ALGEBRA

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Preface

Note: A complete discussion of possible ways of using this text, including suggested course outlines, is given on page xi.

This book is intended to serve as a basic text for an algebra course at the beginning graduate level. Its writing was begun several years ago when I was unable to find a one-volume text which I considered suitable for such a course. My criteria for "suitability," which I hope are met in the present book, are as follows.

(i) A conscious effort has been made to produce a text which an average (but reasonably prepared) graduate student might read by himself without undue difficulty. The stress is on clarity rather than brevity.

(ii) For the reader's convenience the book is essentially self-contained. Consequently it includes much undergraduate level material which may be easily omitted by the better prepared reader.

(iii) Since there is no universal agreement on the content of a first year graduate algebra course we have included more material than could reasonably be covered in a single year. The major areas covered are treated in sufficient breadth and depth for the first year graduate level. Unfortunately reasons of space and economics have forced the omission of certain topics, such as valuation theory. For the most part these omitted subjects are those which seem to be least likely to be covered in a one year course.

(iv) The text is arranged to provide the instructor with maximum flexibility in the choice, order and degree of coverage of topics, without sacrificing readability for the student.

(v) There is an unusually large number of exercises.

There are, in theory, no formal prerequisites other than some elementary facts about sets, functions, the integers, and the real numbers, and a certain amount of "mathematical maturity." In actual practice, however, an undergraduate course in modern algebra is probably a necessity for most students. Indeed the book is written on this assumption, so that a number of concepts with which the typical graduate student may be assumed to be acquainted (for example, matrices) are presented in examples, exercises, and occasional proofs before they are formally treated in the text.

The guiding philosophical principle throughout the book is that the material should be presented in the *maximum useable generality consistent with good pedagogy*. The principle is relatively easy to apply to various technical questions. It is more difficult to apply to broader questions of conceptual organization. On the one hand, for example, the student must be made aware of relatively recent insights into the nature of algebra: the heart of the matter is the study of morphisms (maps); many deep and important concepts are best viewed as universal mapping properties. On the other hand, a high level of abstraction and generality is best appreciated and fully understood only by those who have a firm grounding in the special situations which motivated these abstractions. Consequently, concepts which can be characterized by a universal mapping property are not *defined* via this property if there is available a definition which is more familiar to or comprehensible by the student. In such cases the universal mapping property is then given in a theorem.

Categories are introduced early and some *terminology* of category theory is used frequently thereafter. However, the language of categories is employed chiefly as a useful convenience. A reader who is unfamiliar with categories should have little difficulty reading most of the book, even as a casual reference. Nevertheless, an instructor who so desires may give a substantial categorical flavor to the entire course without difficulty by treating Chapter X (Categories) at an early stage. Since it is essentially independent of the rest of the book it may be read at any time.

Other features of the mathematical exposition are as follows.

Infinite sets, infinite cardinal numbers, and transfinite arguments are used routinely. All of the necessary set theoretic prerequisites, including complete proofs of the relevant facts of cardinal arithmetic, are given in the Introduction.

The proof of the Sylow Theorems suggested by R. J. Nunke seems to clarify an area which is frequently confusing to many students.

Our treatment of Galois theory is based on that of Irving Kaplansky, who has successfully extended certain ideas of Emil Artin. The Galois group and the basic connection between subgroups and subfields are defined in the context of an absolutely general pair of fields. Among other things this permits easy generalization of various results to the infinite dimensional case. The Fundamental Theorem is proved at the beginning, before splitting fields, normality, separability, etc. have been introduced. Consequently the very real danger in many presentations, namely that student will lose sight of the forest for the trees, is minimized and perhaps avoided entirely.

In dealing with separable field extensions we distinguish the algebraic and the transcendental cases. This seems to be far better from a pedogogical standpoint than the Bourbaki method of presenting both cases simultaneously.

If one assumes that all rings have identities, all homomorphisms preserve identities and all modules are unitary, then a very quick treatment of semisimple rings and modules is possible. Unfortunately such an approach does not adequately prepare a student to read much of the literature in the theory of noncommutative rings. Consequently the structure theory of rings (in particular, semisimple left Artinian rings) is presented in a more general context. This treatment includes the situation mentioned above, but also deals fully with rings without identity, the Jacobson radical and related topics. In addition the prime radical and Goldie's Theorem on semiprime rings are discussed.

