An Introduction to Matrices, Vectors, and Linear Programming

HUGH G. CAMPBELL

SECOND EDITION

An Introduction to MATRICES, VECTORS, and LINEAR PROGRAMMING

Second Edition

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Prentice-Hall, Inc., Englewood Cliffs, New Jersey 07632

Library of Congress Cataloging in Publication Data

CAMPBELL, HUGH G (date)
An introduction to matrices, vectors, and linear programming.

Bibliography: p. 272
Includes index.
1. Matrices. 2. Vector algebra. 3. Linear programming. I. Title.
QA188.C35 1977 512.9'43 76-22757
ISBN 0-13-487439-0

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PRENTICE-HALL International, Inc., *London*
PRENTICE-HALL of Australia Pty. Limited, *Sydney*
PRENTICE-HALL of Canada, Ltd., *Toronto*
PRENTICE-HALL of India Private Limited, *New Delhi*
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WHITEHALL BOOKS LIMITED, *Wellington, New Zealand*

Preface

The use of matrix algebra is expanding rapidly within the fields of business administration, accounting, economics, agriculture, engineering, and the social, natural, and biological sciences. The increasing emphasis on quantitative methods in all these fields and the adaptation of computers to perform matrix operations are two of the many reasons for this development. Moreover, many individuals and study groups have advocated that an introduction to matrices, vectors, and linear programming be made at a much earlier stage in the mathematical program than has been customary in the past. Consequently, there is an increasing demand for textbooks on matrix algebra that require only limited mathematical background. Such books must, of necessity, begin in a very elementary manner and attempt to elevate the reader's mathematical maturity as rapidly and gently as possible.

With these needs in mind, the author has attempted to serve the following triple purpose in this text:

1. To introduce matrices and vectors to such an extent that they may be used in a study of systems of linear equations and an introduction to linear programming; both topics lead directly to many applied problems.

2. To help the reader gain further insight into the structure of mathematics.
3. To begin building a vocabulary to enable the reader to read literature of his own discipline that uses the language of matrix algebra.

With these purposes in mind, the author believes that this book is suitable for use in the following ways.

As a text in a freshman or sophomore mathematics course in elementary matrix algebra or linear programming.

As a text for courses in various applied disciplines; such courses presumably will require a text that will introduce the readers to the language of matrices, vectors, and linear programming for use in applications pertinent to their discipline.

As an elementary reference book for students or professional people in fields where matrices are applied.

As a text in high school teacher-training programs at the undergraduate or graduate level.

The book is purposely written in a brief, concise manner because the author has found in his years of teaching that many undergraduate students have a very short interest span when reading textbooks. They seem to grasp the material better when theorems and definitions are emphasized and when there is an example to illustrate each new idea that is presented. The style adopted also has obvious advantages when the book is used for reference purposes. The difficulty that is often experienced when students are exposed to proofs has been recognized, and a conscious effort has been made to overcome this problem. A special effort has been made throughout to interweave the concrete with the abstract. Although many applications are mentioned in the way of motivation, the book does not pretend to reveal the many diverse uses of matrix algebra. Instead the text is designed to give the student a mathematical foundation in matrix algebra on which further study in his or her applied field may be based.

This text differs from many introductory matrix algebra or linear algebra books in that (1) it is more elementary than most; (2) it contains a chapter on algebraic systems that seems to promote abstract thinking by emphasizing how matrices and vectors fit into the structure of mathematics; (3) it contains a chapter on linear programming, which is rapidly becoming a very useful tool in many fields; and (4) it contains numerous examples, exercises, and applications to motivate and assist the student in the learning process.

In this second edition much of the material in Chapters 9 through 12 has been rewritten and rearranged, and a section on the characteristic value problem has been added. One purpose of these changes is to permit more flexibility in the use of the text as outlined in the following paragraph.

The book may be adapted to many types of courses. Some alternatives are listed below.

1. For those interested only in matrix methods for solutions of systems of linear equations, the first eight chapters will suffice.
2. The first ten chapters offer an elementary presentation of matrices, vectors, and linear programming, without reference to the more abstract ideas involved in a study of vector spaces or linear transformations. For those who prefer to get to linear programming as quickly as possible, the following sections are prerequisite: 1.1 to 1.6, 1.8, 2.1, 4.1, 4.5, 4.7, 4.8, 5.1, 5.2, 5.6, 7.1, 7.2, 8.1 to 8.4, and 9.1 to 9.3.
3. Chapters 1 through 8 and 11 and 12 provide an introduction to the use of matrices in a study of vector spaces and linear transformations. (Chapters 9 and 10 are not prerequisite to Chapters 11 and 12.)
4. For those interested in techniques and not in theoretical aspects, some or all of the proofs may be omitted, along with the optional sections 3.5, 3.6, and 3.7.

