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Introduction

The workshop on "Arithmetic of complex manifolds" was held at the Mathematisches Institut der Universität in Erlangen from May 27 to May 31, 1988. It was supported by the Deutsche Forschungsgemeinschaft in the context of its Forschungsschwerpunkt "Komplexe Mannigfaltigkeiten".

It was the aim of the meeting to bring together number theorists and algebraic geometers to discuss problems of common interest, such as moduli problems, complex tori, integral points, rationality questions, automorphic forms etc. During recent years such problems, which are simultaneously of arithmetic and geometric interest, have become more and more important.

This volume contains written versions of some of the lectures given at the workshop as well as papers of other participants.

We are grateful to all those who took part for their contributions to the success of the conference, and in particular to the contributors of this volume. We would also like to thank the DFG for financial support and the Erlangen Mathematical Department for its hospitality.

The Editors.

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Surfaces on quintic threefolds associated to the Horrocks-Mumford bundle.

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§0. Introduction

A Horrocks-Mumford (abbreviated HM) quintic threefold is a quintic hypersurface in $\mathbf{P}^4 = \mathbf{CP}^4$ being invariant under the Heisenberg group of level 5 and with 100 ordinary double points as singularities. We will study the Picard group of a small resolution of a general such quintic, say V , and classify the smooth surfaces on V. In §1 we recall some known facts about the HM-bundle, and in §2 we calculate the defect of V. We show in §3 the existence of three "types" of surfaces of low degree on V; A): Abelian surfaces of degree 10 or 15. B): Determinantal surfaces of degree 10. C): Some surfaces of general type of degree 20. In §4 we use these surfaces to find a basis for the Picard group of a small resolution of V (Prop. 4.6). Finally, we show that a smooth surface on V is either a complete intersection of V and another hypersurface, or it is linked to a surface of type A,B,or C (Theorem 4.7).

§1. Preliminaries

We recall some basic properties of the HM-bundle and the Heisenberg group; for details see [H.M] and [B.H.M] : A minimal abelian surface A in $\mathbf{P}^4$ is of degree 10 and has polarization type $(1,5)$. It is the zerolocus of a section of $\mathcal{F}$,the HM-bundle ; $c_1(\mathcal{F}) = 5$, and we have an exact sequence

(1.1) $$0 \longrightarrow \mathcal{O}_{\mathbf{P}^4} \longrightarrow \mathcal{F} \longrightarrow I_A(5) \longrightarrow 0$$

The Heisenberg group $H =< \sigma, \tau >$ acts on the standard basis of $\mathbf{P}^4 = \mathbf{P}(\mathbf{C}^5)$, by $\sigma : e_k \longmapsto e_{k+1}$, $\tau : e_k \longmapsto \epsilon^k e_k$, where $\epsilon = e^{2\pi i/5}$ and indices are to be read modulo 5.

We have $h^0(\mathcal{F}) = 4$, and a property of $\mathcal{F}$ is

(1.2) $$\textstyle\bigwedge^2 H^0(\mathcal{F}) = H^0(\bigwedge^2 \mathcal{F})^H = H^0(\mathcal{O}_{\mathbf{P}^4}(5))^H.$$

The latter linear system of rank 6 has as base locus 25 lines:

(1.3) $$L_{ij} = \sigma^i \tau^j \{ \; x_0 = x_1 + x_4 = x_2 + x_3 = 0 \; \} \; ,0 \leq i,j \leq 4.$$

After blowing up $\mathbf{P}^4$ in the 25 lines, the linear system gives a morphism ξ

(1.4) $$\begin{array}{ccc} \tilde{\mathbf{P}}^4 & & \\ \downarrow & \searrow^{\xi} & \\ \mathbf{P}^4 & \dashrightarrow & \Omega \subset \mathbf{P}^5 \end{array}$$

Here $\Omega = \mathrm{Im}\xi$ is the Grassmannian of lines in $\mathbf{P}^3$ considered in $\mathbf{P}^5$ via the Plücker embedding. Generically ξ is $100:1$, and it is seen to be given by

$x \in \mathbf{P}^4 \longmapsto$ { The pencil of sections of $\mathcal{F}$ vanishing in x }.

So a point $p \in \Omega$ corresponds to a pencil $s_1 \wedge s_2, s_i \in H^0(\mathcal{F}), i = 1,2$ (compare with (1.2)). Let V_p denote the inverse image under ξ of the tangent hyperplane section of Ω in p; $V_p = \{ x \in \mathbf{P}^4 \mid s_1(x) \wedge s_2(x) = 0 \}$. Then V_p is singular in the , for p general, 100 points $\xi^{-1}(p) = \{ x \in \mathbf{P}^4 \mid s_1(x) = s_2(x) = 0 \}$ (the intersection of two abelian surfaces of degree 10), and V_p has no other singularities when p is general; *V_p is said to*

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be an HM-quintic. So a Zariski open subset of Ω parametrizes the HM-quintics, and an HM-quintic is determined by one of its 100 singular points.
(1.5) The quintic hypersurface in $\mathbf{P}^4$ of C.Schoen [S2] is

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5x_0x_1x_2x_3x_4 = 0.$$

This quintic is Heisenberg invariant, but it has 125 ordinary double points; $\{ (1, \epsilon^a, \epsilon^b, \epsilon^c, \epsilon^{-(a+b+c)}) \mid a, b, c \in Z_5 \}$, so it is not an HM-quintic. The point on the Grassmannian corresponding to this quintic is the (single) image point under ξ of the 100 points obtained by letting the Heisenberg group act on the four points $(1, 1, \epsilon^k, \epsilon^{-k}, 1), k = 1, 2, 3, 4$. We will need this quintic to use it as a specialization of HM-quintics, and for this reason we can only state the main results for a *general* HM-quintic.

§2 A small resolution and the defect of a general HM-quintic.

Let the HM-quintic V be defined by $s_1 \wedge s_2 = 0, s_i \in H^0(\mathcal{F}), i = 1, 2$ and consider $\tilde{V} = \{(x, (\lambda, \mu)) \in \mathbf{P}^4 \times \mathbf{P}^1 \mid \lambda s_1(x) + \mu s_2(x) = 0\}$. Then by projection to the first factor $\tilde{V} \longrightarrow V$ is a small resolution; i.e. $\tilde{V}$ is smooth and a singular point of V, say $p \in \{x \in \mathbf{P}^4 \mid s_1(x) = s_2(x) = 0\}$ is replaced by a $\mathbf{P}^1$; denote it by L_p. For a surface X in V, let $\tilde{X}$ denote the proper transform of X under $\tilde{V} \longrightarrow V$. The tangent cone of V in p has two rulings of planes. We can divide the set of surfaces in V being smooth in p, in two families according to which ruling the tangent plane in p belongs to. One family has the tangent planes of the pencil of abelian surfaces. For a surface X in this family $\tilde{X}$ is the blown up of X locally around p, and L_p becomes the exceptional divisor; $\tilde{X}.L_p = -1$ in $\tilde{V}$. If Y is a surface in the other family, then $\tilde{Y}$ is isomorphic to Y locally around p and $\tilde{Y}.L_p = 1$. If two smooth surfaces on V through p are linked to eachother, then they belong to different families.