There are a large number of exercises of varying scope and difficulty. My experience in attempting to "star" the more difficult ones has thoroughly convinced me of the truth of the old adage: one man's meat is another's poison. Consequently no exercises are starred. The exercises are important in that a student is unlikely to appreciate or to master the material fully if he does not do a reasonable number of exercises. But the exercises are not an integral part of the text in the sense that nontrivial proofs of certain needed results are left entirely to the reader as exercises.

Nevertheless, most students are quite capable of proving nontrivial propositions provided that they are given appropriate guidance. Consequently, some theorems in the text are followed by a "sketch of proof" rather than a complete proof. Sometimes such a sketch is no more than a reference to appropriate theorems. On other occasions it may present the more difficult parts of a proof or a necessary "trick" in full detail and omit the rest. Frequently all the major steps of a proof will be stated, with the reasons or the routine calculational details left to the reader. Some of these latter "sketches" would be considered complete proofs by many people. In such cases the word "sketch" serves to warn the student that the proof in question is somewhat more concise than and possibly not as easy to follow as some of the "complete" proofs given elsewhere in the text.

Seattle, Washington
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Mary, my wife, deserves an accolade for her patience during the (seemingly interminable) time the book was being written. The final word belongs to our daughter Anne, age three, and our son Tom, age two, whose somewhat unexpected arrival after eleven years of marriage substantially prolonged the writing of this book: a small price to pay for such a progeny.

Suggestions on the Use of this Book

GENERAL INFORMATION

Within a given section all definitions, lemmas, theorems, propositions and corollaries are numbered consecutively (for example, in section 3 of some chapter the fourth numbered item is Item 3.4). The exercises in each section are numbered in a separate system. Cross references are given in accordance with the following scheme.

(i) Section 3 of Chapter V is referred to as section 3 throughout Chapter V and as section V.3 elsewhere.

(ii) Exercise 2 of section 3 of Chapter V is referred to as Exercise 2 throughout section V.3, as Exercise 3.2 throughout the other sections of Chapter V, and as Exercise V.3.2 elsewhere.

(iii) The fourth numbered item (Definition, Theorem, Corollary, Proposition, or Lemma) of section 3 of Chapter V is referred to as Item 3.4 throughout Chapter V and as Item V.3.4 elsewhere.

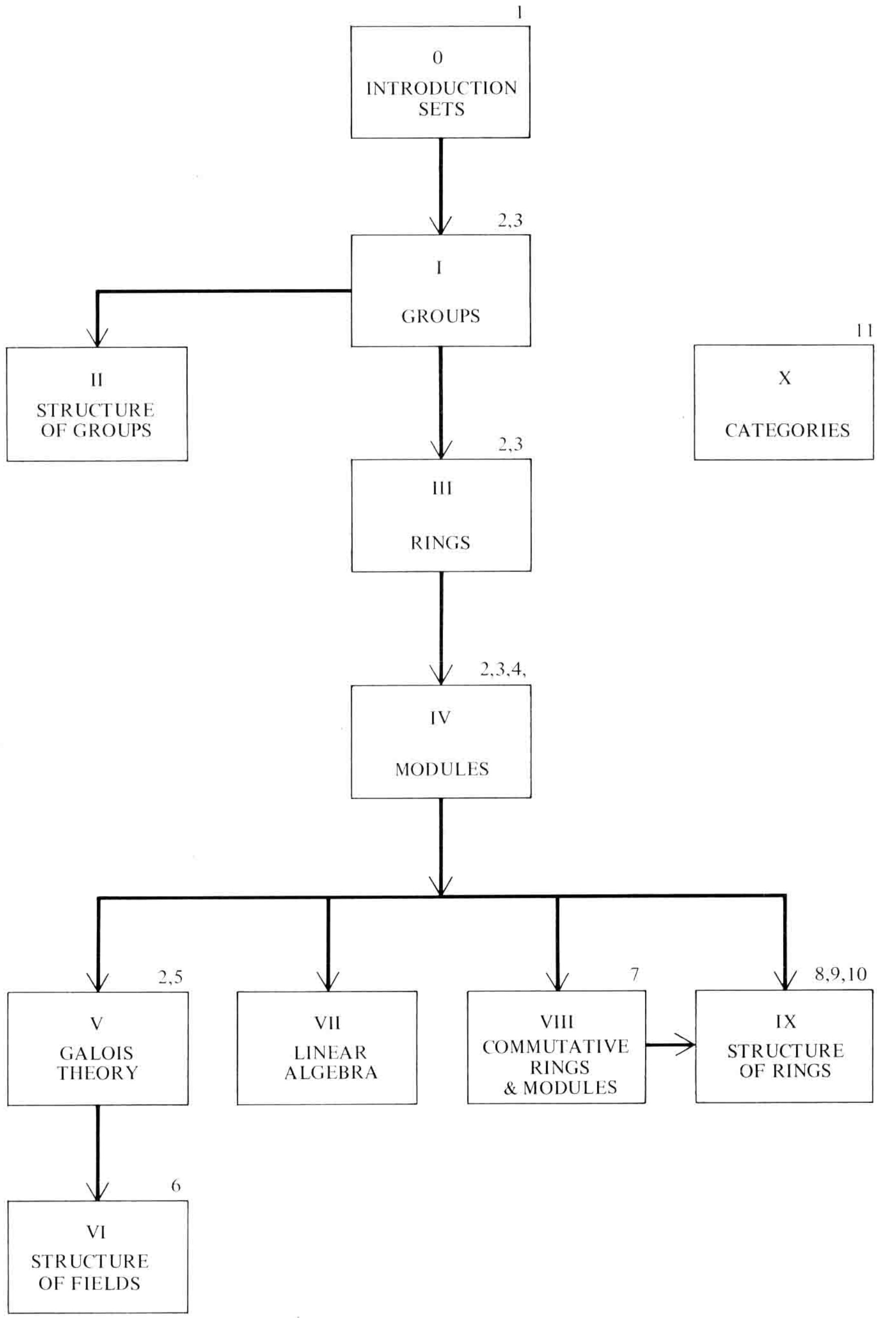
The symbol ■ is used to denote the end of a proof. A complete list of mathematical symbols precedes the index.

