B. Huppert N. Blackburn

Finite Groups III

B. Huppert N. Blackburn

Finite Groups III

Springer-Verlag
Berlin Heidelberg New York 1982

Bertram Huppert
Mathematisches Institut der Universität
Saarstraße 21
D-6500 Mainz

Norman Blackburn
Department of Mathematics
The University
GB-Manchester M13 9 PL

ISBN 3-540-10633-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-10633-2 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Huppert, Bertram, 1927–.
Finite groups III. (Grundlehren der mathematischen Wissenschaften; 243).
Bibliography: p. Includes index. 1. Finite groups. I. Blackburn, N. (Norman).
II. Title. III. Series. QA 171.B578. 512'.22. 81-2288.
ISBN 0-387-10633-2 (U.S.). AACR 2

Printed in Germany

Typesetting: Asco Trade Typesetting Limited, Chai Wan, Hong Kong.
Printing and bookbinding: Konrad Triltsch, Würzburg
2141/3140-543210

Preface

From "Die Meistersinger von Nürnberg", Richard Wagner

This final volume is concerned with some of the developments of the subject in the 1960's. In attempting to determine the simple groups, the first step was to settle the conjecture of Burnside that groups of odd order are soluble. The proof that this conjecture was correct is much too long and complicated for presentation in this text, but a number of ideas in the early stages of it led to a local theory of finite groups, some aspects of which are discussed in Chapter X. Much of this discussion is a continuation of the theory of the transfer (see Chapter IV), but we also introduce the generalized Fitting subgroup, which played a basic role in characterization theorems, that is, in descriptions of specific groups in terms of group-theoretical properties alone. One of the earliest and most important such characterizations was given for Zassenhaus groups; this is presented in Chapter XI. Characterizations in terms of the centralizer of an involution are of particular importance in view of the theorem of Brauer and Fowler. In Chapter XII, one such theorem is given, in which the Mathieu group $\mathfrak{M}_{11}$ and $PSL(3, 3)$ are characterized. This last chapter is mainly concerned with some aspects of multiply transitive permutation groups loosely connected with the Mathieu groups or with sharp n-fold transitivity, and several results from Chapter XI are used in it. The two last chapters are, however, independent of Chapter X.

Again we wish to acknowledge our indebtedness to the many colleagues who have assisted us with this work. In addition to those named in the preface to Volume II, thanks are due to George Glauberman, who read an earlier version of Chapter X. The contributions of all have done a great deal to improve this volume, and it is with the greatest pleasure that we express our gratitude to them.

January, 1982

Bertram Huppert, Mainz
Norman Blackburn, Manchester

Index of Symbols

$\mathbf{J}_e(\mathfrak{P})$ 11
$\mathbf{J}(\mathfrak{P})$ 24
$\mathbf{ZJ}(\mathfrak{P})$ 24
$\underline{\mathbf{K}}(\mathfrak{P}), \overline{\mathbf{K}}(\mathfrak{P})$ 59
$R_p(\mathfrak{G})$ 77
$\hat{\mathbf{J}}(\mathfrak{P})$ 91
$T^g_{\mathfrak{H}}$ 107
$R(\mathfrak{G}, \mathfrak{H})$ 107
$C(\mathfrak{H}, \mathfrak{G})$ 107
$i_{\mathfrak{H},\mathfrak{G}}$ 110
$t_{\mathfrak{H},\mathfrak{G}}$ 110
$\mathbf{F}^*(\mathfrak{G})$ 126
$\mathbf{E}(\mathfrak{G})$ 128
$\mathbf{O}_{p^*}(\mathfrak{G})$ 135
$\mathbf{O}_{p^*,p}(\mathfrak{G})$ 135
$\mathbf{O}_{p',\mathbf{E}}(\mathfrak{G})$ 139
$\mathbf{O}_{p',\mathbf{E},p}(\mathfrak{G})$ 139
$\mathbf{D}(\mathfrak{G})$ 141
$\mathbf{B}(\mathfrak{G})$ 141
$\mathfrak{H} \to \mathfrak{M}$ 141
$И_{\mathbf{S}}(\mathfrak{A})$ 148
$\mathbf{S}(\mathfrak{H})$ 149
$P\Gamma L(n, p)$ 162
$M(p^f)$ 163
$\mathfrak{G}_{(\Lambda)}$ 229
$M(3^2)$ 299
$\mathfrak{M}_{11}, \mathfrak{M}_{12}$ 299
$\mathfrak{M}_{22}, \mathfrak{M}_{23}, \mathfrak{M}_{24}$ 301
$\Omega^{(k)}$ 366
$\Gamma(p^n)$ 378
$\mathfrak{G}_{48}$ 389

Contents

Chapter X. Local Finite Group Theory 1

§ 1. Elementary Lemmas 3
§ 2. Groups of Order Divisible by at Most Two Primes 11
§ 3. The **J**-Subgroup 19
§ 4. Conjugate *p*-Subgroups 28
§ 5. Characteristic *p*-Functors 35
§ 6. Transfer Theorems 39
§ 7. Maximal *p*-Factor Groups 52
§ 8. Glauberman's **K**-Subgroups 57
§ 9. Further Properties of **J**, **ZJ** and **K** 69
§10. The Product Theorem for **J** 76
§11. Fixed Point Free Automorphism Groups 91
§12. Local Methods and Cohomology 106
§13. The Generalized Fitting Subgroup 123
§14. The Generalized p'-Core 131
§15. Applications of the Generalized Fitting Subgroup 142
§16. Signalizer Functors and a Transitivity Theorem 148
Notes on Chapter X 158

Chapter XI. Zassenhaus Groups 160

§ 1. Elementary Theory of Zassenhaus Groups 161
§ 2. Sharply Triply Transitive Permutation Groups 172
§ 3. The Suzuki Groups 182
§ 4. Exceptional Characters 195
§ 5. Characters of Zassenhaus Groups 205
§ 6. Feit's Theorem 219
§ 7. Non-Regular Normal Subgroups of Multiply Transitive Permutation Groups 227
§ 8. Real Characters 234
§ 9. Zassenhaus Groups of Even Degree 246

§10. Zassenhaus Groups of Odd Degree and a Characterization of $PGL(2, 2^f)$ 256
§11. The Characterization of the Suzuki Groups 264
§12. Order Formulae 286
§13. Survey of Ree Groups 291
Notes on Chapter XI 295

Chapter XII. Multiply Transitive Permutation Groups 296

§ 1. The Mathieu Groups 297
§ 2. Transitive Extensions of Groups of Suzuki Type 314
§ 3. Sharply Multiply Transitive Permutation Groups 325
§ 4. On the Existence of 6- and 7-Fold Transitive Permutation Groups 339
§ 5. A Characterization of $\mathfrak{M}_{11}$ and $PSL(3, 3)$ 341
§ 6. Multiply Homogeneous Groups 366
§ 7. Doubly Transitive Soluble Permutation Groups 378
§ 8. A Characterization of $SL(2, 5)$ 387
§ 9. Sharply Doubly Transitive Permutation Groups 413
§10. Permutation Groups of Prime Degree 425
Notes on Chapter XII 438

Bibliography 439

Index of Names 449

Index 451

Chapter X

Local Finite Group Theory

The word *local* is used in finite group-theory in relation to a fixed prime p; thus properties of p-subgroups or their normalisers, for example, are regarded as local. In the case of a soluble group, then, everything is local, but an insoluble group also has global aspects. Now the local behaviour influences the global, that is, there are theorems in which the hypothesis involves only p-subgroups and their normalisers, but the conclusion involves the whole group. This chapter is an introduction to theorems of this sort.

