijective and T^{-1} is also absolutely continuous, we have

$$J(T) = \frac{1}{J(T^{-1}) \circ T}$$

for μ almost all points of X (we admit here $1/0 = +\infty$ and $1/+\infty = 0$).

When X and Y are two Riemannian manifolds without boundary and of the same finite dimension, $f : X \to Y$ is a C^1 diffeomorphism, and λ_X and λ_Y are the respective Lebesgue measures on X and Y induced by the Riemannian metrics, then in this particular case it is easy to see that for any $x \in X$ one has

$$J(f)(x) = \frac{d(\lambda_Y \circ f)}{d\lambda_X}(x) = |\det T_x f|,$$

where $T_x f$ is the derivative of f at x, and for any $h \in L^1(Y,\lambda_Y)$ one has

$$\int_X (h \circ f)(x)|\det T_x f| d\lambda_X(x) = \int_Y h(y) d\lambda_Y(y).$$

Next, we have the following Lebesgue-Vitali theorem on differentiation (see, e.g., [Shi]):

Theorem 1.7. *Let $A \subset \mathbf{R}^n$ be a Borel set, and $h : A \to \mathbf{C}$ an integrable function with respect to the Lebesgue measure λ of $\mathbf{R}^n$. Then the following holds true for λ-almost every $x \in A$:*

$$\lim_{r\to 0} \frac{1}{\lambda(B_r(x)\cap A)} \int_{B_r(x)\cap A} h d\lambda = h(x),$$

where $B_r(x) = \{y \in \mathbf{R}^n : \quad d(x,y) \leq r\}$.

A simple application of the Lebesgue-Vitali theorem yields the following:

Theorem 1.8. *Let $T : (X, \mathcal{B}, \mu) \to (Y, \mathcal{A}, \nu)$ be an absolutely continuous map, where X is a Borel subset of $\mathbf{R}^n$ with $\lambda(X) < +\infty$, $\mathcal{B}$ is the σ-algebra of Borel subsets of X and μ is absolutely continuous with respect to $\lambda|_X$. Then there holds the following formula for μ-almost every $x \in X$:*

$$\lim_{r\to 0} \frac{\mu_T(B_r(x)\cap X)}{\mu(B_r(x)\cap X)} = J(T)(x).$$

Proof. Let $h = d\mu/d\lambda$, then

$$\begin{aligned}
&\lim_{r\to 0} \frac{\mu_T(B_r(x)\cap X)}{\mu(B_r(x)\cap X)} \\
= &\lim_{r\to 0} \frac{\lambda(B_r(x)\cap X)^{-1}\int_{B_r(x)\cap X} J(T) h d\lambda}{\lambda(B_r(x)\cap X)^{-1}\int_{B_r(x)\cap X} h d\lambda} \\
= &J(T)(x), \quad \mu - a.e. x \in X.
\end{aligned}$$

This completes the proof. □

An easy application of Theorem 1.7 also gives

Theorem 1.9. *If $A \subset \mathbf{R}^n$ is a Borel set and μ is a Borel measure on $\mathbf{R}^n$ which is absolutely continuous with respect to λ, then the limit*

$$\lim_{r\to 0} \frac{\mu(B_r(x)\cap A)}{\mu(B_r(x))}$$

exists μ-almost everywhere in $\mathbf{R}^n$, and is equal to 0 if $x \notin A$ and to 1 if $x \in A$.

When the above limit exists for $x \in A$ and is equal to 1, we call x a *density point* of A with respect to μ.