Let S denote the singular locus of V. The defect of V is by definition the rank of the subvector space of $H_2(\tilde{V}; \mathbf{Q})$ spanned by $\{L_p\}_{p \in S}$ (see [W])). A result of C.Schoen [S1.Prop1.3] used on our quintic V says

(2.1) $$\operatorname{defect}(V) = h^1(I_S(5)).$$

Proposition 2.2: *Let $V = \{F = 0\}$ be a general HM-quintic. Then $H^0(I_S(5))$ is generated by $\{x_i \frac{\partial F}{\partial x_j}\}_{0 \leq i,j \leq 4}$ together with 4 Heisenberg invariant quintics linearly independent of F. The defect of V is 3.*

Remark 2.3: $\tilde{V}$ is a threefold with trivial canonical class, so $H^1(T_{\tilde{V}}) \simeq H^1(\Omega^2_{\tilde{V}})$, where $T_{\tilde{V}}$ denotes the tangent bundle of $\tilde{V}$ (see [C]). The space of first order deformations of $\tilde{V}$ is $H^1(T_{\tilde{V}})$, and $H^1(\Omega^2_{\tilde{V}})$ is isomorphic to $H^0(I_S(5))/ < x_i \frac{\partial F}{\partial x_j} >_{0 \leq i,j \leq 4}$ by Griffiths' residues. The proposition tells that these vector spaces have rank 4. It follows that the open subset of the Grassmannian of lines in $\mathbf{P}^3$ which parametrizes non-isomorphic HM-quintics, is the whole moduli space of HM-quintics.

Proof of the proposition: The Heisenberg group act as $\mathbf{Z}_5 \times \mathbf{Z}_5 =: G$ on $\mathbf{P}^4$ and on $H^0(\mathcal{O}_{\mathbf{P}^4}(5))$. An element of $H^0(\mathcal{O}_{\mathbf{P}^4}(5))^G$ is also invariant under the involution $\iota : e_k \longmapsto e_{-k}$. Hence there are points $p_l, l = 1, \ldots, 4$, such that $S = \bigcup_{l=1}^4 \bigcup_{g \in G} \{g(p_l)\}$, and $\iota(p_1) = \iota(p_2)$, $\iota(p_3) = \iota(p_4)$.

The evaluation mapping $\psi : H^0(\mathcal{O}_{\mathbf{P}^4}(5)) \longrightarrow H^0(\mathcal{O}_S)$ is a morphism of G-modules; $H^0(\mathcal{O}_S)$ is 4 times the regular representation of G, and $H^0(\mathcal{O}_{\mathbf{P}^4}(5)) = 6V_{0,0} \bigoplus_{(r,s) \neq (0,0)} 5V_{r,s}$

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where $V_{r,s}$ are the characters of $G =<\sigma> \times <\tau>$. A basis for $5V_{r,s}$ or $6V_{0,0}$ is

$$\mathcal{B}_{r,s} = \{\sum_{i=0}^{4} \epsilon^{-ri} \prod_{j=0}^{4} x_{i+j}^{m_j} \mid \sum j m_j \equiv s \pmod 5, \sum m_j = 5\}$$

By Schur's Lemma ψ decomposes; $\psi = \bigoplus \psi_{r,s}$, where for $(r,s) \neq (0,0), \psi_{r,s} : 5V_{r,s} \longrightarrow 4V_{r,s}$, and $\psi_{0,0} : 6V_{0,0} \longrightarrow 4V_{0,0}$ are given by $A \longmapsto [(A(p_l)]_{l=1,\dots,4}$. Consequently

$$H^0(I_S(5)) = \ker\psi = \bigoplus \ker\psi_{r,s}.$$

The mapping $\xi : \mathbf{P}^4 - - \rightarrow \mathbf{P}^5$ (defined by $6V_{0,0}$) of §1 sends the points $p_1, \dots, p_4$ to a single point p. Hence $\ker\psi_{0,0}$ comes from the set of hyperplanes in $\mathbf{P}^5$ through p; a 5-dimensional vector space. So F together with 4 other Heisenberg invariant quintics constitute a basis for $\ker\psi_{0,0}$.

In the sequel we assume that $(r,s) \neq (0,0)$. We want to show that the mapping $\psi_{r,s}$ is surjective. This is an open condition on S (or the image point $p \in \Omega$ under ξ), so it suffices to check it in a special case because V is assumed to be general. Choose $p_l = (1,1,\epsilon^l,\epsilon^{-l},1), l = 1,\dots,4$; the points considered in (1.5). Write $\mathcal{B}_{r,s} = \{f_{r,s}^j\}_{j=1,\dots,5}$ and let $M_{r,s} = [f_{r,s}^j(p_i)]_{1\le i\le 4, 1\le j\le 5}$. The Galois automorphism $\epsilon \longmapsto \epsilon^l$ sends $M_{r,s}$ to $M_{lr,s}$, and ι sends $M_{r,s}$ to $M_{-r,-s}$ (modulo permutations of rows and of columns). Hence it is enough to check that $M_{1,0}$; $M_{1,1}$, $M_{0,1}$; $M_{1,2}$ and $M_{0,2}$ have maximal rank. This is straight forward using the explicit bases $\mathcal{B}_{r,s}$, so we omit the calculations.

The 24-dimensional space $\bigoplus_{(r,s)\neq(0,0)} \ker\psi_{r,s}$ is contained in the space $< x_i \frac{\partial F}{\partial x_j} >_{0\le i,j\le 4}$. This follows when one considers the action of G on the latter space. From the cohomology of the exact sequence

$$0 \longrightarrow I_S(5) \longrightarrow \mathcal{O}_{\mathbf{P}^4}(5) \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

it follows by (2.1) that the defect of V is 3.

§3. Surfaces of low degree on V.

We will stick to the notation and the small resolution in §2.