Some such theorems are already known from Chapter IV; for example, Burnside's transfer theorem, which asserts that if the centre of the normaliser of a Sylow p-subgroup $\mathfrak{S}$ contains $\mathfrak{S}$, then the whole group is p-nilpotent. This is proved by showing that the transfer into $\mathfrak{S}$ is an epimorphism. An essential lemma (IV, 2.5) states that two $\mathfrak{S}$-invariant subsets of $\mathfrak{S}$ are conjugate in $\mathfrak{G}$ if and only if they are conjugate in $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$. This has many other applications, being a link between the global and local properties. More generally, the situation in which two subsets A, B of $\mathfrak{S}$ are conjugate in $\mathfrak{G}$ frequently arises; such sets A, B are often described as *fused*, particularly when they are not conjugate in $\mathfrak{S}$. In general, fusion can be reduced not to one but to a sequence of local transformations. This is the subject matter of § 4, where the precise way in which A can be transformed into B is investigated. It is shown that if $A^g = B$, then $g = g_1 g_2 \cdots g_n$ where g_i normalises some subgroup $\mathfrak{P}_i$ of $\mathfrak{S}$ and $A^{g_1 \cdots g_{i-1}} \subseteq \mathfrak{P}_i$. Moreover there are certain sets $\mathscr{F}$ of subgroups of $\mathfrak{S}$ for which the additional condition that $\mathfrak{P}_i \in \mathscr{F}$ may be imposed. These sets $\mathscr{F}$ are called *conjugation families*.

Another theorem with a local hypothesis but a global conclusion is the theorem of Thompson (IV, 6.2) that, for p odd, if $\mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{S}))$ and $\mathbf{N}_{\mathfrak{G}}(\mathbf{J}_0(\mathfrak{S}))$ are p-nilpotent, so is $\mathfrak{G}$. Here $\mathbf{J}_0(\mathfrak{S})$ denotes a characteristic subgroup of $\mathfrak{S}$. Certain similarly defined characteristic subgroups are very useful for establishing non-simplicity criteria; this is shown in § 2, where a character-free proof of the solubility of groups of order $p^a q^b$ is given. In § 3, it is shown that there is such a characteristic group $\mathbf{ZJ}(\mathfrak{S})$ which is always normal in $\mathfrak{G}$ whenever $\mathbf{O}_p(\mathfrak{G}) \geq \mathbf{C}_{\mathfrak{G}}(\mathbf{O}_p(\mathfrak{G}))$ and $\mathfrak{G}$ is p-stable. This can be used to give another proof of the above theorem of Thompson; it was also used by BENDER [3] to simplify greatly a section of the proof of the solubility of groups of odd order. For all such applications, criteria for p-stability are of course required, such as those given in Chapter IX.

Now $\mathbf{J}(\mathfrak{P})$ is defined for any p-group $\mathfrak{P}$ by certain rules. To analyse these, we consider first, in § 5, completely general rules, supposing only that there is defined in each p-group $\mathfrak{P}$ a subgroup $\mathbf{W}(\mathfrak{P})$ and that whenever α is an isomorphism of $\mathfrak{P}$ onto $\tilde{\mathfrak{P}}$, $\mathbf{W}(\mathfrak{P})\alpha = \mathbf{W}(\tilde{\mathfrak{P}})$. Such a $\mathbf{W}$ is called a *characteristic p-functor*. In order to study fusion, a conjugation family is defined in § 5 corresponding to any characteristic p-functor $\mathbf{W}$. This enables us to prove, for example, that $\mathfrak{G}$ and $\mathbf{N}_{\mathfrak{G}}(\mathbf{W}(\mathfrak{S}))$ have isomorphic maximal p-factor groups if and only if the same is true in the normaliser of any non-identity p-subgroup (Theorem 7.3). By combining this with some results about the transfer developed in § 6, a commutator condition is obtained which implies that $\mathfrak{G}$ and $\mathbf{N}_{\mathfrak{G}}(\mathbf{W}(\mathfrak{S}))$ have isomorphic maximal p-factor groups. In § 8, two characteristic p-functors $\underline{\mathbf{K}}$, $\overline{\mathbf{K}}$ are defined, and, completely within the context of p-groups, a complementary commutator condition is established (Theorem 8.10). Putting the two together, a theorem of Glauberman, which states that for $p \geq 5$, $\mathfrak{G}$, $\mathbf{N}_{\mathfrak{G}}(\overline{\mathbf{K}}(\mathfrak{S}))$ and $\mathbf{N}_{\mathfrak{G}}(\underline{\mathbf{K}}(\mathfrak{S}))$ all have isomorphic maximal p-factor groups, follows. Grün's second theorem (IV, 3.7) makes a similar assertion, but requires that $\mathfrak{G}$ be p-normal. Glauberman's theorem, however, has no such hypothesis. Among its consequences are the fact that if $\mathfrak{G}$ is not a p-group, there exists a Sylow subgroup $\mathfrak{S}$ of $\mathfrak{G}$ for which $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S}) > \mathfrak{S}$. In § 9, it is shown that $\underline{\mathbf{K}}$, $\overline{\mathbf{K}}$ could be used in place of $\mathbf{J}_0$ in the theorem of Thompson, and before this, it is shown that when every section of $\mathfrak{G}$ is p-stable, $\overline{\mathbf{K}}$, $\underline{\mathbf{K}}$ and, for p odd, $\mathbf{ZJ}$ have a property which is described as *strongly controlling fusion*: whenever A, A^g are contained in $\mathfrak{S}$, there exists $h \in \mathbf{N}_{\mathfrak{G}}(\mathbf{W}(\mathfrak{S}))$ such that $a^g = a^h$ for all $a \in A$.

In § 10, we consider another property of $\mathbf{J}$. If $\mathfrak{G}$ is p-soluble, $\mathbf{O}_{p'}(\mathfrak{G}) = 1$ and $\mathfrak{S} \in S_p(\mathfrak{G})$, then the equation

$$\mathfrak{G} = \mathbf{N}_{\mathfrak{G}}(\mathbf{J}(\mathfrak{S}))\mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{S}))$$

holds under many circumstances; certainly if $p > 3$. This kind of factorization is of considerable importance and made its first appearance (implicitly) in Thompson's theorem. Conditions for its validity when p is 2 or 3 are found in § 10, and in § 11 these are applied to prove a theorem on fixed point free automorphism groups. It is conjectured that if $\mathfrak{A}$ is a fixed point free group of automorphisms of $\mathfrak{G}$ and $(|\mathfrak{A}|, |\mathfrak{G}|) = 1$, then $\mathfrak{G}$ is soluble; in § 11 this is proved when $\mathfrak{A}$ is elementary Abelian.

