Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries: Fondazione C.I.M.E., Firenze
Adviser: Roberto Conti

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A. Ambrosetti F. Gori
R. Lucchetti (Eds.)

Mathematical Economics

Montecatini Terme 1986

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Lectures given at the 2nd 1986 Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held at Montecatini Terme, Italy
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Editors

Antonio Ambrosetti
Scuola Normale Superiore
Piazza dei Cavalieri 7, 56100 Pisa, Italy

Franco Gori
Università di Venezia
Dipartimento di Matematica Applicata e Informatica
Dorsoduro 3825/e, 30123 Venezia, Italy

Roberto Lucchetti
Università di Milano, Dipartimento di Matematica
Via C. Saldini 50, 20133 Milano, Italy

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INTRODUCTION

In the last few years an ever increasing interest has been shown by economists and mathematicians in deepening and multiplying the many links already existing between their areas of research. Economists are looking for more advanced mathematical techniques to be applied to the analysis of formal models of greater complexity; mathematicians have found in problems from economics the stimulus to start new directions of study and to explore different trends within their theories.

The principal aim of the CIME Session on "Mathematical Economics" held at Villa La Querceta in Montecatini Terme, Italy, from June 25 to July 3 1986, has been the one of offering scholars from the two fields an opportunity of meeting and working together.

The common base of discussion was provided by four survey courses - whose texts are contained in the present volume - which were given by I. Ekeland "Some Variational Methods Arising from Mathematical Economics", A. Mas-Colell "Differentiability Techniques in the Theory of General Economic Equilibrium", J. Scheinkman "Dynamic General Equilibrium Models" and S. Zamir "Topics in Non Cooperative Game Theory".

Even if Ekeland's and Zamir's lectures were more "mathematically oriented", whereas Mas-Colell and Scheinkman put a greater emphasis on the economical contents, in every class, the focus of the discussion was placed over the connections naturally arising between problems from the two sciences.

It's our feeling that the Session was very successful in reaching its intended objectives, and we wish to express our gratitude to the four speakers, for the extremely high quality of the lectures delivered and the stimulating atmosphere they were able to create in Montecatini, and to all the participants, who supported the meeting with their interest and their lively discussions.

Our final thanks go to the CIME Scientific Committee for the invitation to organize the courses and to the CIME staff for its very effective job.

Antonio Ambrosetti
Franco Gori
Roberto Lucchetti

C.I.M.E. Session on "Mathematical Economics"

List of Participants

E. ALVONI, Istituto di Matematica Gen. e Finanz., Università, Piazza Scaravilli 2, 40126 Bologna, Italy

A. AMBROSETTI, Scuola Normale Superiore, Piazza dei Cavalieri 7, 56100 Pisa, Italy

E. BALDER, Mathematical Institute, University of Utrecht, Budapestlaan 6, 3584 CD Utrecht, The Netherlands

A. BATTINELLI, Istituto di Matematica Applicata alle Scienze Economiche e Sociali, Università, Via Montebello 7, 50123 Firenze, Italy