First of all some wellknown facts for a smooth surface X in $\mathbf{P}^4$: Let d = the degree of X , $\chi = \chi(\mathcal{O}_X)$, H = the class of a hyperplane section of X , K = a canonical divisor and $\omega_X = \mathcal{O}_X(K)$, π = the sectional genus of X , and $q = h^1(\mathcal{O}_X)$ = the irregularity of X. Then (see [H, p.434])

(3.1) $$d^2 - 10d - 5H.K - 2K^2 + 12\chi = 0$$

and by adjunction

(3.2) $$H.K = 2\pi - 2 - d.$$

If X is contained in a hypersurface of degree n with only ordinary double points, then

(3.3) $$\pi = 1 + d/2\ (d/n + n - 4) - \mu/2n\ , \text{ where } \mu = \#(\mathrm{Sing}(V) \cap X).$$

(Proof: Use $\mu = c_2(I_X/I_X^2\ (n))$ and $d^2 = c_2(I_X/I_X^2)$)

If X and X' are linked by the hypersurfaces V_m and V_n (i.e. $X \cup X' = V_m \cap V_n$), of degree m and n respectively, then we have the exact sequence of linkage ([P.S])

(3.4) $$0 \longrightarrow \omega_X(5-(m+n)) \longrightarrow \mathcal{O}_{X\cup X'} \longrightarrow \mathcal{O}_{X'} \longrightarrow 0$$

We say that X is linked (m,n) to X'.

The following surfaces on V will be used to find generators of Pic$\tilde{V}$. To simplify the reference we will divide them into three "types".

Type A. Abelian surfaces on V:

By definition V contains a pencil of abelian surfaces of degree 10 passing through S, the 100 singular points of V. For such a surface A, the invariants are $d = 10$, $\pi = 6$, $K = 0$, $\chi = 0$, and $\tilde{A}.L_p = -1$ for all $p \in S$.

By (1.1) we have $h^0(I_A(5)) = 3$, spanned by three G-invariant quintics. This ideal is generated by global sections outside $\bigcup L_{ij}$, and we can link A to a smooth surface A' containing the 25 lines ; $A \bigcup A' = V \bigcap V_5$, where V_5 is a G-invariant quintic. From (3.4) we get

(3.5) $$0 \longrightarrow \omega_{A'} \longrightarrow \mathcal{O}_{V \cup V_5}(5) \longrightarrow \mathcal{O}_A(5) \longrightarrow 0.$$

Since $h^0(\mathcal{O}_{V\cup V_5}(5)) = h^0(\mathcal{O}_{\mathbf{P}^4}(5)) - 2 = 124$ and $h^0(\mathcal{O}_A(5)) = 125$ by Riemann-Roch, the fact $h^0(I_A(5)) = 3$ implies $h^0(\omega_{A'}) = 1$ and $h^1(\omega_{A'}) = h^1(\mathcal{O}_{A'}) = 2$; hence $\chi(\mathcal{O}_{A'}) = 0$. By (3.3) and (3.2) we get $H.K = 25$, and by taking global sections of (3.5) we find $K = \sum L_{ij}$. Since $K^2 = -25$ by (3.1), we can blow down the 25 skew lines L_{ij}, and we get an abelian surface by the Enriques' classification. Since A' is linked to A, we have $\tilde{A}'.L_p = 1$ for all $p \in S$. In Corollary 4.11 we will see that A' has polarization type (2,10).

Conclusion: The surfaces of type A constitute two linear systems of divisors in $\tilde{V}$.

Type B. Determinantal surfaces of degree 10 on V:

At first, consider a slightly more general setting: Let $M = M(x)$ be a 5×5 matrix with linear forms in $x_0, \ldots, x_4$ as entries , and let V_M denote the quintic hypersurface in $\mathbf{P}^4$ defined by $\det M = 0$. The degeneracy locus $\{ x \in \mathbf{P}^4 \mid \text{rank} M(x) < 4\}$ is contained in the singular locus of V_M. By Giambelli's formula [F,p.261] the degeneracy locus consists of 50 points or it has dimension one or more. We will assume that M is general in the following sense: V_M has only isolated singularities (but possibly more than 50) and M drops rank (rank $= 3$) in 50 distinct points.

Consider a general rank-4 subvector space of the row/column space of M ; it can be represented by a 4×5 matrix $N = N(x)$. By Giambelli's formula

$$\Delta = \{x \in \mathbf{P}^4 \mid \text{Rank} N(x) = 3\}$$

is a surface of degree 10. This surface is projectively Cohen-Macaulay with resolution

$$0 \longrightarrow 4\mathcal{O}_{\mathbf{P}^4}(-5) \xrightarrow{\;{}^tN\;} 5\mathcal{O}_{\mathbf{P}^4}(-4) \xrightarrow{\;L\;} \mathcal{O}_{\mathbf{P}^4} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0,$$

where L is the dot product with the maximal minors of N. The surface Δ passes through the 50 singular points of V_M considered above, so $\pi = 11$ by (3.3). From the resolution we find $\chi = 5$, and $K^2 = 5$ by (3.1).

A surface Δ_R coming from the row space of M is linked (4,5) to a surface Δ_C from the column space. Restricting the exact sequence of linkage to Δ_C we get

$$0 \longrightarrow \omega_{\Delta_C}(-4) \longrightarrow \mathcal{O}_{\Delta_C} \longrightarrow \mathcal{O}_{\Delta_R \cap \Delta_C} \longrightarrow 0$$

So $K = 4H_{\Delta_C} - \Delta_R \cap \Delta_C$, and the canonical mapping of Δ_C is induced by the Cremona transformation given by 5 independent quartics containing Δ_R (i.e. 5 maximal minors of the matrix defining Δ_R). We have a diagram

$$\begin{array}{ccccc} & & I & & \\ & {}^{p_1}\swarrow & & \searrow^{p_2} & \\ V_M & & - - \rightarrow & & V_{M'} \subset \mathbf{P}^4 \\ \cup & & |K| & & \cup \\ \Delta & & - - \rightarrow & & S_5 \end{array}$$

where $I = \{(x,x') \in \mathbf{P}^4 \times \mathbf{P}^4 \mid M(x)^t[x'] = \mathcal{O}\}$, and M' is the matrix satisfying $M'(x')^t[x] = M(x)^t[x']$. The projections $p_i, i = 1,2$, contract 50 lines; the "kernel" of the matrix where it has rank 3. Hence $V_M - - \rightarrow V_{M'}$ replaces 50 singular points with 50 lines and contracts 50 lines to 50 singular points. V_M has only isolated singularities iff the same holds for $V_{M'}$. The surface S_5 is a hyperplane section of $V_{M'}$, so it is smooth when Δ_C is general.

Back to the HM-quintics: A result of R.Moore is that the equation of an HM-quintic can be expressed as determinants: Let $M_y(x) = [x_{i+j}y_{i-j}]_{0 \le i,j \le 4}$ for $y \in \mathbf{P}^4$, i.e.

$$M_y(x) = \begin{bmatrix} x_0y_0 & x_1y_4 & x_2y_3 & x_3y_2 & x_4y_1 \\ x_1y_1 & x_2y_0 & x_3y_4 & x_4y_3 & x_0y_2 \\ x_2y_2 & x_3y_1 & x_4y_0 & x_0y_4 & x_1y_3 \\ x_3y_3 & x_4y_2 & x_0y_1 & x_1y_0 & x_2y_4 \\ x_4y_4 & x_0y_3 & x_1y_2 & x_2y_1 & x_3y_0 \end{bmatrix}$$

Then $F_y = \det M_y(x)$ is a Heisenberg invariant quintic as is easily checked by the action on the matrix. Furthermore , $\mathrm{Rank} M_y(x) = 3$ in the 50 points $< G, \iota > \{y\}$.