Since $\mathfrak{G}/\mathfrak{G}'$ is in duality with $\operatorname{Hom}(\mathfrak{G}, \mathbb{C}^\times) = \mathbf{H}^1(\mathfrak{G}, \mathbb{C}^\times)$, where $\mathbb{C}^\times$ is regarded as a trivial $\mathbb{Z}\mathfrak{G}$-module, the transfer of $\mathfrak{G}$ into a subgroup $\mathfrak{H}$ gives rise to a homomorphism of $\mathbf{H}^1(\mathfrak{H}, \mathbb{C}^\times)$ into $\mathbf{H}^1(\mathfrak{G}, \mathbb{C}^\times)$. This is a special case of the corestriction homomorphism of $\mathbf{H}^n(\mathfrak{H}, \mathsf{M})$ into $\mathbf{H}^n(\mathfrak{G}, \mathsf{M})$ described in I, 16.17. It is shown in 12.8 that $\mathbf{H}^n(\mathfrak{G}, \mathsf{M})$ and $\mathbf{H}^n(\mathbf{N}_{\mathfrak{G}}(\mathbf{W}(\mathfrak{S})), \mathsf{M})$ have isomorphic Sylow p-subgroups if M is a trivial $\mathfrak{G}$-module and $\mathbf{W}$ is a characteristic p-functor which strongly controls fusion in $\mathfrak{G}$—Grün's second theorem is a special case of this. This is applied to the Schur multiplier of $\mathfrak{G}$ in 12.17; if $\mathfrak{S} \in S_p(\mathfrak{G})$ and the class of $\mathfrak{S}$ is at most $\frac{1}{2}p$, the Sylow p-subgroups of the Schur multipliers of $\mathfrak{G}$ and of $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$ are isomorphic.

In addition to the transfer, a number of results, which have become very familiar in finite group theory, are frequently used in proving these theorems; these include the properties of the centralizers of the Fitting subgroup and $\mathbf{O}_{p',p}(\mathfrak{G})$ and a number of other facts which are collected in § 1. In § 13 and § 14, some of these results are generalized in such a way that solubility hypotheses are removed. In doing this, the role of the nilpotent group is taken by the quasinilpotent group (13.2) and that of the p'-group by the p^*-group (14.2). It is shown, for example, that every group $\mathfrak{G}$ has a unique maximal normal quasinilpotent subgroup $\mathbf{F}^*(\mathfrak{G})$ and that $\mathbf{C}_{\mathfrak{G}}(\mathbf{F}^*(\mathfrak{G})) \le \mathbf{F}(\mathfrak{G})$; again, every group $\mathfrak{G}$ has a generalized p'-core $\mathbf{O}_{p^*}(\mathfrak{G})$, and if $\mathfrak{P}$ is a p-subgroup of $\mathfrak{G}$, $\mathbf{O}_{p^*}(\mathbf{C}_{\mathfrak{G}}(\mathfrak{P})) \le \mathbf{O}_{p^*}(\mathfrak{G})$. Finally, in § 16, another aspect of local properties is briefly considered; this involves the relationship between the various soluble p'-subgroups of a group $\mathfrak{G}$ which are normalised by a fixed Abelian p-subgroup of $\mathfrak{G}$.

§ 1. Elementary Lemmas

In this chapter a number of elementary results will be used frequently. Some of these have already appeared in the previous chapter; the remainder are collected together in this section.

First, we establish a lemma for characteristic subgroups of p-groups analogous to the theorem (III, 7.3) that a maximal normal Abelian subgroup of a p-group is its own centralizer.

1.1 Lemma. *Let $\mathfrak{G}$ be a p-group and let $\mathfrak{A}$ be a characteristic Abelian subgroup of $\mathfrak{G}$. Then there exists a characteristic subgroup $\mathfrak{B}$ of $\mathfrak{G}$ such that*

(i) $\mathfrak{B} \geq \mathbf{C}_{\mathfrak{G}}(\mathfrak{B}) = \mathbf{Z}(\mathfrak{B}) \geq \mathfrak{A}$, *and*

(ii) $\mathfrak{B}/\mathbf{Z}(\mathfrak{B})$ *is an elementary Abelian subgroup of* $\mathbf{Z}(\mathfrak{G}/\mathbf{Z}(\mathfrak{B}))$.

In particular, the class of $\mathfrak{B}$ is at most 2.

Proof. Let $\mathscr{X}$ be the set of characteristic subgroups $\mathfrak{X}$ of $\mathfrak{G}$ such that $\mathbf{Z}(\mathfrak{X}) \geq \mathfrak{A}$ and $\mathfrak{X}/\mathbf{Z}(\mathfrak{X})$ is an elementary Abelian subgroup of $\mathbf{Z}(\mathfrak{G}/\mathbf{Z}(\mathfrak{X}))$. Thus $\mathfrak{A} \in \mathscr{X}$. Let $\mathfrak{B}$ be a maximal element of $\mathscr{X}$. If $\mathfrak{B} \geq \mathbf{C}_{\mathfrak{G}}(\mathfrak{B})$, then $\mathfrak{B}$ has all the required properties. Suppose then that $\mathfrak{B} \ngeq \mathbf{C}_{\mathfrak{G}}(\mathfrak{B})$, that is, $\mathbf{Z}(\mathfrak{B}) < \mathbf{C}_{\mathfrak{G}}(\mathfrak{B})$. Let $\mathfrak{D}/\mathbf{Z}(\mathfrak{B})$ be the set of elements of order at most p in $(\mathbf{C}_{\mathfrak{G}}(\mathfrak{B})/\mathbf{Z}(\mathfrak{B})) \cap \mathbf{Z}(\mathfrak{G}/\mathbf{Z}(\mathfrak{B}))$. Thus $\mathfrak{D} \leq \mathbf{C}_{\mathfrak{G}}(\mathfrak{B})$, $\mathfrak{D}^p \leq \mathbf{Z}(\mathfrak{B})$ and $[\mathfrak{D}, \mathfrak{G}] \leq \mathbf{Z}(\mathfrak{B})$. By III, 7.2, $\mathfrak{D} > \mathbf{Z}(\mathfrak{B})$, so $\mathfrak{D} \nleq \mathfrak{B}$. But $\mathfrak{D}\mathfrak{B} \in \mathscr{X}$, since $[\mathfrak{D}\mathfrak{B}, \mathfrak{G}] = [\mathfrak{D}, \mathfrak{G}][\mathfrak{B}, \mathfrak{G}] \leq \mathbf{Z}(\mathfrak{B})$ and $(\mathfrak{D}\mathfrak{B})^p = \mathfrak{D}^p\mathfrak{B}^p \leq \mathbf{Z}(\mathfrak{B})$. This contradicts the maximality of $\mathfrak{B}$. **q.e.d.**

1.2 Lemma. *Suppose that $\mathfrak{G}$ is a p-group and that α is an automorphism of $\mathfrak{G}$ of order prime to p. If there exists a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ for which $\mathfrak{H}\mathbf{C}_{\mathfrak{G}}(\mathfrak{H}) \leq \mathbf{C}_{\mathfrak{G}}(\alpha)$, then α is the identity automorphism.*

Proof. This is proved by induction on $|\mathfrak{G}|$. There is nothing to prove if $\mathfrak{H}\mathbf{C}_{\mathfrak{G}}(\mathfrak{H}) = \mathfrak{G}$. Otherwise, there exists a maximal α-invariant proper subgroup $\mathfrak{M}$ of $\mathfrak{G}$ such that $\mathfrak{M} \geq \mathfrak{H}\mathbf{C}_{\mathfrak{G}}(\mathfrak{H})$. Since $\mathbf{N}_{\mathfrak{G}}(\mathfrak{M})$ is α-invariant, $\mathbf{N}_{\mathfrak{G}}(\mathfrak{M}) = \mathfrak{G}$, by III, 2.3. Thus $\mathfrak{M} \trianglelefteq \mathfrak{G}$. By the inductive hypothesis, $\mathfrak{M} \leq \mathbf{C}_{\mathfrak{G}}(\alpha)$. By IX, 6.3, α induces the identity mapping on $\mathfrak{G}/\mathbf{C}_{\mathfrak{G}}(\mathfrak{M})$. But $\mathbf{C}_{\mathfrak{G}}(\mathfrak{M}) \leq \mathbf{C}_{\mathfrak{G}}(\mathfrak{H}) \leq \mathfrak{M}$, so α induces the identity mapping on $\mathfrak{G}/\mathfrak{M}$. By I, 4.4, α is the identity mapping. **q.e.d.**

