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with Hints and Solutions

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СБОРНИК КОНКУРСНЫХ ЗАДАЧ ПО МАТЕМАТИКЕ

С методическими указаниями и решениями

Под редакцией А. И. Прилепко

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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PREFACE

The book contains more than three thousand mathematics problems and covers each topic taught at school. The problems were contributed by 120 of the higher schools of the USSR and all the universities.

The book is divided into four parts: algebra and trigonometry, fundamentals of analysis, geometry and vector algebra, and the problems and questions set during oral examinations.

The authors considered it necessary to include some material relating to complex numbers, combinatorics, the binomial theorem, elementary trigonometric inequalities, and set theory and the method of coordinates. The authors believe that this material will help the readers systematize their knowledge in the principal divisions of mathematics.

In writing the book, the authors have used their experience of examining students in mathematics at higher schools and the preparation of television courses designed to help students revise their knowledge for the entrance examinations to higher educational establishments.

To make it easier for readers to grasp the material, some of the sections have been supplemented with explanatory text. The problems are all answered and some have additional hints or complete solutions.

The more difficult problems are marked with asterisks.

Part 4 is entitled "Oral Examination Problems and Questions" and includes samples suggested by the higher schools.

The authors hope that this book will help those who want to enter the various types of higher school, aid the teachers, and be of use to all those who want to deepen and systematize their knowledge of mathematics.

The authors

CONTENTS

ALGEBRA, TRIGONOMETRY, AND ELEMENTARY FUNCTIONS

1.1. Problems on Integers. Criteria for Divisibility

Natural numbers ($N = \{1, 2, 3, \ldots, n, \ldots\}$). Every natural number n can be uniquely factored into a product of prime factors

$$n = p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m},$$

where $p_1, \ldots, p_m$ are elementary divisors of the number n, and $k_1, \ldots, k_m$ are the multiplicities of those divisors ($k_1, \ldots, k_m \in N$).

To calculate the *greatest common divisor* of two natural numbers, each of their elementary common divisor must be raised to a power which is equal to the smallest of the multiplicities with which the divisor appears in the prime factorization of the given numbers and all the resulting numbers must then be multiplied together.

To calculate the *least common multiple* of two natural numbers, every prime divisor appearing in the factorization of at least one of these numbers must be raised to a power which is equal to the largest of the multiplicities with which the divisor appears in the prime factorization of the given numbers and all the resulting numbers must then be multiplied together.

Example. Assume $n_1 = 2^3 \cdot 5^2 \cdot 7 = 1400$, $n_2 = 2^2 \cdot 3^2 \cdot 5 \cdot 11 = 1980$. The greatest common divisor of n_1 and n_2 is $2^2 \cdot 5 = 20$. The least common multiplicity of n_1 and n_2 is $2^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 = 138600$.

If a_0 is a digit in the unit's place, a_1 is a digit in the ten's place, etc., of the natural number n, then this number can be written as

$$n = \overline{a_k a_{k-1} \ldots a_1 a_0} = a_k \cdot 10^k + \ldots + a_1 \cdot 10 + a_0.$$

Criteria for divisibility. The number n can be divided

(1) by 2 (by 5) if and only if a_0 can be divided by 2 (by 5);

(2) by 4 if and only if the number $a_1 \cdot 10 + a_0$ can be divided by 4;

(3) by 3 (by 9) if and only if the sum of all the digits in that number can be divided by 3 (by 9).

1. The sum of the digits in a two-digit number is 6. If we add 18 to that number, we get a number consisting of the same digits written in the reverse order. Find the number.

2. If we multiply a certain two-digit number by the sum of its digits, we get 405. If we multiply the number consisting of the same digits written in the reverse order by the sum of the digits, we get 486. Find the number.

3. Find three numbers, the second of which is as much greater than the first as the third is greater than the second, if the product of the two smaller numbers is 85 and the product of the two larger numbers is 115.