Let $\xi_0, \ldots, \xi_5$ be a basis for $H^0(\mathcal{O}_{\mathbf{P}^4}(5))^G$. Since F_y is also invariant letting G act on y, we have $F_y = \sum a_{ij}\xi_i(y)\xi_j(x)$, $a_{ij} \in \mathbf{C}$. The mapping ξ is defined by $\xi_0, \ldots, \xi_5$, and it is $100 : 1$, so there are points y_1 and y_2 in $\mathbf{P}^4$ such that for our given quintic we have $F = F_{y_1} = F_{y_2}$, and for the singular locus $S = < G, \iota > \{y_1, y_2\}$.

Let Δ_1 (resp. Δ_2) be a determinantal surface from the rowpace of M_{y_1} (resp M_{y_2}). The involution ι transposes $M_y(x)$ (modulo permutations of rows and of columns), so Δ_j is linked to $\iota\Delta_j$, $j = 1,2$. Since ι leaves an abelian surface of type A invariant , we can divide S into four groups each invariant under G, such that (after possibly interchanging Δ_j and $\iota\Delta_j$) we have the intersection numbers

Table 3.6 :

X	$\tilde{X}.L_1$	$\tilde{X}.L_2$	$\tilde{X}.L_3$	$\tilde{X}.L_4$
Δ_1	$+1$	-1	0	0
Δ_2	0	0	$+1$	-1

Here L_k denotes any of the 25 exceptional lines in group k, and $\iota L_1 = L_2$, $\iota L_3 = L_4$.

The matrix $M_y(x)$ gives rise to a matrix $M'_y(x')$ under the Cremona transformation defined above. Let $x' = x$ and $y = z$, then

$$M'_z(x) = \begin{bmatrix} x_0z_0 & x_1z_4 & x_2z_3 & x_3z_2 & x_4z_1 \\ x_4z_2 & x_0z_1 & x_1z_0 & x_2z_4 & x_3z_3 \\ x_3z_4 & x_4z_3 & x_0z_2 & x_1z_1 & x_2z_0 \\ x_2z_1 & x_3z_0 & x_4z_4 & x_0z_3 & x_1z_2 \\ x_1z_3 & x_2z_2 & x_3z_1 & x_4z_0 & x_0z_4 \end{bmatrix}$$

As for $M_y(x)$ one finds $G_z := \det M'_z(x) = \sum b_{ij}\xi_i(z)\xi_j(x)$, $b_{ij} \in \mathbf{C}$. The assignment $F_y \longmapsto G_z$ is well defined and gives a bijection of the set of HM-quintics. So for our given quintic F, there are points z_1 and z_2 in $\mathbf{P}^4$ such that $F = G_{z_1} = G_{z_2}$ and $< G, \iota >\{z_1, z_2\}$ consists of 100 points. Write the singular locus S as $< G, \iota >\{y_1, y_2\}$.

Claim: We can order z_1, z_2 and y_1, y_2 such that

Table 3.7: $\text{Rank}M'_z(y) =$

$z\backslash y$	y_1	ιy_1	y_2	ιy_2
z_1	4	3	3	4
ιz_1	3	4	4	3
z_2	4	3	4	3
ιz_2	3	4	3	4

Proof: $M'_{\iota z}(\iota y)$ is equal to $M'_z(y)$ (modulo permutations of rows and of columns), so it suffices to check row 1 and 3 in the table. Consider $I = \{(y,z) \in \mathbf{P}^4 \times \mathbf{P}^4 \mid F_y = G_z\}$. The rank of $M'_z(y)$ is 4 or less when $(y,z) \in I$ because $G_z(y) = F_y(y) = 0$, and there exist 50 y's such that $\text{Rank}M'_z(y) = 3$. If $\text{Rank}M'_z(y) = 4$,then the same is true in a neighbourhood of $(y,z) \in I$. So for our *general* quintic it suffices to check the table in a special case:

Let $y_1 = z_1 = (1,1,\epsilon,\epsilon^4,1)$ and $y_2 = z_2 = (1,1,\epsilon^2,\epsilon^3,1)$. Then $\text{Rank}M'_{z_1}(y) = 3$ for $y \in \{\iota y_1, y_2\}$ (two rows coincide) ; hence $G_{z_1} = F_{\iota y_1} = F_{y_2}$ is Schoen's quintic (see (1.5)), and $\text{Rank}M'_{z_1} = 4$ for $y \in \{y_1, \iota y_2\}$ since this quintic has isolated singularities. Under the Galois automorphism $\epsilon \longmapsto \epsilon^2$, we have $z_1 \longmapsto z_2$,$\{y_1, \iota y_2\} \longmapsto \{y_1, y_2\}$, $\{\iota y_1, y_2\} \longmapsto \{\iota y_1, \iota y_2\}$, and $G_{z_2} = F_{\iota y_1} = F_{\iota y_2}$. Hence $G_{z_1} = G_{z_2}$, and the table follows.

From the matrices $M'_{z_j}(x)$ and $M'_{\iota z_j}, j = 1,2$, we find 8 divisor classes in $\tilde{V}$ of determinantal surfaces, say represented by $\nabla_1, \iota\nabla_1, \nabla_2, \iota\nabla_2$ together with their linked determinantal surfaces. From Table 3.7 we have

Table 3.8:

X	$\tilde{X}.L_1$	$\tilde{X}.L_2$	$\tilde{X}.L_3$	$\tilde{X}.L_4$
∇_1	0	$\star$	$\star$	0
$\iota\nabla_1$	$\star$	0	0	$\star$
∇_2	0	$\star$	0	$\star$
$\iota\nabla_2$	$\star$	0	$\star$	0

We will see in Lemma 4.3 that $\star = 1$ by choice of ∇_1 and ∇_2.

Conclusion: There are 12 linear systems in $\tilde{V}$ of surfaces of type B. Such a surface in V is isomorphic to a quintic surface in $\mathbf{P}^3$ via the canonical mapping.

Type C. Surfaces of degree 20.