1.3 Lemma. *Suppose that $\mathfrak{G}$ is a p-group and $\mathbf{C}_{\mathfrak{G}}(\mathfrak{N}) \leq \mathfrak{N} \trianglelefteq \mathfrak{G}$. Suppose that*

$$\mathfrak{N} = \mathfrak{N}_0 \geq \mathfrak{N}_1 \geq \cdots \geq \mathfrak{N}_k = 1,$$

where $\mathfrak{N}_i \trianglelefteq \mathfrak{G}$ $(i = 0, 1, \ldots, k)$. Let $\mathfrak{A}$ be a group of automorphisms α of $\mathfrak{G}$ for which $\mathfrak{N}_i\alpha = \mathfrak{N}_i$ $(i = 0, 1, \ldots, k)$. Let

$$\mathfrak{B} = \{\beta \,|\, \beta \in \mathfrak{A}, (x\mathfrak{N}_i)\beta = x\mathfrak{N}_i \quad \text{for all} \quad x \in \mathfrak{N}_{i-1}\ (i = 1, \ldots, k)\}.$$

Then $\mathfrak{B}$ is a normal p-subgroup of $\mathfrak{A}$.

In particular, any non-identity p′-element of $\mathfrak{A}$ induces a non-identity automorphism on $\mathfrak{N}_{i-1}/\mathfrak{N}_i$ for at least one i.

Proof. If $\alpha \in \mathfrak{A}$, let $\rho(\alpha)$ denote the automorphism of $\mathfrak{N}$ induced by α. Then ρ is a homomorphism of $\mathfrak{A}$ into the group of automorphisms of $\mathfrak{N}$. Let $\mathfrak{K} = \ker \rho$, $\mathfrak{A}_0 = \operatorname{im} \rho$, $\mathfrak{B}_0 = \rho(\mathfrak{B})$. Then there is an isomorphism between $\mathfrak{A}/\mathfrak{K}$ and $\mathfrak{A}_0$ in which $\mathfrak{B}/\mathfrak{K}$ and $\mathfrak{B}_0$ correspond. Now

$$\mathfrak{B}_0 = \{\beta \mid \beta \in \mathfrak{A}_0, (x\mathfrak{N}_i)\beta = x\mathfrak{N}_i \quad \text{for all} \quad x \in \mathfrak{N}_{i-1} \ (i = 1, \ldots, k)\}.$$

By IX, 7.3, $\mathfrak{B}_0$ is a normal p-subgroup of $\mathfrak{A}_0$. Hence $\mathfrak{B} \trianglelefteq \mathfrak{A}$ and it only remains to show that $\mathfrak{K}$ is a p-group. If $\gamma \in \mathfrak{K}$, choose r such that the order of $\delta = \gamma^{p^r}$ is prime to p. Then $\mathfrak{N}\mathbf{C}_{\mathfrak{G}}(\mathfrak{N}) = \mathfrak{N} \leq \mathbf{C}_{\mathfrak{G}}(\delta)$, so $\delta = 1$ by 1.2. Thus $\mathfrak{K}$ is a p-group. **q.e.d.**

1.4 Lemma. *Suppose that α is an automorphism of the Abelian p-group $\mathfrak{A}$ and that α leaves fixed every element of $\mathfrak{A}$ of order p. Then the order of α is a power of p.*

Proof. Let $\mathfrak{A}_n = \Omega_n(\mathfrak{A})$ $(n = 0, 1, \ldots)$. Thus $\mathfrak{A}_n\alpha = \mathfrak{A}_n$ and

$$1 = \mathfrak{A}_0 \leq \mathfrak{A}_1 \leq \cdots \leq \mathfrak{A}_m = \mathfrak{A}$$

for some m. If $x \in \mathfrak{A}_n$ and $x\alpha = xy$,

$$y^{p^{n-1}} = x^{-p^{n-1}}(x\alpha)^{p^{n-1}} = x^{-p^{n-1}}(x^{p^{n-1}})\alpha = 1,$$

since the order of $x^{p^{n-1}}$ is at most p. Thus $y \in \mathfrak{A}_{n-1}$, and $(x\mathfrak{A}_{n-1})\alpha = x\mathfrak{A}_{n-1}$. By 1.3, the order of α is a power of p. **q.e.d.**

1.5 Lemma (THOMPSON). *Suppose that $\mathfrak{P}$, $\mathfrak{K}$ are subgroups of the group $\mathfrak{G}$, $\mathfrak{P}$ is a p-group, $\mathfrak{K} = \mathbf{O}^p(\mathfrak{K})$ and $[\mathfrak{K}, \mathfrak{P}] = 1$. Suppose that $\mathfrak{B}$ is a p-subgroup of $\mathfrak{G}$ and $\mathfrak{P}\mathfrak{K} \leq \mathbf{N}_{\mathfrak{G}}(\mathfrak{B})$. If $[\mathfrak{K}, \mathbf{C}_{\mathfrak{B}}(\mathfrak{P})] = 1$, then $[\mathfrak{K}, \mathfrak{B}] = 1$.*

Proof. Let $\mathfrak{H} = \{(u, v) \mid u \in \mathfrak{P}, v \in \mathfrak{B}\}$. Then $\mathfrak{H}$ is a group if we put

$$(u, v)(u', v') = (uu', v^{u'}v')$$

for all u, u' in $\mathfrak{P}$ and v, v' in $\mathfrak{B}$. If x is an element of $\mathfrak{K}$ of order prime to p, then $x \in \mathbf{C}_{\mathfrak{G}}(\mathfrak{P})$, so

$$(u, v^x)(u', v'^x) = (uu', v^{xu'}v'^x) = (uu', (v^{u'}v')^x).$$

Hence if we put $(u, v)\alpha = (u, v^x)$, α is an automorphism of $\mathfrak{H}$. Note that $\mathfrak{H}$ is a p-group and the order of α is prime to p. If $\mathfrak{Q} = \{(u, 1) \mid u \in \mathfrak{P}\}$,

$\mathfrak{Q} \leq \mathfrak{H}$ and $\mathbf{C}_{\mathfrak{H}}(\mathfrak{Q}) = \{(u, v) | u \in \mathbf{Z}(\mathfrak{P}), v \in \mathbf{C}_{\mathfrak{B}}(\mathfrak{P})\}$. Since $[\mathfrak{K}, \mathbf{C}_{\mathfrak{B}}(\mathfrak{P})] = 1$, $\mathfrak{Q}\mathbf{C}_{\mathfrak{H}}(\mathfrak{Q}) \leq \mathbf{C}_{\mathfrak{H}}(\alpha)$. By 1.2, α is the identity automorphism. Thus $x \in \mathbf{C}_{\mathfrak{G}}(\mathfrak{B})$ for every element x of $\mathfrak{K}$ of order prime to p. Since $\mathfrak{K} = \mathbf{O}^p(\mathfrak{K})$, it follows that $\mathfrak{B}$ centralizes $\mathfrak{K}$. **q.e.d.**

