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INTRODUCTION

In the study of n-dimensional knots, i.e. imbedded n-spheres in (n + 2)-space, one encounters a collection of Λ-modules $A_1, \ldots, A_n$ (the Alexander modules), where $\Lambda = Z[t, t^{-1}]$, the ring of integral Laurent polynomials. These modules encompass many of the classical knot invariants.

The more important properties and relations among these modules are more easily stated in terms of the Z-torsion submodules $\{T_i\}$ and the quotients $F_i = A_i/T_i$. An important additional feature is the existence of a product structure on F_q, when $n = 2q + 1$, and T_q, when $n = 2q$. It is now understood exactly which collections $\{T_i, F_i\}$ of Λ-modules, with product structure on the correct term, arise from knots (except for T_1). See [L] for more detail.

In the present work we make an algebraic study of the types of modules and product structures which arise as Alexander modules. In particular, we introduce a collection of invariants which are reasonably tractable but sensitive enough to reflect the panorama of these modules. In some cases, they succeed in classifying but we will be most concerned with determining when a given set of invariants can be realized.

A preliminary reduction of the problem is obtained as follows. Let π be an irreducible element of Λ. For any

Λ-module A we can then consider the π-primary submodule A_π. If A is a Z-torsion module, then we consider π which are integer primes; in this case A splits as the direct sum of the $\{A_\pi\}$. If A is Z-torsion free, we consider π which are irreducible primitive polynomials—but now A only contains the direct sum of the $\{A_\pi\}$. We will, in either case, concentrate on these π-primary modules. A further restriction will be made in the Z-torsion free case. The quotient ring $R = \Lambda/(\pi)$ is, in an obvious manner, a subring of the algebraic number field generated by a root of π. We will restrict our attention to the case when R is integrally closed, i.e. a Dedekind ring. Later on we will determine effective criterion for π to satisfy this condition.

The general setting then is the following. We consider a unique factorization domain Λ with a particular prime π such that the quotient ring $R = \Lambda/(\pi)$ is a Dedekind domain. If A is a π-primary Λ-module, we will derive from A a collection of R-modules $\{A_i, A^i\}$ tied together by means of a family of short exact sequences $0 \to A_{i+1} \to A_i \to A^i \to A^{i+1} \to 0$. It will also be useful to consider $\Delta_i = \text{Cok}\{A_{i+1} \to A_i\} \approx \text{Ker}\{A^i \to A^{i+1}\}$. If R is a Dedekind domain, these modules are all described by "numerical" invariants (rank and ideal class). When A carries a suitable product structure, there is a duality relationship between the $\{A_i\}$ and $\{A^i\}$ and, furthermore, Δ_i (or a closely related $\tilde{\Delta}_i$) inherits a more familiar type of product structure which can be handled by techniques from algebraic number theory.

We now outline in somewhat more detail the implementation of the above program. Our treatment of the Z-torsion case is relatively brief. In this case, R is the principal ideal domain $Z/p[t, t^{-1}]$ and all the derived modules are R-torsion. It is easy to see that the derived modules and sequences fail to classify A (except in trivial situations), but, on the other hand, the realizability problem is easily solved: All possible $\{A_i, A^i\}$, related by the required exact sequences, are realizable. When A has a product structure of the type we are considering, Δ_i inherits a symmetric (or skew-symmetric) bilinear form, as a vector space over Z/p, in which t acts isometrically. Such "isometric structures" are completely understood (see [La], [MI]). The product realizability theorem requires somewhat more work. It turns out that any $\{A_i\}$, with each $A_{i+1} \subseteq A_i$, together with any isometric structure on the $\Delta_i = A_i/A_{i+1}$ can be realized by some A with product structure. As mentioned above, because of duality relations between $\{A_i\}$ and $\{A^i\}$, this is the best one can hope for.