4. The sum of two numbers is equal to 15 and their arithmetic mean is 25 per cent greater than their geometric mean. Find the numbers.

5. The difference between two numbers is 48, the difference between the arithmetic mean and the geometric mean of the numbers is 18. Find the numbers.

6. The geometric mean of two numbers exceeds by 12 the smaller of the numbers, and the arithmetic mean of the same numbers is smaller by 24 than the larger of the numbers. Find the numbers.

7. Find the two-digit number if the number of its units exceeds by 2 the number of its tens and the product of the required number by the sum of its digits is equal to 144.

8. The product of the digits of a two-digit number is twice as large as the sum of its digits. If we subtract 27 from the required number, we get a number consisting of the same digits written in the reverse order. Find the number.

9. The product of the digits of a two-digit number is one-third that number. If we add 18 to the required number, we get a number consisting of the same digits written in the reverse order. Find the number.

10. The sum of the squares of the digits of a two-digit number is 13. If we subtract 9 from that number, we get a number consisting of the same digits written in the reverse order. Find the number.

11. A two-digit number is thrice as large as the sum of its digits, and the square of that sum is equal to the trebled required number. Find the number.

12. Find a two-digit number which exceeds by 12 the sum of the squares of its digits and by 16 the doubled product of its digits.

13. The sum of the squares of the digits constituting a two-digit number is 10, and the product of the required number by the number consisting of the same digits written in the reverse order is 403. Find the number.

14. If we divide a two-digit number by the sum of its digits, we get 4 as a quotient and 3 as a remainder. Now if we divide that two-digit number by the product of its digits, we get 3 as a quotient and 5 as a remainder. Find the two-digit number.

15. If we divide a two-digit number by a number consisting of the same digits written in the reverse order, we get 4 as a quotient and 15 as a remainder; now if we subtract 9 from the given number, we get the sum of the squares of the digits constituting that number. Find the number.

16. Find the two-digit number the quotient of whose division by the product of its digits is equal to 8/3, and the difference between the required number and the number consisting of the same digits written in the reverse order is 18.

17. Find the two-digit number, given: the quotient of the required number and the sum of its digits is 8; the quotient of the product of its digits and that sum is 14/9.

18. If we divide the unknown two-digit number by the number consisting of the same digits written in the reverse order, we get 4 as a quotient and 3 as a remainder. Now if we divide the required number by the sum of its digits, we get] 8 as a quotient and 7 as a remainder. Find the number.

19. If we divide a two-digit number by the sum of its digits, we get 6 as a quotient and 2 as a remainder. Now if we divide it by the product of its digits, we get 5 as a quotient and 2 as a remainder. Find the number.

20. What two-digit number is less than the sum of the squares of its digits by 11 and exceed their doubled product by 5?

21. Find two successive natural numbers if the square of the sum of those numbers exceeds the sum of their squares by 112.

22. First we increase the denominator of a positive fraction by 3 and next time we decrease it by 5. The sum of the resulting fractions proves to be equal to 2/3. Find the denominator of the fraction if its numerator is 2.

23. The denominator of an irreducible fraction is greater than the numerator by 2. If we diminish the numerator of the inverse fraction by 3 and subtract the given fraction from the resulting one, we get 1/15. Find the fraction.

24. Let us consider a fraction whose denominator is smaller than the square of the numerator by unity. If we add 2 to the numerator and the denominator, the fraction will exceed 1/3; now if we subtract 3 from the numerator and the denominator, the fraction remains positive but smaller than 1/10. Find the fraction.

25. There are only three positive two-digit numbers such that each number is equal to the incomplete square of the sum of its digits. Find two of them, being given that the second number exceeds the first one by 50 units.

26. Find the sum of all two-digit numbers which, being divided by 4, leave a remainder of 1.

27. Find the sum of all three-digit numbers which give a remainder of 4 when they are divided by 5.

28. Find the sum of all two-digit numbers which give a remainder of 3 when they are divided by 7.

29. Find the sum of all odd three-digit numbers which are divisible by 5.

30. The product of a two-digit number by a number consisting of the same digits written in the reverse order is equal to 2430. Find the number.