For those whose Latin is a bit rusty, the phrase *mutatis mutandis* may be roughly translated: "by changing the things which (obviously) must be changed (in order that the argument will carry over and make sense in the present situation)."

The title "proposition" is applied in this book only to those results which are *not* used in the sequel (except possibly in occasional exercises and in the proof of other "propositions"). Consequently **a reader who wishes to follow only the main line of the development may omit all propositions (and their lemmas and corollaries) without hindering his progress.** Results labeled as lemmas or theorems are almost always used at some point in the sequel. When a theorem is only needed in one or two places after its initial appearance, this fact is usually noted. The few minior exceptions to this labeling scheme should cause little difficulty.

INTERDEPENDENCE OF CHAPTERS

The table on the opposite page shows chapter interdependence and should be read in conjunction with the Table of Contents and the notes below (indicated by superscripts). In addition the reader should consult the introduction to each chapter for information on the interdependence of the various sections of the chapter.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NOTES

1. Sections 1-7 of the Introduction are essential and are used frequently in the sequel. Except for Section 7 (Zorn's Lemma) this material is almost all elementary. The student should also know a definition of cardinal number (Section 8, through Definition 8.4). The rest of Section 8 is needed only five times. (Theorems II.1.2 and IV.2.6; Lemma V.3.5; Theorems V.3.6 and VI.1.9). Unless one wants to spend a considerable amount of time on cardinal arithmetic, this material may well be postponed until needed or assigned as outside reading for those interested.

2. A student who has had an undergraduate modern algebra course (or its equivalent) and is familiar with the contents of the Introduction can probably begin reading immediately any one of Chapters I, III, IV, or V.

3. A reader who wishes to skip Chapter I is strongly advised to scan Section I.7 to insure that he is familiar with the language of category theory introduced there.

4. With one exception, the only things from Chapter III needed in Chapter IV are the basic definitions of Section III.1. However Section III.3 is a prerequisite for Section IV.6.

5. Some knowledge of solvable groups (Section II.7, II.8) is needed for the of study radical field extensions (Section V.9).

6. Chapter VI requires only the first six sections of Chapter V.

7. The proof of the Hilbert Nullstellensatz (Section VIII.7) requires some knowledge transcendence degrees (Section VI.1) as well as material from Section V.3.

8. Section VIII.1 (Chain Conditions) is used extensively in Chapter IX, but Chapter IX is independent of the rest of Chapter VIII.

9. The basic connection between matrices and endomorphisms of free modules (Section VII.1, through Theorem VII.1.4) is used in studying the structure of rings (Chapter IX).

10. Section V.3 is a prerequisite for Section IX.6.

11. Section I.7 is a prerequisite for Chapter X; otherwise Chapter X is essentially independent of the rest of the book.

SUGGESTED COURSE OUTLINES

The information given above, together with the introductions to the various chapters, is sufficient for designing a wide variety of courses of varying content and length. Here are some of the possible one quarter courses (30 class meetings) on specific topics.