Z-torsion free $Z[t, t^{-1}]$-modules are treated as a special case of π-primary Λ-modules, where Λ is a unique factorization domain, $R = \Lambda/(\pi)$ is Dedekind, and the module has "π-only torsion," i.e. its annihilator is the principal ideal generated by some power of π. This corresponds precisely to demanding that the $\{A_i\}$ (or, in fact, just A_0) is R-torsion free. The realization theorem then states that any

$\{A_i, A^i\}$, where A_0 is R-torsion free, can be realized. The proof is long. As a first step, we consider the simplest case in which $A_0 = A_1 = \ldots = A_{d-1}$ and $A_d = 0$, for some d. These turn out (when A_0 is R-torsion free) to be exactly the projective $\Lambda/(\pi^d)$-modules. Realization of these modules reduces to the construction of invertible ideals in $S = \Lambda/(\pi^d)$, with a given reduction in $R = S/\pi S$. Once these elementary modules are realized, the general case is treated by amalgamating elementary modules together according to instructions read from the sequences: $0 \to A_{i+1} \to A_i \to A^i \to A^{i+1} \to 0$.

The ability of the derived modules and sequences to classify π-primary Λ-modules depends on the degree—the degree of A is the smallest d such that $\pi^{d+1}A = 0$. For modules of degree ≤ 3, classification is successful, but it is shown, by an example, that nonisomorphic modules of degree 4 can have isomorphic derived modules and sequences.

The product structures we consider are (skew)-Hermitian bilinear forms with values in Q/Λ, where Q is the quotient field, or, equivalently, in $S = \Lambda/\pi^{d+1}$, where d is the degree of the module. Such a structure will induce a (skew)-Hermitian form on $\tilde{\Delta}_i$ = R-torsion free quotient of Δ_i, with values in R. In the case $\Lambda = Z[t, t^{-1}]$, we are thus dealing with integral (skew)-Hermitian forms over algebraic number fields (see [J]). The classification question is handled by the following result. Two π-primary Λ-modules (satisfying an extra technical condition which is always true for knot modules) are isometric if and only if they are isomorphic in such a way that the induced isomorphisms on $\tilde{\Delta}_i$ are isometries. Thus the classification result above for degree ≤ 3 extends immediately to a classification result for modules with product structure.

To deal with the product realizability question, we restrict our attention to those modules which are the direct sum of "homogeneous" modules. A module of degree d is homogeneous if every nonzero element α arises from one of degree d in the sense that, for some $\lambda \in \Lambda$ relatively prime to π, we can write $\lambda\alpha = \pi^s\beta$, for some β with $\pi^d\beta \neq 0$. If $\Lambda = Z[t, t^{-1}]$, A is homogeneous of degree d if $A \otimes_Z Q$ is a free $Q[t, t^{-1}]/(\pi^{d+1})$-module—this condition can be easily expressed in terms of the Alexander polynomials. All modules of degree ≤ 2 are semi-homogeneous (direct sums of homogeneous modules), but modules of degree 3 are not necessarily. If A is homogeneous of degree d, we have $\{A_i\}$ all of the same rank—they can therefore be usefully considered to be "lattices" in the vector space $V = A_0 \otimes_R F$ - F is the quotient field of R. Furthermore $\tilde{\Delta}_{d-1}$ is the only nonzero $\tilde{\Delta}_i$, and since $\tilde{\Delta}_{d-1} = A_{d-1}$ is a lattice in V, the induced (skew)-Hermitian form on $\tilde{\Delta}_{d-1}$ determines such a form on V.

It turns out that <u>nonsingularity</u> of the original form is equivalent to the condition that A_i is dual to A_{d-1-i} in V for each i. Thus we can consider our invariants to consist of a nonsingular (skew)-Hermitian form over F and a nest of integral lattices $A_{d-1} \subseteq \cdots \subseteq A_k$ where $d = 2k$ or $2k + 1$

and A_k is self-dual if d is odd. Our realizability theorem then states that any such nest of integral lattices in the space of a nonsingular (skew)-Hermitian form can arise from a homogeneous module of degree d with a nonsingular product structure.