Let $X_j \in \{\Delta_j, \iota\Delta_j\}$ with Δ_j from Table 3.6, $j = 1,2$. Consider the linear system $\mid \tilde{X}_1 + \tilde{X}_2 \mid$ in $\tilde{V}$. Let $X'_j = \iota X_j$, so $X_j + X'_j = 4H$ where H denotes a hyperplane section of V. Let ψ be the natural mapping $H^0(\mathcal{O}_{\tilde{V}}(4H - \tilde{X}'_1)) \otimes H^0(\mathcal{O}_{\tilde{V}}(4H - \tilde{X}'_2)) \longrightarrow H^0(\mathcal{O}_{\tilde{V}}(8H - \tilde{X}'_1 - \tilde{X}'_2)) = H^0(\mathcal{O}_{\tilde{V}}(\tilde{X}_1 + \tilde{X}_2))$. Since X'_1 and X'_2 are both cut out by quartics, Bertini's theorem implies that a general member $\tilde{X}$ of $\text{Im}\psi$ is a smooth surface. The image X of $\tilde{X}$ in V is also smooth since $X_1 \bigcup X_2$ is smooth in the 100 singular points of V.

Let $X' = \iota X$,then X' is linked (8,5) to X and $X' \simeq X$. By (3.3) $\pi_X = 41$, and we have by (3.4)

$$0 \longrightarrow \omega_X(-8) \longrightarrow \mathcal{O}_{X \cup X'} \longrightarrow \mathcal{O}_{X'} \longrightarrow 0.$$

Riemann-Roch yields $\chi(\omega_X(-8)) = \chi(\mathcal{O}_X) + 400$, and $\chi(\mathcal{O}_{X \cup X'}) = 460$. Since $\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_X)$, we find $\chi(\mathcal{O}_X) = 30$, and by (3.1) $K_X^2 = 130$.

The surface X is regular: The Koszul complex defining the (connected) complete intersection $\tilde{X}_1 \bigcap \tilde{X}_2$ in $\tilde{V}$ induces the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}_1 \cup \tilde{X}_2} \longrightarrow \mathcal{O}_{\tilde{X}_1} \oplus \mathcal{O}_{\tilde{X}_2} \longrightarrow \mathcal{O}_{\tilde{X}_1 \cap \tilde{X}_2} \longrightarrow 0$$

Hence regularity of $\tilde{X}_1$ and $\tilde{X}_2$, implies regularity of $\tilde{X}_1 \bigcup \tilde{X}_2$. By semicontinuity, $\tilde{X}$ and then X are both regular.

Conclusion: There are four linear systems of surfaces of type C. The invariants of such a surface are $d = 20$, $\pi = 41$, $\chi = 30$, $K^2 = 130$, and $q = 0$, so the surface is of general type.

§4. The Picard group of $\tilde{V}$ and the smooth surfaces on V.

The Picard group of a small resolution of a nodal hypersurface in $\mathbf{P}^4$ is torsion free by Lefschetz' theorem, and the rank is the defect + 1 (see [W,p.7]). Hence Pic$\tilde{V} \simeq \mathbf{Z}^4$.

Lemma 4.1: *Let Δ_1 and Δ_2 be the surfaces of Table 3.6, A an abelian surface of degree 10, and H the pullback of a hyperplane section of V. Then $\{H, \tilde{A}, \tilde{\Delta}_1, \tilde{\Delta}_2\}$ is a basis for* Pic$\tilde{V} \otimes \mathbf{Q}$.

Proof: We have the following intersections

Table 4.2:

X	$\tilde{X}.L_1$	$\tilde{X}.L_2$	$\tilde{X}.L_3$	$\tilde{X}.L_4$
H	0	0	0	0
A	−1	−1	−1	−1
Δ_1	+1	−1	0	0
Δ_2	0	0	+1	−1

It follows immediately that the four divisors are independent using this table.

For a determinantal surface Y let Y' denote a linked determinantal surface. Recall that $\Delta'_j = \iota\Delta_j = 4H - \Delta_j, j = 1,2$.

Lemma 4.3: *Let $\nabla_1, \iota\nabla_1, \nabla_2$, and $\iota\nabla_2$ be the surfaces of Table 3.8. After possibly interchanging ∇_1 and ∇'_1 or ∇_2 and ∇'_2 ,we have the intersections*

Table 4.4:

X	$\tilde{X}.L_1$	$\tilde{X}.L_2$	$\tilde{X}.L_3$	$\tilde{X}.L_4$
∇_1	0	+1	+1	0
$\iota\nabla_1$	+1	0	0	+1
∇_2	0	+1	0	+1
$\iota\nabla_2$	+1	0	+1	0

Proof: Choose ∇_j such that the intersection of $\tilde{\nabla}_j$ with one of the lines is +1. Since the surfaces have degree 10 , we find by Lemma 4.1 and Table 4.2:

$$\textbf{(4.5)}\quad \begin{array}{ll} \tilde{\nabla}_1 = -1/2\,\tilde{A} - 1/2\,\tilde{\Delta}_1 + 1/2\,\tilde{\Delta}_2 + 3H\,, & \iota\tilde{\nabla}_1 = -1/2\,\tilde{A} + 1/2\,\tilde{\Delta}_1 - 1/2\,\tilde{\Delta}_2 + 3H \\ \tilde{\nabla}_2 = -1/2\,\tilde{A} - 1/2\,\tilde{\Delta}_1 - 1/2\,\tilde{\Delta}_2 + 5H\,, & \iota\tilde{\nabla}_2 = -1/2\,\tilde{A} + 1/2\,\tilde{\Delta}_1 + 1/2\,\tilde{\Delta}_2 + H \end{array}$$

and the table follows.

Proposition 4.6: $\{H, \tilde{A}, \tilde{\nabla}_1, \tilde{\nabla}_2\}$ *is a basis for* Pic$\tilde{V}$.

Proof: The set is clearly a basis for Pic$\tilde{V} \otimes \mathbf{Q}$, and if $\tilde{X} = nH + a\tilde{A} + b_1\tilde{\nabla}_1 + b_2\tilde{\nabla}_2 \in$Pic$\tilde{V}$, then $\tilde{X}.L_k \in \mathbf{Z}$ implies a, b_1, and $b_2 \in \mathbf{Z}$. That $n \in \mathbf{Z}$ follows by intersecting X with a line in V (f.i. L_{00}).

Theorem 4.7: *Let X be a smooth surface in V. Then X is either a complete intersection of V and another hypersurface in $\mathbf{P}^4$, or X is linked to a surface of type A,B, or C, by V and another hypersurface.*

Proof: We will work in Pic$\tilde{V}/ < H >$ since we are only interested in linkage. Write $\tilde{X} \in$ Pic$\tilde{V}/ < H >$ as $a\tilde{A} + b_1\tilde{\nabla}_1 + b_2\tilde{\nabla}_2$,then $\tilde{X}.L_1 = -a$, $\tilde{X}.L_2 = -a + b_1 + b_2$, $\tilde{X}.L_3 = -a + b_1$, $\tilde{X}.L_4 = -a + b_2$, and $\tilde{X}.L_k \in \{-1, 0, 1\}$ since X is smooth.