F. BIRARDI, Via Bertani 26, 50139 Firenze, Italy

P. CANNARSA, Via O. Tommasini 34, 00162 Roma, Italy

A. CARBONE, Dipartimento di Matematica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy

E. CAVAZZUTI, Dipartimento di Matematica, Università, Via G. Campi 213/B, 41100 Modena, Italy

G. CIMATTI, Dipartimento di Matematica,Università, Via Buonarroti 2, 56100 Pisa,Italy

G. COSTA, Istituto di Economia e Finanza, Facoltà di Scienze Politiche, Via Serafini 3, 56100 Pisa, Italy

R.-A. DANA, 9 Square Port-Royal, Paris 13e, France

L. DE BIASE, Istituto Matematico, Università, Via C. Saldini 50, 20133 Milano, Italy

G. DECIMA, Dipartimento di Matematica Appl. e Inf., Università, Ca' Dolfini, 30123 Venezia, Italy

S. DELAGUICHE, 19 avenue d'Eylau, 75116 Paris, France

F. DELBONO, Linacre College, Oxford OX1 3JA, U.K.

V. DENICOLO', Dipartimento di Scienze Economiche, Piazza Scaravilli 2, 40126 Bologna, Italy

D. DI GIOACCHINO, Via Campo Ligure 30, 00168 Roma, Italy

B. D'ONOFRIO, Dipartimento di Matematica, Università dell'Aquila, Via Roma 33, 67100 L'Aquila, Italy

E. EINY, Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel

I. EKELAND, CEREMADE, Université de Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris 16, France

P. FABBRI, Via P. Palagi 36, 40138 Bologna, Italy

L. FERRARA, Via Roccaraso 44, 00135 Roma, Italy

M. GALEOTTI, Istituto di Matematica, Facoltà di Architettura, Piazza Brunelleschi 6, 50100 Firenze, Italy

M. GILLI, Via dei Benedettini 4, 20146 Milano, Italy

F. GORI, Università di Venezia, Dipartimento di Matematica Applicata e Informatica, Dorsoduro 3825/e, 30123 Venezia, Italy

G. GOZZI, Via Carducci 5, 46100 Mantova, Italy

V. GUIDI, Dipartimento di Scienze Economiche, Università, Via Curtatone 1, 50123 Firenze, Italy

J. HERNANDEZ, Universidad Autonoma, Dep. de Matematica, 28036 Madrid, Spain

D. HOMANN, IMW, Universität Bielefeld, Postfach, 4800 Bielefeld 1, West Germany

E. LEHRER, Hebrew University, Institute of Mathematics, Givat Ram, 91904 Jerusalem, Israel

R. LEONCINI, c/o Cassola, Via del Proconsolo 5, 50122 Firenze, Italy

R. LUCCHETTI, Università di Milano, Dipartimento di Matematica, Via C. Saldini 50, 20133 Milano, Italy

L. MALAGUTI, Via G. Benassi 5, 41012 Carpi (MO), Italy

A. MAS-COLELL, Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720, USA

F. MIGNANEGO, Dipartimento di Matematica, Università, Via L.B. Alberti 4, 16132 Genova, Italy

S. MODICA, Via M. Rutelli 38, 90143 Palermo, Italy

D. MONDERER, Department of Mathematics, Everyman's University, Ramat Aviv, Tel Aviv 61392, Israel

L. MONTRUCCHIO, Dipartimento di Matematica, Politecnico, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

J. MORGAN, Dipartimento di Matematica e Applicazioni, Università, Via Mezzocannone 8, 80134 Napoli, Italy

A. MORI, Via S. Martino 284, 55049 Viareggio, Italy

I. NADA, Tel Aviv University, Faculty of Management, University Campus, Ramat Aviv, Tel Aviv 69978, Israel

F. NARDINI, Dipartimento Matematico, Università, Piazza di Porta S. Donato 5, 40127 Bologna, Italy

K. OLSEN, European University Institute, Badia Fiesolana, Via dei Roccettini 9, 50016 San Domenico di Fiesole, Firenze, Italy

N. PACCHIAROTTI, Dipartimento di Matematica, Università, Via G. Campi 213/B, 41100 Modena, Italy

P.-M. PACINI, Istituto Universitario Europeo, Badia Fiesolana, Via Roccettini 9, 50016 San Domenico di Fiesole, Firenze, Italy