It is assumed that the student has mastered elementary algebra and plane geometry. A knowledge of certain topics from elementary analytic geometry and trigonometry is also assumed in a few parts of the book; these topics, however, can be explained sufficiently by the teacher if necessary.

The author wishes to express his sincere gratitude to his wife, Allen, for her help and encouragement. Special appreciation is expressed to the author's friend and colleague Robert E. Spencer for his invaluable advice and assistance throughout the preparation of both this and the first edition of the book; he deserves considerable credit for any success that this book may enjoy. The author is also grateful for the efforts of the late Dr. R. W. Brink in the publication of the first edition.

Blacksburg, Virginia H. G. C.

Contents

Chapter 1

An Introduction to Matrices

1.1 AN ORIENTATION

The primary purpose of this book is to provide the reader with an introduction to two new algebraic systems—the algebra of matrices and the algebra of vectors. We shall assume that the reader is already acquainted with the algebra of complex numbers and the algebra of real numbers. There are many similarities and differences in these old and new systems; these similarities and differences will be emphasized in detail as we proceed.

Over the years the elements found useful by mathematicians have gradually been extended. First there were the counting numbers or positive integers (1, 2, 3, . . .). They were needed to count objects. Zero and the negative integers (0, −1, −2, −3, . . .) were introduced in order to make it possible to subtract any integer from any other integer. The number system was eventually expanded to include all rational numbers ($\frac{2}{3}, \frac{4}{5}, \frac{6}{5}$, etc.). This expansion was necessary in order to permit division. The solution to certain problems and equations led to irrational numbers, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, π, and to imaginary numbers like $2 + 3i$, $0 - 4i$, $7 + 10i$. Historically, these events probably did not occur in the order in which they are listed above. In fact, it is probable that fractions preceded negative numbers in most civilizations. The set or collection of all irrational and rational numbers is called the set of real numbers. The set or collection of all real and imaginary numbers is called the set of complex numbers.

The sets of numbers discussed above are not the only mathematical elements in use today, and indeed the purpose of the first two chapters of this book is to introduce two new kinds of elements—the matrix and the vector. As men increase their knowledge, they seek and find more and more ways to express their ideas by using the language of mathematics. This trend has been quite apparent since the end of World War II. For example, certain problems in decision making that have faced business executives for years are now resolved quickly by computers after a translation has been made to the language of matrix algebra. Such solutions have been conceived only in recent years. Thus vectors and matrices are becoming increasingly important to scholars in all branches of social and natural sciences and engineering. The need is great for trained personnel who can bring the power of matrix and vector methods to bear on problems in one or more of these disciplines.

Actually, the elements vector and matrix are not new to mankind. The notion of a matrix is more than a hundred years old, but the recent technological explosion and the advent of high-speed computers have caused a flurry of interest in this element.

Three objectives of this book are to help the reader

1. study matrices and vectors to such an extent that they may be used in a study of systems of equations, linear programming, linear transformations, and the characteristic value problem (all these topics lead directly to many applied problems);
2. gain further insight into the structure of mathematics;
3. begin building a vocabulary to enable the reader to read literature that uses the language of matrix algebra.

It should be pointed out that this is not a text in applied matrix algebra; however, a few of the many diverse applications will be pointed out along the way.

EXERCISES

Suggested *minimum assignment*: Exercise 1.

1. Tell whether each of the following statements is true or false.
(*a*) 15 is a complex number. (*b*) $\frac{3}{2}$ is a real number.
(*c*) $\sqrt{2}$ is a rational number. (*d*) 0 is a rational number.
(*e*) The solutions of $x^2 - 3 = 0$ are irrational numbers.
(*f*) The solutions of $x^2 + 3 = 0$ are real numbers.

2. Given the numbers $\frac{3}{8}$, $-2i$, $2 - 3i$, 0, $\sqrt{5}$, $\sqrt{16}$, π, 7, $-\frac{4}{2}$, $\sqrt{2}\,i$,

(*a*) which of these numbers are rational numbers?
(*b*) which of these numbers are imaginary numbers?
(*c*) which of these numbers are integers?
(*d*) which of these numbers are irrational numbers?
(*e*) which of these numbers are complex numbers?
(*f*) which of these numbers are real numbers?