To obtain a more comprehensive realization theorem for nonsemihomogeneous modules we consider the "rational" invariants. When $\Lambda = Z[t, t^{-1}]$ this means we pass to $\overline{A} = A \otimes_Z Q$, considered as a module over the principal ideal domain $Q[t, t^{-1}]$; in our more general context, we pass to $\overline{A} = A \otimes_\Lambda \Lambda_\pi$, where Λ_π is the discrete valuation ring obtained by localizing Λ at (π). The derived invariants of $\overline{A}$ are, obviously, also invariants of A. The trivial nature of Λ_π immediately tells us that these invariants classify $\overline{A}$. When there is a product structure, the results of [MI] can be interpreted to state that the derived invariants (with the forms on $\overline{\Delta}_i$) classify $\overline{A}$ isometrically; the derived forms are (skew)-Hermitian forms over the algebraic number field F, when $\Lambda = Z[t, t^{-1}]$, which are well understood (see [La]). Realizability of these invariants by Λ_π-module is easily established ([MI]), so the problem is to pass from $\overline{A}$ to A. It turns out that realization corresponds to the existence of a self-dual lattice in $\oplus_i \overline{\Delta}_{2i}$, and this condition can be expressed in terms of the classical invariants of forms over F. As a by-product of this, one sees easily that semihomogeneous modules are relatively sparse, since semihomogenity requires that <u>each</u> $\overline{\Delta}_{2i}$ contain a self-dual lattice.

The final sections of this work are concerned with the ring $R = \Lambda/(\pi)$. The first problem is to determine, from π, whether R is integrally closed. We are, in fact, able to find a completely effective procedure involving prime factorization of π over Z/p, for each p dividing the discriminant of π, to resolve this issue. Once we know that R is Dedekind, we have the problem of computing the ideal class group of R. This is not the same as computing the ideal class group of an algebraic number field, since, if π is not monic, R contains nonintegers. R does, however, come close to being of the form $\mathcal{O}[\frac{1}{m}]$, where $\mathcal{O}$ is the ring of algebraic integers in F and m an integer. In fact, when π satisfies a condition first considered in [C], $R = \mathcal{O}[\frac{1}{m}]$, where m is the product of the first and last coefficients of π. In this case, the ideal class group of R can be determined from that of $\mathcal{O}$. This computation is then actually carried out, for some quadratic π, using the tables in [B].

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§1. The derived exact sequences

Let Λ be an integral domain, and $\Pi \in \Lambda$ a prime element, i.e., if $\Pi = \Pi_1 \Pi_2$, then either Π_1 or Π_2 is a unit of Λ. Let A be a Λ-module. Define $K_i = K_i(A)$ to be the submodule of all elements killed by Π^i, i.e., $K_i = \{\alpha \in A: \Pi^i\alpha = 0\}$. Define $L_i \subseteq A$ to be $\Pi^i A$. We have inclusions:

$$0 = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$$

$$A = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$$

Finally define $A_i = K_{i+1}/K_i$, the <u>i-th lower Π-derivative</u> of A, and $A^i = L_i/L_{i+1}$, the <u>i-th upper Π-derivative</u> of A, for $i \geq 0$. Since $\Pi K_{i+1} \subseteq K_i$ and $\Pi L_i \subseteq L_{i+1}$, we conclude that A_i and A^i are modules over $\Lambda/(\Pi)$. Furthermore, multiplication by Π induces homomorphisms $\Pi_i: A_{i+1} \to A_i$ and $\Pi^i: A^i \to A^{i+1}$. We can also construct a homomorphism $\Delta_i: A_i \to A^i$ by multiplication by Π^i, since $\Pi^i K_{i+1} \subseteq \Pi^i A = L^i$, while $\Pi^i K_i = 0$.

These constructions are clearly functorial. Given a map $A \to B$ of Λ-modules, there are obvious induced maps $A_i \to B_i$, $A^i \to B^i$ commuting with Π_i, Π^i and Δ_i.