F. PATRONE, Dipartimento di Matematica, Università, Strada Nuova 65, 27100 Pavia, Italy

E. PETAZZONI, Via Ognibene 2, 40135 Bologna, Italy

G. PIERI, Dipartimento di Matematica, Università, Via L.B. Alberti 4, 16132 Genova, Italy

C. van der PLOEG, Department of Mathematics, University of Sussex, Falmer, Brighton BN1 9QE, East Sussex, England

N. RICCIARDI, CEREMADE, Université de Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris 16, France

G. ROSSINI, Dipartimento di Scienze Economiche, Strada Maggiore 45, 40125 Bologna, Italy

M. SABATINI, Dipartimento di Matematica pura ed applicata, Università, Via Roma 33, 67110 L'Aquila, Italy

M. SCARSINI, Istituto di Matematica Finanziaria, Via Kennedy 6, 43100 Parma, Italy

J. SCHEINKMAN, The University of Chicago, Department of Economics, 1126 East 59th Street, Chicago, Ill. 60637, USA

A. SICONOLFI, Dipartimento di Matematica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy

P. SUDHOLTER, Institut für Mathematische Wirtschartsforschung, Universitätstrasse, D-4800 Bielefeld 1, West Germany

B. TERRENI, Dipartimento di Matematica, Università, Via F. Buonarroti 2, 56100 Pisa, Italy

A. TORRE, Dipartimento di Matematica, Università, Strada Nuova 65, 27100 Pavia, Italy

A. VILLANACCI, Via Pasquini 2, 50127 Firenze, Italy

S. ZAMIR, The Hebrew University of Jerusalem, Department of Statistics, Mount Scopus, Jerusalem 91905

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SOME VARIATIONAL PROBLEMS ARISING FROM

MATHEMATICAL ECONOMICS.

Ivar EKELAND, CEREMADE, Paris.

I. Ramsey problems.

Many intertemporal problems in mathematical economics can be written as infinite-horizon optimization problems :

$$(\mathcal{P}) \qquad \begin{cases} \text{Sup} \int_0^\infty e^{-\delta t} \, u(t,x,\dot{x}) \, dt \\ (x(t),\dot{x}(t)) \in A_t \quad \text{a.e.} \\ x(0) = x_0 \quad \text{and} \quad \dot{x} \in L^1_{loc} \end{cases}$$

Here $\delta > 0$ is the discount rate and $u(t,\cdot,\cdot)$ the utility function, so that the integral to be maximized is the aggregated utility over time of the path $x : [0,\infty) \to \mathbb{R}^n$. One usually thinks of $x(t)$ as the capital stock at time t , so that $\dot{x}(t)$ is the rate of (dis-) investment. The set $A_t \subset \mathbb{R}^n x \mathbb{R}^n$ embodies the various constraints (production technology, availability of resources) which the system has to satisfy.

This model contains seemingly more complicated ones. For instance, if one introduces the consumption $c(t)$, so that the criterion becomes

$$\int_0^\infty e^{-\delta t} \, u(t,c) \, dt$$

and the constraints :

$$(x(t),\dot{x}(t),c(t)) \in B_t \qquad \text{a.e.},$$

one would simply define $A_t = \{(x,y) \mid (x,y,c) \in B_t \text{ for some } c\}$, and maximize $u(t,\cdot)$ over all c such that $(x,y,c) \in B_t$. Assuming the maximum is attained at a single point $\bar{c}(t,x,y)$, and setting

$$\bar{u}(t,\wedge,y) = u(t,\bar{c}(t,x,y))$$

brings the problem into the standard form (P) .

The first model of this kind is due to Ramsey towards the end of the last century. In the years of plenty - the sixties - very many variants of this basic model appeared, emphasizing various aspects of the theory of economic growth. We refer to the books by Intriligator [In] and by Arrow and Kurz [AK] for an introduction to this kind of literature. Unfortunately, none of the mathematical problems raised by the Ramsey problem (P) were adequately treated, or even realized at the time. A notable exception is the special issue of JET [1976] , which gives the state of the art until that time.

The main problems connected with (P) are the following :

(1) When does (P) have a solution ? In other words, what conditions on u and A_t are needed for an optimal path $\bar{x}$ to exist ?

(2) What are the necessary conditions for optimality ? In other words, does $\bar{x}$ satisfy some version of the Euler-Lagrange equation in $(0,\infty)$, and what boundary condition must $\bar{x}(t)$ satisfy when $t \to \infty$?

(3) What is the behaviour of $\bar{x}(t)$ when $t \to \infty$? Does it converge to some equilibrium state $\bar{x}(\infty)$, or can it oscillate more or less wildly ?

I don't know how to answer these questions in the full generality of problem (P). I will therefore, as the need arises, restrict myself to simpler models where I know the answer, and leave the general case to others. As a first - and considerable - simplification, let us assume that the problem is autonomous, i.e. t does not appear explicitely. It becomes :

$$(P) \qquad \begin{cases} \operatorname{Sup} \int_0^\infty e^{-\delta t} u(x,\dot{x})\, dt \\ (x,\dot{x}) \in A \\ x(0) = x_0 \end{cases}$$

I.1 Existence.