For $a = 1$ one gets the following solutions for $\bar{X}$ using (4.5) :
$\tilde{A}$, $\quad \tilde{A} + \tilde{\nabla}_2 \equiv -\iota\tilde{\nabla}_2$ (in $\mathrm{Pic}\tilde{V}/< H >$), $\quad \tilde{A} + 2\tilde{\nabla}_2 \equiv -\tilde{\Delta}_1 - \tilde{\Delta}_2$,
$\tilde{A} + \tilde{\nabla}_1 \equiv -\iota\tilde{\nabla}_1$, $\quad \tilde{A} + \tilde{\nabla}_1 + \tilde{\nabla}_2 \equiv \tilde{\Delta}_1$, $\quad \tilde{A} + 2\tilde{\nabla}_1 \equiv -\tilde{\Delta}_1 + \tilde{\Delta}_2$.

For $a = 0$: $\pm\tilde{\nabla}_1$, $\quad -\tilde{\nabla}_1 + \tilde{\nabla}_2 \equiv -\tilde{\Delta}_2$, $\quad \pm\tilde{\nabla}_2$, $\quad 0$, and $\tilde{\nabla}_1 - \tilde{\nabla}_2 \equiv \tilde{\Delta}_2$.

For $a = -1$ one has the negative of the solutions for $a = 1$, and the theorem follows.

Remark 4.8: The minimal degree of a hypersurface used to link a surface of type A (resp. B) to a smooth surface is 5 (resp. 4). This number is 8 or possibly 7 for a surface X of type C: The linkage $X \bigcup X' = V \bigcap V_8$ yields

(4.9) $$0 \longrightarrow \omega_{X'}(-1) \longrightarrow \mathcal{O}_{V\cap V_8}(7) \longrightarrow \mathcal{O}_X(7) \longrightarrow 0$$

By Riemann-Roch and Severi's theorem $h^0(\omega_{X'}(-1)) \geq \chi(\omega_{X'}(-1)) - h^2(\omega_{X'}(-1)) = 10 - 5$; hence by (4.9) , X is contained in at least 5 septics (not being a multiple of V). It is not known if one can link X to a *smooth* surface of degree 15 on V by such a septic (if possible, then the invariants are $\chi = 5, K^2 = 5$). It is easy to check that a sextic cannot be used.

Corollary 4.10: *The only irregular surfaces on V are the abelian surfaces of type A and possibly the surfaces of degree 15 in Remark 4.8.*

Proof: This follows from a more general fact. Suppose a surface X is linked (m, n) to a surface X'. Let $a = m + n - 5$, then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_{X'}(s) & \longrightarrow & \mathcal{O}_{X\cup X'}(a+s) & \xrightarrow{\lambda} & \mathcal{O}_X(a+s) & \longrightarrow & 0 \\
 & & & & \uparrow \psi & & \| & & \\
0 & \longrightarrow & I_X(a+s) & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}(a+s) & \xrightarrow{\gamma} & \mathcal{O}_X(a+s) & \longrightarrow & 0
\end{array}$$

$H^0(\psi)$ is surjective and $H^1(\mathcal{O}_{X\cup X'}(a+s)) = 0$; hence
$H^1(\omega_{X'}(s)) = \mathrm{coker} H^0(\lambda) = \mathrm{coker} H^0(\gamma) = H^1(I_X(a+s))$.
By the Kodaira vanishing theorem we then get $H^1(I_X(a+s)) = 0$ when $s > 0$. So if X is linked (m', n') to a surface X'', then for $a' = m'+n'-5$, we get $H^1(\omega_{X''}) = H^1(I_X(a')) = 0$, when $a' > a$. If we use a fixed hypersurface in the linkage ($m = m'$), then it follows that only a surface of minimal degree linked to X can be irregular.

Corollary 4.11: *A degree 15 abelian surface of type A has polarization (2,10).*

Proof: Consider a Cremona transformation between two HM-quintics as defined in §3; $\psi : V' - - \to V$. Let $X \subset V$ be abelian of degree 15. It is easily checked using the matrix representation of ψ, that the 25 lines $\bigcup L_{ij}$ in V are "blow ups" of 25 points in V', and that the lines $\bigcup L_{ij}$ in V' are *not* contracted by ψ. Hence $X' = \psi^{-1}(X)$ is a surface passing smoothly through 75 of the singular points of V'. So X' is smooth by Proposition 4.6 and the intersection tables; hence by Remark 4.8 and Corollary 4.9, X' is of type A. Since the lines $\bigcup L_{ij}$ in V' are not contracted, X' is minimal so it has degree 10.

ψ is given by a linear system $|\Delta|$, where Δ is a determinantal surface. Since $\Delta + \iota\Delta = 4H$, the mapping ψ restricts as a subsystem of $2H|_{X'}$ (assuming X' is general with Picard number 1). So X has polarization (2,10) because X' has polarization (1,5).

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Curves of genus three on a general abelian threefold and the non–finite generation of the Griffiths group

Fabio Bardelli

Introduction Let A be a general principally polarized abelian threefold defined over $\mathbb{C}$, and $\mathcal{G}(A)$ its Griffiths group defined as

$$\mathcal{G}(A) = \frac{\{\text{ algebraic 1-cycles on } A \text{ homologous to } 0\}}{\{\text{ algebraic 1-cycles on } A \text{ algebraically equivalent to } 0\}}$$

We denote by $[-1] : A \to A$ the "multiplication by -1 " map. The main result of this paper is the following:

Theorem: *A contains infinitely many birationally distinct irreducible curves $\{C_k\}_{k\in\mathbb{N}}$ of geometric genus 3. Furthermore there are infinitely many algebraically independent 1-cycles $C_k - [-1]C_k$ on A. In particular $\mathcal{G}(A)$ is not finitely generated.*

Let $\mathcal{H}_3$ be the upper-half Siegel space of genus 3 and $\pi : \mathcal{A} \to \mathcal{H}_3$ the "universal" abelian threefold over it. Here is an outline of the paper. In section 1 we associate to each element g of the rational symplectic group $G_{\mathbb{Q}} = \mathrm{Sp}(6, \mathbb{Q})$ an isogeny $i_g : A_{g(p)} \to A_p$, where $A_Q = \pi^{-1}(Q)$; this association is compatible with the $G_{\mathbb{Q}}$-action over $\mathcal{H}_3$. Since $G_{\mathbb{Q}}$ is dense in the real symplectic group $G_{\mathbb{R}}$ (which acts transitively on $\mathcal{H}_3$), one can easily produce the infinitely many curves $\{C_k\}_{k\in\mathbb{N}}$ of the statement of the theorem by considering the image under i_g of some Abel-Jacobi embedded curve C in $A_{g(p)} = J(C)$ (see prop. 3.2). We prove in section 3 that the algebraic equivalence class of the cycle $C - [-1]C$ in a general jacobian $J(C)$ is not of finite order (thus giving another proof of G. Ceresa's theorem (see [Ce.])), and in section 4 that this algebraic equivalence class is carried by the monodromy action around the hyperelliptic locus $\mathcal{K} \subset \mathcal{H}_3$ into the class of the cycle $[-1]C - C$. We extend this result to each cycle $C_k - [-1]C_k$ by considering the monodromy action around its corresponding translate of $\mathcal{K}$ in $\mathcal{H}_3$. We observe that the cycle $C_k - [-1]C_k$ is defined only locally over $\mathcal{H}_3$ and therefore, to circumvent the problems arising from this ambiguity (and its density in $\mathcal{H}_3$!), we rely on the study of the local normal functions associated to each $C_k - [-1]C_k$ with values in the primitive intermediate jacobian bundle of the family $\pi : \mathcal{A} \to \mathcal{H}_3$. Since finally we have produced infinitely many cycles $C_k - [-1]C_k$ with infinitely many *distinct* branching divisors in $\mathcal{H}_3$ (which are the translates of $\mathcal{K}$ under the $G_{\mathbb{Q}}$-action) we can easily conclude the proof of the theorem in section 5.