<u>Proposition 1.1</u>: The sequence

$$0 \to A_{i+1} \xrightarrow{\Pi_i} A_i \xrightarrow{\Delta_i} A^i \xrightarrow{\Pi^i} A^{i+1} \to 0$$

is exact for $i \geq 0$.

The proof is straightforward and will be omitted. We refer to this exact sequence as the <u>i-th Π-primary sequence</u> of A.

Note that $K = \bigcup_i K_i$ is the Π-primary submodule of A. If Λ is Noetherian and K is finitely generated, the nested sequence of $\{K_i\}$ terminates after a finite number of steps. The criterion for termination of $\{L_i\}$ is as follows:

<u>Proposition 1.2</u>: Let Λ be a Noetherian domain, Π a prime element of Λ and A a finitely generated Λ-torsion module. The following three conditions are equivalent.

i) $\Pi^m A = \Pi^{m+1} A$, for sufficiently large integer m.

ii) $A = A_\Pi \oplus \Pi^m A$, for sufficiently large integer m, where A_Π is the Π-primary submodule of A.

iii) There is an element $\phi \in \Lambda$ coprime to Π (i.e., $(\Pi, \phi) = \Lambda$) and an integer m, such that $(\Pi^m \phi)A = 0$ (and, therefore, $A = A_\Pi \oplus A_\phi$).

<u>Proof</u>:

<u>(i) ⟹ (iii)</u>: Choose m large enough so that $\Pi^m A = \Pi^{m+1} A$ and $\Pi^m A_\Pi = 0$. Then $\Pi^m A \cap A_\Pi = 0$. Given $\alpha \in A$ we may find $\beta \in A$ such that $\Pi^m \alpha = \Pi^{2m} \beta$; the decomposition $\alpha = (\alpha - \Pi^m \beta) + \Pi^m \beta$ establishes $A = \mathrm{Ker}\Pi^m + \Pi^m A$.

<u>(ii) ⟹ (iii)</u>: Choose m again so that $\Pi^m A_\Pi = 0$. Let $\alpha_1, \ldots, \alpha_k$ generate $\Pi^m A$; then $\alpha_i = \Pi^m(\sum_j \lambda_{ij}\alpha_j)$ for some $\lambda_{ij} \in \Lambda$. Rewriting this as $\sum_j (\delta_{ij} - \Pi^m \lambda_{ij})\alpha_j = 0$, we conclude that $\phi\lambda_i = 0$, for $\phi = \text{determinant } (\delta_{ij} - \Pi^m \lambda_{ij})$--see proof of [L, Cor. (1.3)]. Clearly $(\phi, \Pi^m) = \Lambda$, which implies $(\phi, \Pi) = \Lambda$.

(iii) ⟹ (i): Choose m so that $\Pi^m A = 0$. Consider $\Pi^m \alpha$ an arbitrary element of $\Pi^m A$. If we write:

$1 = \lambda\phi + \mu\Pi$, then $\alpha = \lambda\phi\alpha + \mu\Pi\alpha$ and so $\Pi^m\alpha = \lambda\phi\Pi^m\alpha + \mu\Pi^{m+1}\alpha = \mu\Pi^{m+1}\alpha$, which completes the proof.

The following propositions are of interest because of the definition of a module of type K (see [L]).

Proposition 1.3: If A is finitely generated and Π-primary, and λ is any element of Λ, then the following statements are equivalent:

i) Multiplication by λ defines an automorphism of A.

ii) Multiplication by λ defines an automorphism of every A_i.

iii) Multiplication by λ defines an automorphism of every A^i.

Proof: This follows by repeated use of the five lemma and the observations above.

Proposition 1.4: If A is Π-primary and Λ Noetherian, the following statements are equivalent:

i) A is finitely generated.

ii) A_i is finitely generated, for every i, and some $A_k = 0$.

iii) A^i is finitely generated, for every i, and some $A^k = 0$.

The proof follows immediately from the above observations.