These descriptions are somewhat elastic depending on how much is assumed, the level of the class, etc. Under the heading *Review* we list background material (often of an elementary nature) which is frequently used in the course. This material may be assumed or covered briefly or assigned as outside reading or treated in detail if necessary, depending on the background of the class. It is assumed without explicit mention that the student is familiar with the appropriate parts of the Introduction (see note 1, p. xiii). Almost all of these courses can be shortened by omitting all *Propositions* and their associated *Lemmas* and *Corollaries* (see page xi).

GROUP THEORY

Review: Introduction, omitting most of Section 8 (see note 1, p. xiii). **Basic Course:** Chapters I and II, with the possible omission of Sections I.9, II.3 and the last half of II.7. It is also possible to omit Sections II.1 and II.2 or at least postpone them until after the Sylow Theorems (Section II.5).

MODULES AND THE STRUCTURE OF RINGS

Review: Sections III.1 and III.2 (through Theorem III.2.13). **Basic Course:** the rest of Section III.2; Sections 1-5 of Chapter IV[1]; Section VII.1 (through Theorem VII.1.4); Section VIII.1; Sections 1-4 of Chapter IX. **Additional Topics:** Sections III.4, IV.7, IX.5; Section IV.5 if not covered earlier; Section IX.6; material from Chapter VIII.

FIELDS AND GALOIS THEORY

Review: polynomials, modules, vector spaces (Sections III.5, III.6, IV.1, IV.2). Solvable groups (Sections II.7, II.8) are used in Section V.9. **Basic Course**[2]**:** Sections 1-3 of Chapter V, omitting the appendices; Definition V.4.1 and Theorems V.4.2 and V.4.12; Section V.5 (through Theorem 5.3); Section V.6 (through Theorem V.6.2); Section V.7, omitting Proposition V.7.7 — Corollary V.7.9; Theorem V.8.1; Section V.9 (through Corollary V.9.5); Section VI.1. **Additional Topics:** the rest of Sections V.5 and V.6 (at least through Definition V.6.10); the appendices to Sections V.1 - V.3; the rest of Sections V.4, V.9, and V.7; Section V.8; Section VI.2.

LINEAR ALGEBRA

Review: Sections 3-6 of Chapter III and Section IV.1; selected parts of Section IV.2 (finite dimensional vector spaces). **Basic Course:** structure of torsion modules over a PID (Section IV.6, omitting material on free modules); Sections 1-5 of Chapter VII, omitting appendices and possibly the *Propositions*.

[1]If the stress is primarily on rings, one may omit most of Chapter IV. Specifically, one need only cover Section IV.1; Section IV.2 (through Theorem IV.2.4); Definition I.V2.8; and Section IV.3 (through Definition IV.3.6).

[2]The outline given here is designed so that the solvability of polynomial equations can be discussed quickly after the Fundamental Theorem and splitting fields are presented; it requires using Theorem V.7.2 as a definition, in place of Definition V.7.1. The discussion may be further shortened if one considers only finite dimensional extensions and omits algebraic closures, as indicated in the note preceding Theorem V.3.3.

COMMUTATIVE ALGEBRA

Review: Sections III.1, III.2 (through Theorem III.2.13). **Basic Course:** the rest of Section III.2; Sections III.3 and III.4; Section IV.1; Section IV.2 (through Corollary IV.2.2); Section IV.3 (through Proposition IV.3.5); Sections 1-6 of Chapter VIII, with the possible omission of *Propositions*. **Additional topics:** Section VIII.7 (which also requires background from Sections V.3 and VI.1).

Table of Contents

INTRODUCTION

PREREQUISITES AND PRELIMINARIES

In Sections 1–6 we summarize for the reader's convenience some basic material with which he is assumed to be thoroughly familiar (with the possible exception of the distinction between sets and proper classes (Section 2), the characterization of the Cartesian product by a universal mapping property (Theorem 5.2) and the Recursion Theorem 6.2). The definition of cardinal number (first part of Section 8) will be used frequently. The Axiom of Choice and its equivalents (Section 7) and cardinal arithmetic (last part of Section 8) may be postponed until this information is actually used. Finally the reader is presumed to have some familiarity with the fields **Q**, **R**, and **C** of rational, real, and complex numbers respectively.