Some years ago Nori had proved the non-finite generation of $\mathcal{G}(A)$ by using representation theory (letter to H. Clemens). As yet his proof has not appeared and so we decided to work out all the details of our proof in which the explicit construction of the infinitely many cycles $C_k - [-1]C_k$ is given and their behaviour with respect to the monodromy action is studied.

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Notations

p.p.a.v.	stands for	principally polarized abelian variety
c.u.p.a.s.	stands for	countable union of proper analytic subvarieties

$h^i(X,\mathcal{I}) = \dim H^i(X,\mathcal{I})$

Very often the letter denoting a certain abelian variety will also be used to denote the point associated to it in $\mathcal{H}_3$ = Siegel upper half space. Same remark for pairs (C,X) C curve contained in a variety X and the corresponding point in the appropriate Chow variety.

All varieties will be defined over $\mathbb{C}$ and considered with their complex manifold topology.

1 Isogenies and the rational symplectic group

1.1. Let L be a lattice of rank $2n$, ϑ an alternating, integral valued, unimodular bilinear form on L. We will denote by $G_{\mathbb{Z}}$, $G_{\mathbb{Q}}$, $G_{\mathbb{R}}$ respectively the integral, rational, real symplectic group $\mathrm{Sp}(\vartheta,\cdot)$ of the form ϑ. Set $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ and consider a linear map $g : L_{\mathbb{Q}} \to L_{\mathbb{Q}}$; $g \in G_{\mathbb{Q}}$. Clearly $g(L) \not\subset L$ in general, so we consider the set

$$I = \{h \in \mathbb{Z} : h \cdot g(L) \subset L\}.$$

I is an ideal; therefore $I = \langle k \rangle$ for $k \in \mathbb{Z}$ and we define $L_g = k \cdot g(L) \subset L$:
L_g is a sublattice of finite index in L and we obviously have

1.1.1 for each $m \in \mathbb{Z}$ $\quad m \cdot kg(L) = mL_g$

1.1.2 for any $x,\ y \in L$ $\quad \vartheta(k \cdot g(x), k \cdot g(y)) = k^2\vartheta(x,y)$; so $(kg)^*\vartheta = k^2\vartheta$ and therefore $\vartheta|_{L_g}$ is a multiple of an integral valued unimodular form on L_g: we will denote this form by ϑ_g.

1.2. We recall that we can construct any p.p.a.v. of dimension n by assigning an isomorphism $L \otimes_{\mathbf{Z}} \mathbf{R} =: L_{\mathbf{R}} \xrightarrow{i} \mathbf{C}^n$ such that ϑ is of type (1,1) and positive when read over $\mathbf{C}^n$; and in fact the abelian variety obtained in this way is $A = \mathbf{C}^n/i(L) = \mathbf{C}^n/L$.

The construction in 1.1. allows to associate to each element $g \in G_{\mathbf{Q}}$ the abelian variety $A_g =: \mathbf{C}^n/L_g$ (principally polarized by ϑ_g) together with the isogeny i_g

$$A_g = \mathbf{C}^n/L_g \xrightarrow{i_g} \mathbf{C}^n/L = A$$

where i_g is induced by the inclusion $L_g \subset L$. By 1.1.2. $i_g^*(\vartheta(A)) = k^2\vartheta_g(A_g)$ for $k \in \mathbf{Z}$, where $\vartheta(A)$ and $\vartheta_g(A_g)$ are the principal polarizations induced by ϑ and ϑ_g on A and A_g respectively; furthermore by looking at all the isogenies induced by $m \cdot kg$, $m \in \mathbf{Z}$, we obtain factorizations

$$A_g \xrightarrow{\cdot m} A_g \xrightarrow{i_g} A$$

so that the abelian variety A_g does not depend on the choice of the element $m \cdot kg$ such that $mkg(L) \subset L$, whereas the isogeny i_g does and is in fact modified by a multiplication by m. In the sequel we will always refer to the unique isogeny $A_g \to A$ determined by the inclusion $L_g \subset L$.

1.3. In order to see how the period matrices of A and A_g are related it is enough to choose a symplectic basis $\{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n\}$ of L and a (uniquely determined) basis $\{e_1, \ldots, e_n\}$ of $\mathbf{C}^n$ such that the period matrix of A is

$$\begin{pmatrix} \Omega \\ I \end{pmatrix}$$

where Ω is an $n \times n$ matrix with $\Omega = {}^t\Omega$ $\mathrm{Im}\Omega > 0$, I the identity matrix.

Let $g \in G_{\mathbf{Q}}$, then $g(\alpha_r) = \sum_i a_{ri}\alpha_i + \sum_i b_{ri}\beta_i$; $g(\beta_r) = \sum_i c_{ri}\alpha_i + \sum_i d_{ri}\beta_i$ for $r = 1, \ldots, n$; so g is represented by a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A = \{a_{ri}\}$ and so on; kg can be computed consequently and therefore the period matrix of A_g with respect to the basis $\{kg(\alpha_1), \ldots, kg(\alpha_n); kg(\beta_1), \ldots, kg(\beta_n)\}$ and $\{e_1, \ldots, e_n\}$ is

$$\begin{pmatrix} k\,(A\Omega + B) \\ k\,(C\Omega + D) \end{pmatrix}$$

which can be obviously transformed (by changing the basis $\{e_1, \ldots, e_n\}$) into

$$\begin{pmatrix} (A\Omega + B) \cdot (C\Omega + D)^{-1} \\ \mathrm{Id} \end{pmatrix}$$

It follows that

1.3.1. *The period matrices of A and A_g are related by a Siegel modular substitution of degree n (see [Si.] p.128) defined over* $\mathbb{Q}$.

We therefore introduce $\mathcal{H}_n = \{\Omega$ matrices $n \times n$ with complex entries with $\Omega = {}^t\Omega; \operatorname{Im}\Omega > 0\}$ = the Siegel upper half space.