We assume the following :

(H1) $u : \mathbb{R}^{2n} \to \mathbb{R} \cup \{-\infty\}$ is upper semi-continuous and $A \subset \mathbb{R}^{2n}$ is closed.

(H2) $\forall\, x \in \mathbb{R}^n$, $y \to u(x,y)$ is concave
$\forall\, x \in \mathbb{R}^n$, $A_x = \{y \mid (x,y) \in A\}$ is convex.

(H3) $\begin{cases} \exists\, \varphi : [0,\infty) \to \mathbb{R}, \text{ with } \varphi(t)t^{-1} \to +\infty \text{ when } t \to \infty, \\ \text{such that } u(x,y) \leqslant -\varphi(\|y\|) \text{ for all } (x,y) \in A. \end{cases}$

THM. Assume (H1),(H2),(H3). Then $(\mathcal{P})$ has at least one solution.

Proof. We refer to the books [ET], [C] or [G] for a proof in the general case. The proof in [ET] contains a mistake.

Let us just sketch the proof in the case where the criterion and the constraints split :

$$u(x,y) = u_1(x) + u_2(y) \quad \text{and} \quad A = A_1 \times A_2$$

Then u_2 is concave, $u_2(y) \leqslant -\varphi(\|y\|)$, and u_1 is bounded from above. Take a maximizing sequence :

$$\int_0^\infty e^{-\delta t}\,[\,u_1(x_n)+u_2(\dot{x}_n)\,]\,dt \to \text{Sup}$$

Then there is some large constant C such that :

$$C \leqslant \int_0^\infty e^{-\delta t} u_2(\dot{x}_n)\,dt \leqslant -\int_0^\infty \varphi(\|\dot{x}_n\|)e^{-\delta t}\,dt \quad .$$

Since $[0,\infty)$ endowed with $e^{-\delta t}dt$ has finite measure, we may apply the Dunford-Pettis criterion for weak compactness in L^1, and we conclude that the sequence $\dot{x}_n$ has a weakly convergent subsequence in $L^1(0,\infty\, ;\, e^{-\delta t}dt)$. Denote this subsequence by $\dot{x}_n$ again, and its limit by y :

$$\dot{x}_n \to y \qquad \text{in } L^1(0,\infty\, ;\, e^{-\delta t}dt)$$

Set

$$\bar{x}(t) = x_o + \int_o^t y(s)\, ds$$

so that $y = \frac{d\bar{x}}{dt}$, and $x_n(t) \to \bar{x}(t)$ uniformly on compact subsets of $[0,\infty)$.

Using Fatou's lemma, we have :

$$\limsup_{n\to\infty} \int_o^\infty e^{-\delta t} u_1(x_n(t))\, dt \leqslant \int_o^\infty e^{-\delta t} u_1(\bar{x}(t))\, dt$$

The map $y \to \int_o^\infty e^{-\delta t} u_2(y(t))\, dt$ is concave and upper semicontinuous. By the Hahn-Banach theorem, it must also be <u>weakly</u> u.s.c. and therefore :

$$\limsup_{n\to\infty} \int_o^\infty e^{-\delta t} u_2\left(\frac{dx_n}{dt}\right) dt \leqslant \int_o^\infty e^{-\delta t} u_2\left(\frac{d\bar{x}}{dt}\right) dt$$

Adding up, we get

$$\int_o^\infty e^{-\delta t} \left(u_1(\bar{x}) + u_2\left(\frac{d\bar{x}}{dt}\right)\right) dt \geqslant \text{Sup}$$

All we have to check now is that $\bar{x}$ is admissible, that is, $\bar{x}(t) \in A_1$ and $\frac{d\bar{x}}{dt}(t) \in A_2$ for almost every t . This follows easily from the facts that :

$$x_n(t) \to \bar{x}(t) \quad \text{pointwise and } A_1 \text{ is closed}$$

$$\frac{dx_n}{dt} \to \frac{d\bar{x}}{dt} \quad \text{weakly in } L^1(e^{-\delta t}dt) \text{ and } A_2 \text{ is convex closed.}$$

■

Note that the result holds also in the general (nonautonomous) case, as the proofs show. Note also that convexity is required with respect to the last variable $\dot{x}$ only.

I.2 Euler-Lagrange.