1.6 Lemma. *If $\mathfrak{P}$ is a p-subgroup of the p-constrained group $\mathfrak{G}$,*

$$\mathbf{O}_{p'}(\mathbf{N}_{\mathfrak{G}}(\mathfrak{P})) \leq \mathbf{O}_{p'}(\mathfrak{G}).$$

Proof. Let $\overline{\mathfrak{G}} = \mathfrak{G}/\mathbf{O}_{p'}(\mathfrak{G})$, $\overline{\mathfrak{P}} = \mathfrak{P}\mathbf{O}_{p'}(\mathfrak{G})/\mathbf{O}_{p'}(\mathfrak{G})$, $\overline{\mathfrak{N}} = \mathbf{N}_{\overline{\mathfrak{G}}}(\overline{\mathfrak{P}})$, $\overline{\mathfrak{K}} = \mathbf{O}_{p'}(\overline{\mathfrak{N}})$. Then $\overline{\mathfrak{K}} = \mathbf{O}^p(\overline{\mathfrak{K}})$ since $\overline{\mathfrak{K}}$ is a p'-group, and $[\overline{\mathfrak{P}}, \overline{\mathfrak{K}}] = 1$ since $\overline{\mathfrak{P}}$, $\overline{\mathfrak{K}}$ are normal subgroups of $\overline{\mathfrak{N}}$ of coprime orders. Also $\overline{\mathfrak{P}}\overline{\mathfrak{K}} \leq \mathbf{N}_{\overline{\mathfrak{G}}}(\mathbf{O}_p(\overline{\mathfrak{G}}))$ and

$$[\overline{\mathfrak{K}}, \mathbf{C}_{\mathbf{O}_p(\overline{\mathfrak{G}})}(\overline{\mathfrak{P}})] \leq \mathbf{O}_p(\overline{\mathfrak{G}}) \cap [\overline{\mathfrak{K}}, \overline{\mathfrak{N}}] \leq \mathbf{O}_p(\overline{\mathfrak{G}}) \cap \overline{\mathfrak{K}} = 1.$$

Hence by 1.5, $[\overline{\mathfrak{K}}, \mathbf{O}_p(\overline{\mathfrak{G}})] = 1$. But $\mathbf{O}_{p'}(\overline{\mathfrak{G}}) = 1$ and $\overline{\mathfrak{G}}$ is p-constrained (VII, 13.3), so $\overline{\mathfrak{K}} \leq \mathbf{O}_p(\overline{\mathfrak{G}})$. Since $\overline{\mathfrak{K}}$ is a p'-group, $\overline{\mathfrak{K}} = 1$. Now by IX, 6.11, $\overline{\mathfrak{N}} = \mathbf{N}_{\mathfrak{G}}(\mathfrak{P})\mathbf{O}_{p'}(\mathfrak{G})/\mathbf{O}_{p'}(\mathfrak{G})$, so $\overline{\mathfrak{N}} \cong \mathbf{N}_{\mathfrak{G}}(\mathfrak{P})/\mathfrak{M}$, where $\mathfrak{M} = \mathbf{N}_{\mathfrak{G}}(\mathfrak{P}) \cap \mathbf{O}_{p'}(\mathfrak{G})$ is a normal p'-subgroup of $\mathbf{N}_{\mathfrak{G}}(\mathfrak{P})$. Since $\mathbf{O}_{p'}(\overline{\mathfrak{N}}) = \overline{\mathfrak{K}} = 1$, it follows that $\mathbf{O}_{p'}(\mathbf{N}_{\mathfrak{G}}(\mathfrak{P})) = \mathfrak{M} \leq \mathbf{O}_{p'}(\mathfrak{G})$. **q.e.d.**

1.7 Corollary. *Suppose that $\mathfrak{P}$ is a p-subgroup of the p-constrained group $\mathfrak{G}$. If $\mathfrak{P}$ contains every p-element of $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P})$, then*

$$\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) = \mathbf{Z}(\mathfrak{P}) \times (\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) \cap \mathbf{O}_{p'}(\mathfrak{G})).$$

Proof. By hypothesis, $\mathbf{Z}(\mathfrak{P})$ is the only Sylow p-subgroup of $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P})$. Since $\mathbf{Z}(\mathfrak{P}) \leq \mathbf{Z}(\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}))$, $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) = \mathbf{Z}(\mathfrak{P}) \times \mathfrak{N}$ for some $\mathfrak{N}$, by IV, 2.6. Thus $\mathfrak{N} = \mathbf{O}_{p'}(\mathbf{C}_{\mathfrak{G}}(\mathfrak{P})) \leq \mathbf{O}_{p'}(\mathbf{N}_{\mathfrak{G}}(\mathfrak{P})) \leq \mathbf{O}_{p'}(\mathfrak{G})$ by 1.6, and $\mathfrak{N} = \mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) \cap \mathbf{O}_{p'}(\mathfrak{G})$. **q.e.d.**

We now turn to a generalization of part of IX, 6.11.

1.8 Lemma. *Suppose that $\mathfrak{A}$ is a group of operators on a group $\mathfrak{G}$ and that either $\mathfrak{A}$ or $\mathfrak{G}$ is soluble.*

a) *If $\mathfrak{N}$ is a normal $\mathfrak{A}$-invariant subgroup of $\mathfrak{G}$ and $(|\mathfrak{N}|, |\mathfrak{A}|) = 1$, then*

$$\mathbf{C}_{\mathfrak{G}/\mathfrak{N}}(\mathfrak{A}) = \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})\mathfrak{N}/\mathfrak{N}.$$

b) *If $(|[\mathfrak{G}, \mathfrak{A}]|, |\mathfrak{A}|) = 1$, then $\mathfrak{G} = \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})[\mathfrak{G}, \mathfrak{A}]$.*

Proof. a) Obviously $\mathbf{C}_{\mathfrak{G}}(\mathfrak{A})\mathfrak{N}/\mathfrak{N} \leq \mathbf{C}_{\mathfrak{G}/\mathfrak{N}}(\mathfrak{A})$. If $x\mathfrak{N} \in \mathbf{C}_{\mathfrak{G}/\mathfrak{N}}(\mathfrak{A})$, then by I, 18.6, there exists $y \in \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})$ such that $y\mathfrak{N} = x\mathfrak{N}$. Hence $x \in \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})\mathfrak{N}$.

b) By III, 1.6b), $\mathfrak{M} = [\mathfrak{G}, \mathfrak{A}]$ is a normal $\mathfrak{A}$-invariant subgroup of $\mathfrak{G}$. Clearly $\mathbf{C}_{\mathfrak{G}/\mathfrak{M}}(\mathfrak{A}) = \mathfrak{G}/\mathfrak{M}$. But by a),

$$\mathbf{C}_{\mathfrak{G}/\mathfrak{M}}(\mathfrak{A}) = \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})\mathfrak{M}/\mathfrak{M}.$$

Thus $\mathfrak{G} = \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})\mathfrak{M} = \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})[\mathfrak{G}, \mathfrak{A}]$. **q.e.d.**

Note that on account of the solubility of groups of odd order, the hypothesis that either $\mathfrak{A}$ or $\mathfrak{G}$ be soluble is unnecessary.