We conclude this section with the notion of Lebesgue spaces. A map between two measure spaces is called an invertible measure-preserving transformation if it is bijective and measure-preserving and so is its inversion. Two measure spaces $(X_i, \mathcal{B}_i, \mu_i), i = 1, 2$ are said to be isomorphic mod 0 if there exist $Y_1 \in \mathcal{B}_1, Y_2 \in \mathcal{B}_2$ with $\mu_1(X_1 \backslash Y_1) = 0 = \mu_2(X_2 \backslash Y_2)$ and there exists an invertible measure-preserving transformation $\phi : (Y_1, \mathcal{B}_1|_{Y_1}, \mu_1|_{Y_1}) \to (Y_2, \mathcal{B}_2|_{Y_2}, \mu_2|_{Y_2})$. Given a probability space $(X, \mathcal{B}, \mu)$, let $X_0 = X \backslash \{x : x \in X$ with $\{x\} \in \mathcal{B}$ and $\mu(\{x\}) > 0\}$ and $s = \mu(X_0)$. We call $(X, \mathcal{B}, \mu)$ a *Lebesgue space* if $(X_0, \mathcal{B}|_{X_0}, \mu|_{X_0})$ is isomorphic mod 0 to the space $([0, s], \mathcal{L}([0, s]), l)$, where $\mathcal{L}([0, s])$ is the σ-algebra of Lebesgue measurable subsets of $[0, s]$ and l is the usual Lebesgue measure. There is now the following important theorem (see [Roy]):

Theorem 1.10. *Let X be a Polish space, μ a Borel probability measure on X, and $\mathcal{B}_\mu(X)$ the completion of the Borel σ-algebra of X with respect to μ. Then $(X, \mathcal{B}_\mu(X), \mu)$ is a Lebesgue space.*

Throughout the remaining sections of this chapter it is always assumed that $(X, \mathcal{B}, \mu)$ is a Lebesgue space.

§2 Measurable Partitions

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space.

Any collection of non-empty disjoint sets that covers X is said to be a partition of X. Subsets of X that are unions of elements of a partition ξ are called ξ-sets.

A countable system $\{B_\alpha : \alpha \in \mathcal{A}\}$ of measurable ξ-sets is said to be a basis of ξ if, for any two elements C and C' of ξ, there exists an $\alpha \in \mathcal{A}$ such that either $C \subset B_\alpha, C' \not\subset B_\alpha$ or $C \not\subset B_\alpha, C' \subset B_\alpha$. A partition with a basis is said to be *measurable*. Obviously, every element of a measurable partition is a measurable set.

For $x \in X$ we will denote by $\xi(x)$ the element of a partition ξ which contains x. If ξ, ξ' are measurable partitions of X, we write $\xi \le \xi'$ if $\xi'(x) \subset \xi(x)$ for μ-almost every $x \in X$, $\xi = \xi'$ is also considered up to mod 0.

For any system of measurable partitions $\{\xi_\alpha\}$ of X there exists a product $\bigvee_\alpha \xi_\alpha$, defined as the measurable partition ξ satisfying the following two conditions : 1) $\xi_\alpha \le \xi$ for all α; 2) if $\xi_\alpha \le \xi'$ for all α, then $\xi \le \xi'$.

For any system of measurable partitions $\{\xi_\alpha\}$ of X there exists an intersection $\bigwedge_\alpha \xi_\alpha$, defined as the measurable partition ξ satisfying the conditions : 1) $\xi_\alpha \ge \xi$ for all α; 2) if $\xi_\alpha \ge \xi'$ for all α, then $\xi \ge \xi'$.

For measurable partitions ξ_n, $n \in \mathbf{N}$ and ξ of X, the symbol $\xi_n \nearrow \xi$ indicates that $\xi_1 \le \xi_2 \le \cdots$ and $\bigvee_{n=1}^{+\infty} \xi_n = \xi$, the symbol $\xi_n \searrow \xi$ indicates that $\xi_1 \ge \xi_2 \ge \cdots$ and $\bigwedge_{n=1}^{+\infty} \xi_n = \xi$.

If $\{B_1, B_2, \cdots\}$ is a basis for the partition ξ and β_n is the partition of X into the sets B_n and $X \backslash B_n$, then the partitions $\xi_n = \bigvee_{i=1}^{n} \beta_i$ form an increasing