We recall that $\mathcal{H}_n$ carries a "universal" family of abelian varieties that we will denote by $\pi : \mathcal{A} \to \mathcal{H}_n$. We set $A_P = \pi^{-1}(P)$. $\mathcal{H}_n$ in fact may be thought of as a fine moduli space for the set of pairs $(A, \mathcal{B})$ where A is a p.p.a.v. and $\mathcal{B}$ a symplectic basis of $H_1(A, \mathbb{Z})$. Therefore from now on $G_{\mathbb{Z}}$, $G_{\mathbb{Q}}$, $G_{\mathbb{R}}$ will be the usual integral, rational, real symplectic groups acting on $\mathcal{H}_n$ by

$$\text{if } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{\mathbb{Q}},\ \Omega \in \mathcal{H}_n$$

$$g(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}.$$

We recall that:

1.3.2.

1. $G_{\mathbb{R}}$ acts on $\mathcal{H}_n$ transitively (see [Gr. I])
2. $G_{\mathbb{Q}}$ is dense in $G_{\mathbb{R}}$.

In view of 1.3.1 and 1.3.2 we can conclude that:

1.3.3. For each $P \in \mathcal{H}_n$ and each $g \in G_{\mathbb{Q}}$ we have $(A_P)_g = A_{g(P)}$, therefore for a fixed $P \in \mathcal{H}_n$ *the set of* p.p.a.v. *isogenous to A_P by the construction* 1.2. *is parametrized in $\mathcal{H}_n$ by the $G_{\mathbb{Q}}$-orbit of P which is a* dense *countable subset of $\mathcal{H}_n$.*

1.3.4. For each p.p.a.v. A one can consider $g_1 \in G_{\mathbb{Z}}$, $g_2 \in G_{\mathbb{Q}}$ and get a diagram:

$$A_{g_2 \cdot g_1} = \left(A_{g_1}\right)_{g_2} \xrightarrow{i_{g_2}} A_{g_1} \underset{\sim}{\xrightarrow{i_{g_1}}} A$$

Clearly i_{g_1} is an isomorphism of p.p.a.v. and therefore the isogenies $i_{g_2 \cdot g_1}$ and i_{g_2} correspond under the isomorphism i_{g_1}, that is $i_{g_2 \cdot g_1} = i_{g_1} \cdot i_{g_2}$.

1.4. We let $\mathcal{K}$ be the locus of jacobians of hyperelliptic curves in $\mathcal{H}_3$, and we let

$$S_{\mathcal{K}} = \left\{ g \in G_{\mathbb{Q}} \quad : \quad g(\mathcal{K}) \subset \mathcal{K} \right\}$$

the stabilizeur of $\mathcal{K}$ in $G_{\mathbb{Q}}$. $S_{\mathcal{K}}$ is an infinite group (it contains for instance $G_{\mathbb{Z}}$). We consider the set (quotient set)

$$\mathcal{Q} = G_{\mathbb{Q}} / S_{\mathcal{K}}$$

that is the set of equivalence classes of elements of $G_{\mathbb{Q}}$ under the relation $g_1 \sim g_2 \Leftrightarrow g_1^{-1} g_2 \in S_{\mathcal{K}}$. $\mathcal{Q}$ is an infinite set because the $G_{\mathbb{Q}}$-orbit of a point $P \in \mathcal{K}$ is dense in $\mathcal{H}_3$. We will denote by $\tilde{g}$ the equivalence class in $\mathcal{Q}$ of the element $g \in G_{\mathbb{Q}}$.

Clearly it makes sense to talk about $\tilde{g}(\mathcal{K})$ (that is this loci are well defined) and one has

$$\tilde{g_1} \neq \tilde{g_2} \qquad \Leftrightarrow \qquad \tilde{g_1}(\mathcal{K}) \neq \tilde{g_2}(\mathcal{K})$$

so that:

1.4.1. *$\mathcal{Q}$ is in a one to one correspondence with the set of* distinct *translates of* $\mathcal{K}$ *under the action of* $G_{\mathbb{Q}}$.

2 Some recalls: normal funcions

We want to recall several well known facts. We consider the intermediate jacobian bundle $\tilde{\pi} : J \to \mathcal{H}_3$ of the universal family $\pi : \mathcal{A} \to \mathcal{H}_3$. The fibre $\tilde{\pi}^{-1}(P)$ is just the intermediate jacobian $J(A_P)$ of the abelian threefold A_P.

2.1. For each $P \in \mathcal{H}_3$ we let ϑ be the cohomology class in $H^2(A_P, \mathbb{Z})$ of the polarizing form of A_P (or of its theta-divisor). The Lefschetz map

$$L : H^1(A_P, \mathbb{Q}) \xrightarrow{\cdot \wedge \vartheta} H^3(A_P, \mathbb{Q})$$

defines a mapping $\mathcal{L} : \mathrm{Pic}^0(A_P) \to J(A_P)$ whose kernel is finite. We define $J_{pr}(A_P) = J(A_P)/\mathrm{Im}\mathcal{L}$. Here pr stands for primitive. We can perform this construction uniformly over $\mathcal{H}_3$ and so a "primitive jacobian bundle" $J_{pr} \to \mathcal{H}_3$ is defined and its fibre over $P \in \mathcal{H}_3$ is $J_{pr}(A_P)$.

2.2. For each $P \in \mathcal{H}_3$ we denote by $J_H(A_P)$ the maximal compact complex subtorus of $J(A_P)$ all of whose lattice vectors are annihilated by $H^{3,0}(A_P)$. By using the derivative of the period mapping of our universal family $\pi : \mathcal{A} \to \mathcal{H}_3$ at the point P and specifically the part of it giving the map

$$H^{3,0}(A_P) \otimes H^1(A_P, \mathcal{T}_{A_P}) \longrightarrow H^{2,1}(A_P),$$

plus the fact that $T_P(\mathcal{H}_3) = H^1(A_P, \mathcal{T}_{A_P})_{(\vartheta)} = \{$ cohomology classes whose cup product with ϑ is 0 $\} = \{$ first order deformations of A_P for which ϑ is preserved of type $(1,1)$ in $H^2(A, \mathbb{C})\}$ see [Gr. II], one can see easily that for a general point $P \in \mathcal{H}_3$

$$J_H(A_P) = \mathrm{Im}\mathcal{L}.$$

Thus if for each open set U in $\mathcal{H}_3$ we define

$$H(U) = \{u \in U : \dim J_H(A_u) > 3\}$$

$H(U)$ is a c.u.p.a.s.. in U; and $J_{pr}(A_u) = J(A_u)/J_H(A_u)$ for each $u \in U \setminus H(U)$.
We also recall that the image under the Abel-Jacobi map of any algebraic 1-cycle $\mathcal{Z}$ algebraically equivalent to 0 in A_P lies in $J_H(A_P)$.

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