1.9 Lemma. *Suppose that $\mathfrak{G}$ is a p'-group and $\mathfrak{A}$ is an Abelian p-group of operators on $\mathfrak{G}$. Then*

$$\mathfrak{G} = \langle \mathbf{C}_{\mathfrak{G}}(\mathfrak{B}) \mid \mathfrak{B} \leq \mathfrak{A},\ \mathfrak{A}/\mathfrak{B} \textit{ is cyclic}\rangle.$$

If also $\mathfrak{A}$ is not cyclic,

$$\mathfrak{G} = \langle \mathbf{C}_{\mathfrak{G}}(x) \mid x \in \mathfrak{A},\ x \neq 1\rangle.$$

Proof. The first assertion is proved by induction on $|\mathfrak{G}|$. If q is any prime other than p, $\mathfrak{G}$ possesses an $\mathfrak{A}$-invariant Sylow q-subgroup $\mathfrak{Q}$, by IX, 1.11, and $\mathfrak{G}$ is generated by all such Sylow subgroups. Thus it suffices to prove that $\mathfrak{Q} = \mathfrak{C}$, where

$$\mathfrak{C} = \langle \mathbf{C}_{\mathfrak{Q}}(\mathfrak{B}) \mid \mathfrak{B} \leq \mathfrak{A},\ \mathfrak{A}/\mathfrak{B} \text{ is cyclic}\rangle.$$

Suppose that $\mathfrak{C} < \mathfrak{Q}$. Then $\Phi(\mathfrak{Q})\mathfrak{C} < \mathfrak{Q}$. Let $\mathfrak{M}$ be a maximal $\mathfrak{A}$-invariant subgroup of $\mathfrak{Q}$ for which $\Phi(\mathfrak{Q})\mathfrak{C} \leq \mathfrak{M} < \mathfrak{Q}$. Then $\mathfrak{Q}/\mathfrak{M}$ is irreducible under $\mathfrak{A}$. If $\mathfrak{B}_0$ is the kernel of the representation of $\mathfrak{A}$ on $\mathfrak{Q}/\mathfrak{M}$, then $\mathfrak{A}/\mathfrak{B}_0$ is cyclic, by II, 3.10. Also

$$\mathfrak{Q}/\mathfrak{M} = \mathbf{C}_{\mathfrak{Q}/\mathfrak{M}}(\mathfrak{B}_0) = \mathbf{C}_{\mathfrak{Q}}(\mathfrak{B}_0)\mathfrak{M}/\mathfrak{M},$$

by 1.8a), so

$$\mathfrak{Q} = \mathbf{C}_{\mathfrak{Q}}(\mathfrak{B}_0)\mathfrak{M} = \mathfrak{M},$$

a contradiction.

Suppose that $\mathfrak{A}$ is not cyclic. If $\mathfrak{A}/\mathfrak{B}$ is cyclic, then $\mathfrak{B} \neq 1$ and $\mathbf{C}_{\mathfrak{G}}(\mathfrak{B}) \leq \mathbf{C}_{\mathfrak{G}}(x)$ for some $x \in \mathfrak{A} - \{1\}$. Hence $\mathfrak{G} = \langle \mathbf{C}_{\mathfrak{G}}(x) \mid x \in \mathfrak{A},\ x \neq 1\rangle$. **q.e.d.**

1.10 Lemma. a) *If $\mathfrak{S} \in S_p(\mathfrak{G})$ and $\mathfrak{A}$ is a maximal normal Abelian subgroup of $\mathfrak{S}$, $\mathbf{C}_{\mathfrak{G}}(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$ for some p'-subgroup $\mathfrak{D}$.*

b) *If p is odd, $\mathfrak{S} \in S_p(\mathfrak{G})$ and $\mathfrak{A}$ is a maximal normal elementary Abelian subgroup of $\mathfrak{S}$, every element of order p in $\mathbf{C}_{\mathfrak{G}}(\mathfrak{A})$ lies in $\mathfrak{A}$.*

c) *Suppose that $\mathfrak{A}$ is a p-subgroup of $\mathfrak{G}$ and every element of order p in $\mathbf{C}_{\mathfrak{G}}(\mathfrak{A})$ lies in $\mathfrak{A}$. If $\mathfrak{K} \leq \mathfrak{G}$, $\mathfrak{A} \leq \mathbf{N}_{\mathfrak{G}}(\mathfrak{K})$ and $\mathfrak{K} \cap \mathfrak{A} = 1$, $\mathfrak{K}$ is a p'-group.*

Proof. a) See IX, 5.9.

b) Since $\mathfrak{S} \in S_p(\mathbf{N}_{\mathfrak{G}}(\mathfrak{A}))$ and $\mathbf{C}_{\mathfrak{G}}(\mathfrak{A}) \trianglelefteq \mathbf{N}_{\mathfrak{G}}(\mathfrak{A})$, $\mathbf{C}_{\mathfrak{S}}(\mathfrak{A}) \in S_p(\mathbf{C}_{\mathfrak{G}}(\mathfrak{A}))$. Hence if x is an element of $\mathbf{C}_{\mathfrak{G}}(\mathfrak{A})$ of order p, $x^c \in \mathbf{C}_{\mathfrak{S}}(\mathfrak{A})$ for some $c \in \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})$. But x^c is of order p, so by III, 12.1, $x^c \in \mathfrak{A}$. Since $c \in \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})$, it follows that $x \in \mathfrak{A}$.

c) Suppose $\mathfrak{P} \in S_p(\mathfrak{A}\mathfrak{K})$ and $\mathfrak{P} \geq \mathfrak{A}$. Then $\mathfrak{P} = \mathfrak{A}\mathfrak{P}_0$, where $\mathfrak{P}_0 = \mathfrak{P} \cap \mathfrak{K} \in S_p(\mathfrak{K})$. Also $\mathfrak{P}_0 \trianglelefteq \mathfrak{P}$, so if $\mathfrak{P}_0 \neq 1$, $\mathfrak{P}_0$ contains an element z of $\mathbf{Z}(\mathfrak{P})$ of order p. But then $z \in \mathbf{C}_{\mathfrak{G}}(\mathfrak{A})$, so by hypothesis, $z \in \mathfrak{A}$. Since $\mathfrak{A} \cap \mathfrak{K} = 1$, it follows that $z = 1$, a contradiction. Thus $\mathfrak{P}_0 = 1$ and $\mathfrak{K}$ is a p'-group. **q.e.d.**

1.11 Theorem. *Suppose that $\mathfrak{G}$ is p-constrained, $\mathfrak{S} \in S_p(\mathfrak{G})$ and $\mathfrak{A}$ is a maximal normal Abelian subgroup of $\mathfrak{S}$. If $\mathfrak{K}$ is a subgroup of $\mathfrak{G}$ for which $\mathfrak{A} \leq \mathbf{N}_{\mathfrak{G}}(\mathfrak{K})$ and $\mathfrak{A} \cap \mathfrak{K} = 1$, then $\mathfrak{K} \leq \mathbf{O}_{p'}(\mathfrak{G})$.*

Proof. Suppose that this is false and that $\mathfrak{G}$ is a counterexample of minimal order. By 1.10a) and c), $\mathfrak{K}$ is a p'-group.

a) $\mathbf{O}_{p'}(\mathfrak{G}) = 1$.