The derivation of necessary conditions for optimality, including some version of the Euler-Lagrange equations, requires an a priori estimates : it must first be shown that $\bar{x}$ is locally Lipschitz, that is, $\frac{d\bar{x}}{dt}$ is uniformly bounded on compact intervals of $[0,\infty)$, before anything further can be said. This delicate point is sadly missing from the literature of the sixties and seventies, although Tonelli had delved on it in his classical treatise [T] of 1921-23 . Cesari resurrected it in his recent book [C], and it was taken up again by Ball and Mizel [BM1], [BM2], and later by Clarke and Vinter [CV1], [CV2].

THM. Assume $u(x,y)$ is continuous and satisfies (H3). Let the slice $A_x = \{y \mid (x,y) \in A\}$ be closed and star-shaped with respect to the brigin, for every x . Then, if $\bar{x}$ solves (P), for any T there will be some $K > 0$ such that

$$0 \leqslant t \leqslant T \Rightarrow \left\|\frac{d\bar{x}}{dt}(t)\right\| \leqslant K \quad .$$ ■

Proof. Pick $T > 0$. To simplify notations, write x instead of $\bar{x}$.

Note first that x is uniformly bounded on $[0,T]$. Indeed, setting $\inf_{t \geqslant 0} \varphi(t) = -c$, we have :

$$\begin{aligned}\int_0^T [\varphi(\|\dot{x}\|)+c\,]e^{-\delta t}dt &\leqslant \int_0^\infty [\varphi(\|\dot{x}\|)+c\,]e^{-\delta t}dt \\ &\leqslant -\int_0^\infty u(x,\dot{x})e^{-\delta t}dt + c/\delta \\ &\leqslant \frac{1}{\delta}(c - u(x_o,0))\end{aligned}$$

Since $\varphi(t)t^{-1} \to +\infty$, it follows that $\dot{x} \in L^1(0,T)$, so that $x(t)$ stays in a bounded subset, say :

$$\|x(t)\| \leqslant A \qquad \text{for} \quad 0 \leqslant t \leqslant T$$

For any $M > 0$ large enough, we can define a change of time variable $s = \sigma(t)$ by the conditions :

$$\begin{cases} \sigma(0) = 0 \\ \dfrac{d\sigma}{dt} = \|\dot{x}(t)\| & \text{if } t \in L_M \\ \dfrac{d\sigma}{dt} = 1 & \text{if } t \notin L_M \end{cases}$$

$$L_M = \{t \mid \|\dot{x}(t)\| \geq M \text{ and } 0 \leq t \leq T\} .$$

Define $x_M(s) = x \circ \sigma^{-1}(s)$.

Let us first check that the path x_M is admissible, that is $(x_M(s), \dot{x}_M(s)) \in A$ for almost every s . If $t \notin L_M$, we have $(x_M(s), \dot{x}_M(s)) = (x(t), \dot{x}(t)) \in A$. If $t \in L_M$, we have, with $s = \sigma(t)$:

$$(x_M(s), \dot{x}_M(s)) = (x(t) , \dot{x}(t)\|\dot{x}(t)\|^{-1})$$

which belongs to A , since $A_{x(t)}$ is star-shaped with respect to the origin.

Since x_M is admissible, we must have

$$\int_0^{\infty} u(x_M, \dot{x}_M) e^{-\delta s} \, ds \leq \int_0^{\infty} u(x, \dot{x}) e^{-\delta t} \, dt$$

We may assume that u is non-positive (otherwise replace $u(x,y)$ by $u(x,y)-c$). Set $s = \sigma(t)$, so that $s \geq t$; we have :

$$0 \leq \sigma(t)-t \leq \int_{L_M} (\|\dot{x}(t)\|-1) \, dt$$

Writing $s = \sigma(t)$ in the preceding inequality, we get :

$$\int_0^{\infty} u(x, \dot{x}) e^{-\delta t} \, dt \geq \int_0^{\infty} u(x_M \circ \sigma(t) , \dot{x}_M \circ \sigma(t)) \, e^{-\delta\sigma(t)} \, d\sigma(t)$$

$$\geq \int_0^{\infty} u(x_M \circ \sigma(t) , \dot{x}_M \circ \sigma(t)) \, e^{-\delta t} \frac{d\sigma}{dt} \, dt$$