1.2 BASIC DEFINITIONS

Undoubtedly the reader often sees various types of data presented in the form of rectangular arrays of numbers, such as the box score of a baseball game or weather statistics for several locations. Consider the following price chart for a length of pipe of given diameters.

	2 cm.	4 cm.	6 cm.
Copper	50¢	65¢	75¢
Tin	30¢	40¢	45¢

Such rectangular arrays are examples of matrices. They are usually designated by capital letters A, B, C, and so on.

DEFINITION 1 Let m and n represent positive integers; a rectangular array of entries, arranged in m rows and n columns,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called an ***m by n matrix.***

The *entries* of a matrix often are called the *elements* of the matrix. In this text we shall assume, unless it is stated otherwise, that the entries of a matrix are *scalars*; the term *scalar* will be defined explicitly in Chapter 3. In the meantime, the reader can *interpret the word scalar as a complex number, a real number, or a functional value thereof.*

EXAMPLE 1 The following are examples of matrices.

(*a*) $\begin{bmatrix} 2 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$; the entries are integers, which are real numbers.

(*b*) $\begin{bmatrix} x^2 & 2 \\ 2x & x+2 \end{bmatrix}$; the entries are values of functions of a real variable x.

(c) $\begin{bmatrix} 2i & i \\ 3 & i-1 \end{bmatrix}$; the entries are complex numbers.

(d) $\begin{bmatrix} \pi \\ \frac{2}{3} \\ \sqrt{2} \end{bmatrix}$; the entries are real numbers.

One use of matrix notation is to enable us to manipulate large rectangular arrays of numbers as single entities. This step frequently simplifies the statements of various operations and relationships.

EXAMPLE 2 Let the matrix $A = \begin{bmatrix} 8 & 2 \\ 1 & 1 \\ 2 & 7 \end{bmatrix}$ represent the number of gadgets R, S, and T that factories P and Q can produce in a day; that is, matrix A represents the following production capacity.

	Factory P	Factory Q
Gadget R	8 per day	2 per day
Gadget S	1 per day	1 per day
Gadget T	2 per day	7 per day

Later in this book, using the entity "matrix A," we will extend this example to accomplish some very useful results. First, however, we must become familiar with the terminology and manipulations of matrix algebra.

If all the entries of a matrix are real numbers, the matrix is called a ***real matrix***, like Examples 1(a), 1(b), and 1(d). If all the entries of a matrix are zero, the matrix is called a ***zero matrix*** or ***null matrix*** and is denoted by **0**. Boldface type will be used to distinguish a zero matrix from the zero scalar.

The horizontal lines of the array are called ***rows***. The vertical lines are ***columns***. Each entry of a matrix A is designated, in general, as a_{ij}, where i represents the row number and j is the column number; thus a_{31} is the entry in the third row and first column. The double subscript can be called the ***address*** of the entry.

The dimensions of the array (number of rows stated first) determine the ***order*** of the matrix, designated "m by n". Example 1(a) had a total of two rows and three columns of entries. We say that the order of this matrix is 2 by 3. The order of Example 1(d) is 3 by 1. Such a matrix with a single column is called a ***column matrix***. When a matrix consists of a single row (that is, is of order 1 by n), it is called a ***row matrix***. When the dimensions of a matrix are equal, it is called a ***square matrix***. The ***main diagonal*** of a square matrix consists of the entries $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$.

EXAMPLE 3 In this example the entries of the main diagonal of a square matrix are enclosed in parentheses for emphasis.

$$\begin{bmatrix} (3) & 4 & 7 \\ 0 & (\sqrt{2}) & 6 \\ \pi & 0 & (0) \end{bmatrix}$$

This real matrix is said to be of order 3 by 3.

Matrices are denoted in several different ways by different authors. Some use parentheses; some use double vertical lines. In this book an m by n matrix will be written in any one of the three following ways.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{(m,\, n)}.$$

The last notation is an abbreviation in which it is understood that the row subscript i assumes all integral values from 1 to m and the column subscript j assumes all integral values from 1 to n.

EXAMPLE 4 The abbreviated notation can be used to represent the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{by} \quad [a_{ij}]_{(2,\, 3)}.$$

We conclude this section with one illustration of the use of matrix notation.