Let $\overline{\mathfrak{G}} = \mathfrak{G}/\mathbf{O}_{p'}(\mathfrak{G})$, $\overline{\mathfrak{S}} = \mathfrak{S}\mathbf{O}_{p'}(\mathfrak{G})/\mathbf{O}_{p'}(\mathfrak{G})$; thus $\overline{\mathfrak{S}} \in S_p(\overline{\mathfrak{G}})$ and $\overline{\mathfrak{A}} = \mathfrak{A}\mathbf{O}_{p'}(\mathfrak{G})/\mathbf{O}_{p'}(\mathfrak{G})$ is a maximal normal Abelian subgroup of $\overline{\mathfrak{S}}$. Also $\overline{\mathfrak{A}} \leq \mathbf{N}_{\overline{\mathfrak{G}}}(\overline{\mathfrak{K}})$, where $\overline{\mathfrak{K}} = \mathfrak{K}\mathbf{O}_{p'}(\mathfrak{G})/\mathbf{O}_{p'}(\mathfrak{G})$. Since $\mathfrak{K}$ is a p'-group, $\overline{\mathfrak{A}} \cap \overline{\mathfrak{K}} = 1$. But $\overline{\mathfrak{K}} \neq 1 = \mathbf{O}_{p'}(\overline{\mathfrak{G}})$, so $\overline{\mathfrak{G}}$ is a counterexample to the theorem, and $\mathbf{O}_{p'}(\mathfrak{G}) = 1$ on account of the minimality of $|\mathfrak{G}|$.

Let $\mathfrak{T}$ be a minimal non-identity $\mathfrak{A}$-invariant subgroup of $\mathfrak{K}$. Thus $\mathfrak{T}$ is a p'-group.

b) $\mathfrak{G} = \mathbf{O}_p(\mathfrak{G})\mathfrak{T}\mathfrak{A}$.

Let $\mathfrak{H} = \mathbf{O}_p(\mathfrak{G})\mathfrak{T}\mathfrak{A}$. Thus $\mathfrak{H} \leq \mathfrak{G}$ and $\mathbf{O}_p(\mathfrak{G})\mathfrak{A} \in S_p(\mathfrak{H})$. Since $\mathbf{O}_p(\mathfrak{G})\mathfrak{A} \leq \mathfrak{S}$ and $\mathbf{C}_{\mathfrak{S}}(\mathfrak{A}) = \mathfrak{A}$, $\mathfrak{A}$ is a maximal normal Abelian subgroup of $\mathbf{O}_p(\mathfrak{G})\mathfrak{A}$. Since $\mathbf{O}_{p'}(\mathfrak{G}) = 1$ and $\mathfrak{G}$ is p-constrained, $\mathbf{O}_p(\mathfrak{G}) \geq \mathbf{C}_{\mathfrak{G}}(\mathbf{O}_p(\mathfrak{G}))$, so $[\mathbf{O}_p(\mathfrak{G}), \mathfrak{T}] \neq 1$ and $\mathfrak{T} \nleq \mathbf{O}_{p'}(\mathfrak{H})$. Thus $\mathfrak{H}$ is a counterexample to the theorem, and $\mathfrak{H} = \mathfrak{G}$ on account of the minimality of $|\mathfrak{G}|$.

c) $[\mathfrak{T}, \mathfrak{A}] \neq 1$.

This is clear if $\mathfrak{T} \nleq \mathbf{N}_{\mathfrak{G}}(\mathfrak{A})$, so we suppose that $\mathfrak{T} \leq \mathbf{N}_{\mathfrak{G}}(\mathfrak{A})$. Then $\mathfrak{A} \trianglelefteq \mathfrak{G}$ since $\mathfrak{G} = \mathfrak{S}\mathfrak{T}$ by b). By 1.10a), $\mathbf{C}_{\mathfrak{G}}(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$ for some

p'-subgroup $\mathfrak{D}$. Thus $\mathfrak{D}$ is a characteristic subgroup of $C_{\mathfrak{G}}(\mathfrak{A})$ and $C_{\mathfrak{G}}(\mathfrak{A}) \trianglelefteq \mathfrak{G}$. Hence $\mathfrak{D} \trianglelefteq \mathfrak{G}$ and $\mathfrak{D} \leq O_{p'}(\mathfrak{G}) = 1$. Thus $C_{\mathfrak{G}}(\mathfrak{A}) = \mathfrak{A}$ and $[\mathfrak{T}, \mathfrak{A}] \neq 1$.

d) Let $V = O_p(\mathfrak{G})/\Phi(O_p(\mathfrak{G}))$. Then $\mathfrak{G}$ is a group of operators on V, and by III, 13.4b),

$$V = [V, \mathfrak{T}] \times C_V(\mathfrak{T}).$$

Now $[V, \mathfrak{T}]$ is $\mathfrak{A}$-invariant, so $[[V, \mathfrak{T}], \mathfrak{A}] \leq [V, \mathfrak{T}]$. But also $[[V, \mathfrak{T}], \mathfrak{A}] \leq [V, \mathfrak{A}] \leq C_V(\mathfrak{T})$, since

$$[O_p(\mathfrak{G}), \mathfrak{A}, \mathfrak{T}] \leq [\mathfrak{A}, \mathfrak{T}] \cap O_p(\mathfrak{G}) \leq \mathfrak{T} \cap O_p(\mathfrak{G}) = 1.$$

Hence $[[V, \mathfrak{T}], \mathfrak{A}] = 1$ and $[V, \mathfrak{T}] \leq C_V(\mathfrak{A})$. Since $[V, \mathfrak{T}]$ is $\mathfrak{T}$-invariant, it follows that $[V, \mathfrak{T}] \leq C_V(\mathfrak{A}^t)$ for all $t \in \mathfrak{T}$; hence

$$[V, \mathfrak{T}] \leq C_V([\mathfrak{T}, \mathfrak{A}]).$$

Now $[\mathfrak{T}, \mathfrak{A}]$ is an $\mathfrak{A}$-invariant subgroup of $\mathfrak{T}$, and by c), $[\mathfrak{T}, \mathfrak{A}] \neq 1$. By minimality of $\mathfrak{T}$, $[\mathfrak{T}, \mathfrak{A}] = \mathfrak{T}$. Thus $[V, \mathfrak{T}] \leq C_V(\mathfrak{T})$. Hence $[V, \mathfrak{T}] = 1$ and

$$\mathfrak{T} \leq C_{\mathfrak{G}}(O_p(\mathfrak{G})/\Phi(O_p(\mathfrak{G}))).$$

By IX, 1.6, $\mathfrak{T} \leq O_p(\mathfrak{G})$, contrary to $\mathfrak{T} \neq 1$. **q.e.d.**

We show that for p odd, the conclusion of 1.11 holds under weaker assumptions.

1.12 Theorem (THOMPSON-BENDER). *For p odd, suppose that $\mathfrak{A}$ is a p-subgroup of the p-constrained group $\mathfrak{G}$ and that every element of order p in $C_{\mathfrak{G}}(\mathfrak{A})$ lies in $\mathfrak{A}$. If $\mathfrak{K} \leq \mathfrak{G}$, $\mathfrak{A} \leq N_{\mathfrak{G}}(\mathfrak{K})$ and $\mathfrak{K} \cap \mathfrak{A} = 1$, then $\mathfrak{K} \leq O_{p'}(\mathfrak{G})$.*

Proof. Suppose that this is false and let $\mathfrak{G}$ be a counterexample of minimal order. By 1.10, $\mathfrak{K}$ is a p'-group.

a) $O_{p'}(\mathfrak{G}) = 1$.