Replacing $x_N \circ \sigma$ and $\dot{x}_M \circ \sigma$ by their value, we get

$$\int_{L_M} u(x,\dot{x})e^{-\delta t}\,dt \geqslant \int_{L_M} u\left(x, \frac{\dot{x}}{\|\dot{x}\|}\right) \|\dot{x}\| e^{-\delta t}\,dt$$

Hence :

$$\int_{L_M} \left[\varphi(\|\dot{x}\|) + u\left(x, \frac{\dot{x}}{\|\dot{x}\|}\right) \|\dot{x}\|\right] e^{-\delta t}\,dt \leqslant 0$$

Set $\operatorname{Max}\{|u(x,y)| \ \|x\| \leqslant A \text{ and } \|y\| \leqslant 1\} = B$. The preceding inequality reads :

$$\int_{L_M} [\varphi(\|\dot{x}\|) - B\|\dot{x}\|]\, e^{-\delta t}\,dt \leqslant 0$$

which is wrong as soon as $\varphi(\|\dot{x}\|)\|\dot{x}\|^{-1} \geqslant B$. This happens on L_M when M is large enough. ■

Note that, by the preceding proof, if $u(x,y)$ does not depend on x , then $\left\|\frac{d\bar{x}}{dt}(t)\right\| \leqslant K$ on $[0,\infty)$, that is, $\bar{x}$ is Lipschitz on the whole of $\mathbb{R}_+$. More generally, if u depends on x and y , but $\|\bar{x}(t)\|$ is bounded (by A) on $[0,+\infty)$, then so is $\left\|\frac{d\bar{x}}{dt}(t)\right\|$ (by K).

If $u(x,y)$ is C^1 , this a priori bound will enable us to differentiate under the integral, and the Euler-Lagrange equations will follow.

THM. Assume $u(x,y)$ is C^1 and satisfies (H3). Assume A is convex and $A_x \neq \emptyset \Rightarrow A_x \ni 0$. Let $\bar{x}$ be an optimal path, and y an admissible path such that, for some $T > 0$, we have $y(t) = \bar{x}(t)$ if $t \geqslant T$, and $\left\|\frac{dy}{dt}\right\| \leqslant K$ for $t \leqslant T$. Then

$$(E) \qquad \int_0^T \left[\frac{\partial u}{\partial x}\left(\bar{x}, \frac{d\bar{x}}{dt}\right)(x-\bar{x}) + \frac{\partial u}{\partial y}\left(\bar{x}, \frac{d\bar{x}}{dt}\right)\left(\frac{dx}{dt} - \frac{d\bar{x}}{dt}\right)\right] e^{-\delta t}\,dt \leqslant 0$$

<u>Proof.</u> We just write :

$$0 \leqslant \int_0^\infty u\left(\bar{x}, \frac{d\bar{x}}{dt}\right) e^{-\delta t}dt - \int_0^\infty u\left(\bar{x}+h(x-\bar{x}), \frac{d\bar{x}}{dt} + h\left(\frac{dx}{dt} - \frac{d\bar{x}}{dt}\right)\right) e^{-\delta t}\,dt$$

divide by h and let $h \to 0$. ■

We shall discuss the interpretation of (E) in the particular case when there are only state constraints : $A = A_1 \times \mathbb{R}^n$, with A convex and $\text{Int } A \neq \emptyset$.

If $\bar{x}(t_o)$ belongs to the interior of A_1 , then so does $\bar{x}(t)$ for $|t-t_o| \leqslant \eta$, and (E) gives the familiar Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial u}{\partial y}\left(\bar{x}, \frac{d\bar{x}}{dt}\right) e^{-\delta t}\right) = \frac{\partial u}{\partial x}\left(\bar{x}, \frac{d\bar{x}}{dt}\right) e^{-\delta t}$$