1. LOGIC

We adopt the usual logical conventions, and consider only statements that have a truth value of either true or false (not both). If P and Q are statements, then the statement "P and Q" is true if both P and Q are true and false otherwise. The statement "P or Q" is true in all cases except when both P and Q are false. An implication is a statement of the form "P implies Q" or "if P, then Q" (written symbolically as $P \Rightarrow Q$). An implication is false if P is true and Q is false; it is true in all other cases. In particular, *an implication with a false premise is always a true implication.* An equivalence or biconditional is a statement of the form "P implies Q and Q implies P." This is generally abbreviated to "P if and only if Q" (symbolically $P \Leftrightarrow Q$). The biconditional "$P \Leftrightarrow Q$" is true exactly when P and Q are both true or both false; otherwise it is false. The negation of the statement P is the statement "it is not the case that P." It is true if and only if P is false.

2. SETS AND CLASSES

Our approach to the theory of sets will be quite informal. Nevertheless in order to define adequately both cardinal numbers (Section 8) and categories (Section I.7) it will be necessary to introduce at least the rudiments of a formal axiomatization of set theory. In fact the entire discussion may, if desired, be made rigorously precise; see Eisenberg [8] or Suppes [10]. An axiomatic approach to set theory is also useful in order to avoid certain paradoxes that are apt to cause difficulty in a purely intuitive treatment of the subject. A paradox occurs in an axiom system when both a statement and its negation are deducible from the axioms. This in turn implies (by an exercise in elementary logic) that *every* statement in the system is true, which is hardly a very desirable state of affairs.

In the Gödel-Bernays form of axiomatic set theory, which we shall follow, the primitive (undefined) notions are **class, membership,** and **equality.** Intuitively we consider a class to be a collection A of objects (elements) such that given any object x it is possible to determine whether or not x is a member (or element) of A. We write $x \in A$ for "x is an element of A" and $x \notin A$ for "x is not an element of A." The axioms are formulated in terms of these primitive notions and the first-order predicate calculus (that is, the language of sentences built up by using the connectives *and*, *or*, *not*, *implies* and the quantifiers *there exists* and *for all*). For instance, equality is assumed to have the following properties for all classes A, B, C : $A = A$; $A = B \Rightarrow B = A$; $A = B$ and $B = C \Rightarrow A = C$; $A = B$ and $x \in A \Rightarrow x \in B$. The **axiom of extensionality** asserts that two classes with the same elements are equal (formally, $[x \in A \Leftrightarrow x \in B] \Rightarrow A = B$).

A class A is defined to be a **set** if and only if there exists a class B such that $A \in B$. Thus a set is a particular kind of class. A class that is not a set is called a **proper class.** Intuitively the distinction between sets and proper classes is not too clear. Roughly speaking a set is a "small" class and a proper class is exceptionally "large." The **axiom of class formation** asserts that for any statement $P(y)$ in the first-order predicate calculus involving a variable y, there exists a class A such that $x \in A$ if and only if the statement $P(x)$ is true. We denote this class A by $\{x \mid P(x)\}$, and refer to "the class of all x such that $P(x)$." Sometimes a class is described simply by listing its elements in brackets, for example, $\{a,b,c\}$.

EXAMPLE.[1] Consider the class $M = \{X \mid X \text{ is a set and } X \notin X\}$. The statement $X \notin X$ is not unreasonable since many sets satisfy it (for example, the set of all books is not a book). M is a proper class. For if M were a set, then either $M \in M$ or $M \notin M$. But by the definition of M, $M \in M$ implies $M \notin M$ and $M \notin M$ implies $M \in M$. Thus in either case the assumption that M is a set leads to an untenable paradox: $M \in M$ and $M \notin M$.

We shall now review a number of familiar topics (unions, intersections, functions, relations, Cartesian products, etc.). The presentation will be informal with the mention of axioms omitted for the most part. However, it is also to be understood that there are sufficient axioms to guarantee that when one of these constructions is performed on *sets*, the result is also a set (for example, the union of sets is a set; a subclass of a set is a set). The usual way of proving that a given class is a set is to show that it may be obtained from a set by a sequence of these admissible constructions.

A class A is a **subclass** of a class B (written $A \subset B$) provided:

$$\text{for all } x \in A, \quad x \in A \quad \Rightarrow \quad x \in B. \tag{1}$$

[1]This was first propounded (in somewhat different form) by Bertrand Russell in 1902 as a paradox that indicated the necessity of a formal axiomatization of set theory.