Suppose that $\mathfrak{N} = O_{p'}(\mathfrak{G}) \neq 1$. By IX, 6.11, $C_{\mathfrak{G}/\mathfrak{N}}(\mathfrak{A}\mathfrak{N}/\mathfrak{N}) = C_{\mathfrak{G}}(\mathfrak{A})\mathfrak{N}/\mathfrak{N}$. Thus any element of order p in $C_{\mathfrak{G}/\mathfrak{N}}(\mathfrak{A}\mathfrak{N}/\mathfrak{N})$ is of the form $x\mathfrak{N}$ with $x \in C_{\mathfrak{G}}(\mathfrak{A})$ and $x^p \in \mathfrak{N}$. If $x = x_1x_2 = x_2x_1$, where x_1 is a p-element and x_2 is a p'-element, then $x_2 \in \mathfrak{N}$ and $x_1^p = 1$. Since $x_1 \in C_{\mathfrak{G}}(\mathfrak{A})$, it follows from the hypothesis that $x_1 \in \mathfrak{A}$. Hence $x\mathfrak{N} \in \mathfrak{A}\mathfrak{N}/\mathfrak{N}$. Thus $\mathfrak{G}/\mathfrak{N}$

satisfies the conditions of the theorem. Since $\mathfrak{G}$ is a minimal counterexample, it follows that $\mathfrak{K}\mathfrak{N}/\mathfrak{N} \leq \mathbf{O}_{p'}(\mathfrak{G}/\mathfrak{N}) = 1$ and $\mathfrak{K} \leq \mathfrak{N} = \mathbf{O}_{p'}(\mathfrak{G})$.

Let $\mathscr{S}$ be the set of subgroups $\mathfrak{X}$ of $\mathbf{O}_p(\mathfrak{G})$ such that $\mathfrak{A}\mathfrak{K} \leq \mathbf{N}_{\mathfrak{G}}(\mathfrak{X})$ but $\mathfrak{K} \not\leq \mathbf{C}_{\mathfrak{G}}(\mathfrak{X})$. Since $\mathbf{O}_{p'}(\mathfrak{G}) = 1$, $\mathfrak{G}$ is p-constrained and $\mathfrak{K}$ is a non-identity p'-group, $\mathbf{O}_p(\mathfrak{G}) \in \mathscr{S}$. Thus $\mathscr{S}$ is non-empty. Let $\mathfrak{H}$ be a minimal element of $\mathscr{S}$. Thus

b) $\mathfrak{H}$ is a subgroup of $\mathbf{O}_p(\mathfrak{G})$, $\mathfrak{A}\mathfrak{K} \leq \mathbf{N}_{\mathfrak{G}}(\mathfrak{H})$, $\mathfrak{K} \not\leq \mathbf{C}_{\mathfrak{G}}(\mathfrak{H})$.

We prove next that

c) the class of $\mathfrak{H}$ is at most 2 (cf. III, 13.5).

Indeed, by minimality of $\mathfrak{H}$, $\mathfrak{K} \leq \mathbf{C}_{\mathfrak{G}}(\mathfrak{H}')$. Thus $[\mathfrak{H}', \mathfrak{H}, \mathfrak{K}] = [\mathfrak{K}, \mathfrak{H}', \mathfrak{H}] = 1$. By III, 1.10, $[\mathfrak{H}, \mathfrak{K}, \mathfrak{H}'] = 1$. Now since $\mathfrak{K} \not\leq \mathbf{C}_{\mathfrak{G}}(\mathfrak{H})$, there exists $y \in \mathfrak{K}$ such that y induces a non-trivial automorphism η on $\mathfrak{H}$. In fact η is not of order a power of p, since $\mathfrak{K}$ is a p'-group. But $\mathfrak{K} \leq \mathbf{N}_{\mathfrak{G}}([\mathfrak{H}, \mathfrak{K}])$, so η leaves $[\mathfrak{H}, \mathfrak{K}]$ fixed and induces the identity automorphism on $\mathfrak{H}/[\mathfrak{H}, \mathfrak{K}]$. It follows from I, 4.4 that η induces a non-identity automorphism on $[\mathfrak{H}, \mathfrak{K}]$. Thus $\mathfrak{K} \not\leq \mathbf{C}_{\mathfrak{G}}([\mathfrak{H}, \mathfrak{K}])$. But $\mathfrak{A}\mathfrak{K} \leq \mathbf{N}_{\mathfrak{G}}([\mathfrak{H}, \mathfrak{K}])$, so $[\mathfrak{H}, \mathfrak{K}] \in \mathscr{S}$. By minimality of $\mathfrak{H}$, $[\mathfrak{H}, \mathfrak{K}] = \mathfrak{H}$. Thus

$$[\mathfrak{H}, \mathfrak{H}'] = [\mathfrak{H}, \mathfrak{K}, \mathfrak{H}'] = 1$$

and the class of $\mathfrak{H}$ is at most 2.

Since $|\mathfrak{H}|$ is odd, each element of $\mathfrak{H}$ has a unique square root. Thus by c) and VIII, 9.16, there exists an addition on $\mathfrak{H}$ with respect to which $\mathfrak{H}$ is an Abelian group $\tilde{\mathfrak{H}}$ and $\mathfrak{A}\mathfrak{K}$ is a group of operators on $\tilde{\mathfrak{H}}$. By III, 13.4b),

$$\tilde{\mathfrak{H}} = [\tilde{\mathfrak{H}}, \mathfrak{K}] \oplus \mathbf{C}_{\tilde{\mathfrak{H}}}(\mathfrak{K});$$

here $[\tilde{\mathfrak{H}}, \mathfrak{K}]$ is understood in the sense of the additive structure of $\tilde{\mathfrak{H}}$ and $\mathbf{C}_{\tilde{\mathfrak{H}}}(\mathfrak{K}) = \mathbf{C}_{\mathfrak{H}}(\mathfrak{K})$. Thus $[\tilde{\mathfrak{H}}, \mathfrak{K}]$ is an $\mathfrak{A}$-invariant subgroup of $\tilde{\mathfrak{H}}$, and by b), $[\tilde{\mathfrak{H}}, \mathfrak{K}] \neq 0$. Since $[\tilde{\mathfrak{H}}, \mathfrak{K}]$ and $\mathfrak{A}$ are p-groups, there exists an element u of order p in $[\tilde{\mathfrak{H}}, \mathfrak{K}]$ such that $u \in \mathbf{C}_{\tilde{\mathfrak{H}}}(\mathfrak{A}) = \mathbf{C}_{\mathfrak{H}}(\mathfrak{A})$. But then, by hypothesis, $u \in \mathfrak{A}$. Thus if $g \in \mathfrak{K}$ and the commutator $[u, g]$ is now understood in the ordinary sense,

$$[u, g] \in \mathfrak{H} \cap [\mathfrak{A}, \mathfrak{K}] \leq \mathfrak{H} \cap \mathfrak{K} = 1.$$

Thus $u \in \mathbf{C}_{\mathfrak{H}}(\mathfrak{K}) = \mathbf{C}_{\tilde{\mathfrak{H}}}(\mathfrak{K})$ and

$$u \in \mathbf{C}_{\tilde{\mathfrak{H}}}(\mathfrak{K}) \cap [\tilde{\mathfrak{H}}, \mathfrak{K}] = 0,$$

a contradiction. **q.e.d.**