Corollary 1.5: If $\Lambda = Z[t, t^{-1}]$ and A is Π-primary, the following statements are equivalent:

i) A is of type K.

ii) A_i is of type K, for every i, and $A_k = 0$ for some k.

iii) A^i is of type K, for every i, and $A^k = 0$ for some k.

§2. Finite modules

From now on we assume $\Lambda = Z[t, t^{-1}]$. We turn first to the case of finite Λ-modules. As usual any such can be decomposed into the direct sum of its p-primary components, p running over scalar primes. Each of these p-primary components is a Λ-module and so it suffices to study finite p-primary Λ-modules.

If we apply the considerations of §1 for $\Pi = p$, we have the family of p-primary sequences

$$(2.1) \qquad 0 \to A_{i+1} \xrightarrow{\Pi_i} A_i \xrightarrow{\Delta_i} A^i \xrightarrow{\Pi^i} A^{i+1} \to 0$$

where each A_i, A^i is a $\Lambda_p = \Lambda/(p) = Z_p[t, t^{-1}]$-module. Since Λ_p is a principal ideal domain, we may describe the modules A_i, A^i by polynomial invariants. The condition that A be a module of type K is equivalent, by Corollary 1.4, to the condition that $t = 1$ not be a root of any of these polynomial invariants.

It is easy to see that the p-primary sequences (2.1) are not generally sufficient to classify A. For example, define two Λ-module structures on Z/p^2 by

i) $t\alpha = 2\alpha$,

ii) $t\alpha = (p + 2)\alpha \quad (p \neq 2)$.

It is easy to check that the p-primary sequences (2.1) are isomorphic, but the modules themselves, are not.

It is of interest to compare the p-primary sequences (2.1) of A and $e^2(A) = \mathrm{Ext}^2_\Lambda(A, \Lambda)$ in light of the duality relation ([L, 3.4(i)]). For any Λ_p-module B, define $B^* = \mathrm{Hom}_{Z_p}(B, Z_p)$ with Λ_p-module structure induced from that on B, i.e., if $\phi \in B^*$, $\lambda \in \Lambda_p$, then $\lambda\phi = \phi \circ \lambda$ (perhaps one really should set $\lambda\phi = \phi \circ \bar{\lambda}$). Then it is not difficult to check that $B^* \approx B$, if B is a finitely generated Λ_p-torsion module. The interest of $*$ is that it defines a contravariant functor.

Proposition 2.2: Let A be a p-primary Λ-module of type K. Then $e^2(A)_i \approx A^i$, $e^2(A)^i \approx A_i$ and the i-th p-primary sequence for $e^2(A)$ is the "dual" of that for A: $0 \to (A^{i+1})^* \xrightarrow{(\Pi^i)^*} (A^i)^* \xrightarrow{(\Delta_i)^*} A_i^* \xrightarrow{(\Pi_i)^*} A_{i+1}^* \to 0$.

Proof: A homomorphism $e^2(A)_i \to (A^i)^*$ is defined as follows. Let $\phi \in \mathrm{Hom}_Z(A, Q/Z) \approx e^2(A)$ ([L, 4.2]) satisfy $p^{i+1}\phi = 0$. Then $\phi(p^{i+1}A) = 0$ and so ϕ induces a homomorphism $p^iA/p^{i+1}A \to Z_p \subseteq Q/Z$. It is a straightforward exercise to check it is bijective.

Similarly we define an isomorphism $e^2(A)^i \to (A_i)^*$. If $\phi \in p^i\mathrm{Hom}_Z(A, Q/Z)$, then $p^i\alpha = 0$ implies $\phi(\alpha) = 0$--i.e., $\phi(K_i) = 0$ where $K_i = \mathrm{Ker}\, p^i$ and we can define a homomorphism $K_{i+1}/K_i \to Z_p \subseteq Q/Z$. If $\phi \in p^{i+1}\mathrm{Hom}_Z(A, Q/Z)$, then $\phi(K_{i+1}) = 0$ and we get a well defined homomorphism $e^2(A)^i \to (A_i)^*$. Again, it is straightforward to check this is an isomorphism.