EXAMPLE 5 Suppose that there exists a certain defined relation between persons, nations, numbers, or biological characteristics. Let us call this relation "dominance," and in Figure 1.1 suppose that an arrow from point i to point j denotes the domi-

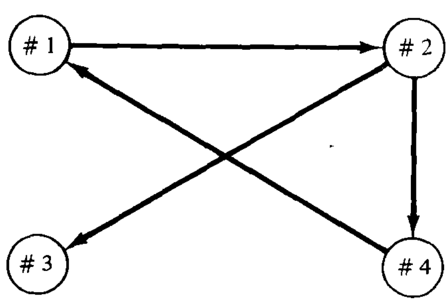

Figure 1.1

nance of i over j. This dominance can be indicated by the following matrix

$$\begin{array}{c} \\ \#1 \\ \#2 \\ \#3 \\ \#4 \end{array} \begin{array}{c} \begin{array}{cccc} \#1 & \#2 & \#3 & \#4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

where $a_{ij} = 1$ if i dominates j and $a_{ij} = 0$ if i does not dominate j. No point dominates itself.

A variation of this example occurs when each point either dominates or is dominated by each of the other points. A very interesting use of this model was made in a study of influence among the justices of the Michigan Supreme Court.[1] The use of the concept of dominance may pertain to biological characteristics, sociological behavior, and other areas, as well as to political influence.

EXERCISES

Suggested minimum assignment: Exercises 1, 3, 4, 7, and 9.

1. Let $A = \begin{bmatrix} 2 & 5 \\ \pi & 0 \end{bmatrix}$, $B = [8 \quad 3 \quad 4]$, $C = \begin{bmatrix} 2 - 3i & 5 & -3 \\ 1 + i & 2 & -4 \end{bmatrix}$,

$D = [3]$, $E = \begin{bmatrix} i & 2i \\ 4 & 1 \\ 3 & 2 \end{bmatrix}$.

(*a*) Which are real matrices?
(*b*) Which are not real matrices?
(*c*) Which are square matrices?
(*d*) Which are row matrices?
(*e*) Which are column matrices?
(*f*) State the order of each matrix.

2. Let $A = [i \quad 2i \quad 3i \quad 4i]$, $B = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ \sqrt{4} \end{bmatrix}$, $C = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

$D = [0]$, $E = \begin{bmatrix} 1 - 2i & 3 \\ 4i & 4 \end{bmatrix}$. Answer the same questions as in Exercise 1 above.

3. Why is the following array not a matrix? $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

[1] S. Ulmer, "Leadership in the Michigan Supreme Court" in *Judicial Decision Making*, G. Shubert (Ed.), New York, Free Press, 1963.

(*a*) What is a_{22}?
(*b*) What is the address of the entry 6?
(*c*) What is the order of A?

5. Let $A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \\ 8 & 9 \end{bmatrix}$.

(*a*) What is a_{21}?
(*b*) Is a_{23} defined?
(*c*) What is the address of the entry 8?
(*d*) What is the order of A?

6. Write a 2 by 3 zero matrix.

7. Display the matrix $A = [a_{ij}]_{(3,2)}$ with entries $a_{12} = 4$, $a_{22} = 5$, $a_{31} = 6$, $a_{11} = 2$, $a_{32} = 8$, $a_{21} = 3$.

8. List the entries on the main diagonal of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

9. Does every matrix have a main diagonal? Why?

10. One variation of the dominance concept illustrated in Example 5 is that every object either dominates or is dominated by every other object. Suppose that we have this situation among four objects. Let the matrix representation be

$$\begin{array}{c} \\ \#1 \\ \#2 \\ \#3 \\ \#4 \end{array} \begin{array}{c} \begin{array}{cccc} \#1 & \#2 & \#3 & \#4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array},$$

where $a_{ij} = 1$ if i dominates j and $a_{ij} = 0$ if i does not dominate j. Draw a diagram representing this dominance.

1.3 EQUALITIES AND INEQUALITIES

Now that a matrix has been defined, it is reasonable to define certain relations and operations for matrices.

DEFINITION 2 Two matrices A and B are said to be ***equal*** *(designated $A = B$) when they are of the same order and all their corresponding entries are equal; that is, $a_{ij} = b_{ij}$ for all i and j.*

By the notation $A \neq B$, we mean that A is *not* equal to B.