In the general case, the allowable variations $y = x-\bar{x}$ must satisfy $y(t) \in T(\bar{x}(t),A)$ (tangent cone to A at $\bar{x}(t)$) for every t . We then have :

$$\int_o^T \left|\frac{\partial u}{\partial x}\left(\bar{x}, \frac{d\bar{x}}{dt}\right) y + \frac{\partial u}{\partial y}\left(\bar{x}, \frac{d\bar{x}}{dt}\right) \frac{dy}{dt}\right| e^{-\delta t}\, dt \leqslant 0$$

for every Lipschitz function y such that $u(t) \in T(\bar{x}(t),A)$, $y(0) = 0$ and $y(t) = 0$ for $t \geqslant T$. Since $\frac{d\bar{x}}{dt} \in L^\infty$, so do $\frac{\partial u}{\partial x}\left(\bar{x}, \frac{d\bar{x}}{dt}\right)$ and $\frac{\partial u}{\partial y}\left(\bar{x}, \frac{d\bar{x}}{dt}\right)$, and the inequality holds in fact for all y such that $\frac{dy}{dt} \in L^1$, $y(t) \in$ $\in T(\bar{x},t),A)$, $y(0) = 0$ and $y(t) = 0$ for $t \geqslant T$. Integrate by parts :

$$\int_o^T \left|- \int_o^t \frac{\partial u}{\partial x}\left(\bar{x}, \frac{d\bar{x}}{dt}\right) e^{-\delta s}\, ds + \frac{\partial u}{\partial y}\left(\bar{x}, \frac{d\bar{x}}{dt}\right) e^{-\delta t}\right| \frac{dy}{dt}\, dt \leqslant 0$$

Set :

$$f(t) = - \int_o^t \frac{\partial u}{\partial u}\left(\bar{x}, \frac{d\bar{x}}{dt}\right) e^{-\delta s}\, ds + \frac{\partial u}{\partial y}\left(\bar{x}, \frac{d\bar{x}}{dt}\right) e^{-\delta t} \in L^\infty$$

$$Au(t) = \int_o^t u(s)ds \in C^o , \quad \text{with } u \in L^1(0,T) \text{ and } \int_o^T u(s)ds = 0$$

$$C = \{v \in C^o([0,T]) \mid v(t) \in T(\bar{x}(t),A) \quad \forall\, t\}$$

Then C is a cone with non-empty interior in C^o . Letting $C^\perp$ be its polar cone, which is a subset of U , the set of all Radon measures on $[t_o-\eta, t_o+\eta]$:

$$C^\perp = \left\{\mu \in U \mid \int v\, d\mu \leqslant 0 \quad \forall\, v \in C\right\}$$

we have the standard formula from convex analysis :

$$[A^{-1}C]^{\perp} = A^{\star}C^{\perp}$$

So $f \in A^{\star}C^{\perp}$. This means that there is some $\bar{\mu} \in C^{\perp}$ and some constant $\xi \in \mathbb{R}^{2n}$ such that :

$$f(t) - \xi = \bar{\mu}([0,t]) - \mathrm{Supp}\,\bar{\mu}$$

This relation is equivalent to the following (use the theory of desintegration of measures) : there is a measurable vector-valued function $\nu : [0,\infty) \to \mathbb{R}^n$, with $\nu(t) \in N(\bar{x}(t),A)$ (the normal cone to A at $\bar{x}(t)$), and a scalar-valued non-negative Radon measure ρ on $[0,\infty)$, such that :

$$f(t) - \xi = \int_o^t \nu(s)\, e^{-\delta\tau}\, d\rho(s)$$

In other words :

$$e^{\delta t}\frac{df}{dt} = \frac{d}{dt}\left(\frac{\partial u}{\partial y}\left(\bar{x}, \frac{d\bar{x}}{dt}\right)\right) - \delta\frac{\partial u}{\partial y}\left(\bar{x}, \frac{d\bar{x}}{dt}\right) - \frac{\partial u}{\partial x}\left(\bar{x}, \frac{d\bar{x}}{dt}\right)$$

$$= \nu(t)\, d\rho$$

where the right-hand side is to be understood as a measure.

I.3 Transversality condition at infinity.

A very original approach to this problem is due to Ph. Michel [M]. Here, I will favour the Ekeland-Scheinkman approach [ES], which has a broader scope.

THM. Assume that

(1) $u(x,y)$ is C^1, concave in y, and satisfies (H3).

(2) A is convex and $A_x \neq \emptyset \Rightarrow 0 \in A_x$.

Let $\bar{x}$ be a solution of problem ($\mathcal{P}$), and let x be another admissible path such that :