31. Find the pairs of natural numbers the difference of whose squares is 45.

32. There is a natural number which becomes equal to the square of a natural number when 100 is added to it, and to the square of another natural number when 168 is added to it. Find the number.

33. Find two natural numbers, whose sum is 85 and whose least common multiple is 102.

34. Find all pairs of natural numbers whose greatest common divisor is 5 and the least common multiple is 105.

35. Find two three-digit numbers whose sum is a multiple of 504 and the quotient is a multiple of 6.

36. Represent the number 19 as the difference between the cubes of natural numbers.

37. Find three numbers if the cube of the first number exceeds their product by 2, the cube of the second number is smaller than their product by 3, and the cube of the third number exceeds their product by 3.

38. Find all two-digit numbers such that the sum of the digits constituting the number is not less than 7; the sum of the squares of the digits is not greater than 30; the number consisting of the same digits written in the reverse order is not larger than half the given number.

39. In a four-digit number the sum of the digits in the thousands, hundreds and tens is equal to 14, and the sum of the digits in the units, tens and hundreds is equal to 15, the digit of the tens being greater by 4 than the digit of the tens. Among all the numbers satisfying these conditions, find the number the sum of the squares of whose digits is the greatest.

40. In a four-digit number the sum of the digits of the thousands and tens is equal to 4, the sum of the digits of the hundreds and the units is 15, and the digit of the units exceeds by 7 the digit of the thousands. Among all the numbers satisfying these conditions find the number the sum of the product of whose digit of the thousands by the digit of the units and the product of the digit of the hundreds by that of the tens assumes the least value.

41. Prove that if the number a is equal to the sum of the squares of two unequal natural numbers, then $2a$ is also equal to the sum of the squares of two unequal natural numbers.

42. Find the sum of all irreducible fractions between 10 and 20 with a denominator of 3.

43. There are natural numbers m and n. Find all the fractions m/n whose denominator is smaller than the numerator by 16, the fraction itself is smaller than the sum of the trebled inverse and 2, and the numerator is not greater than 30.

44. Given a sequence

$$u_n = \frac{(1+(-1)^n)+1}{5n+6}.$$

Find the number of the terms of the sequence (u_n) which satisfy the condition $u_n \in (1/100,\ 39/100)$.

1.2. Real Numbers. Transformation of Algebraic Expressions

Rational numbers (Q). Every rational number p/q ($p \in Z$ is an integer, $q \in N$ is a natural number) can be represented as an *infinite periodic decimal fraction* (possibly with a zero period)

$$p/q = \pm a,\ \alpha_1 \ldots \alpha_n\ (\beta_1\beta_2 \ldots \beta_m).$$

A reverse representation also holds

$$\pm a,\ \alpha_1 \ldots \alpha_n\ (\beta_1\beta_2 \ldots \beta_m) = \pm a \pm 0,\ \alpha_1 \ldots \alpha_n\ (\beta_1\beta_2 \ldots \beta_m)$$

$$= \pm a \pm \frac{\overline{\alpha_1 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m} - \overline{\alpha_1 \ldots \alpha_n}}{\underbrace{99 \ldots 9}_{n}\underbrace{0 \ldots 0}_{n}}.$$

Real numbers (R). The numbers which can be represented as various decimal fractions are called *real* numbers.

Formulas for abbreviated multiplication. The following equations hold for any numbers $a,\ b \in R$:

$$(a+b)^2 = a^2 + 2ab + b^2;$$
$$(a-b)^2 = a^2 - 2ab + b^2;$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3;$$
$$a^2 - b^2 = (a-b)(a+b);$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2);$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2);$$
$$a^4 - b^4 = (a-b)(a+b)(a^2 + b^2);$$
$$a^4 + b^4 = (a^2 - \sqrt{2}ab + b^2)(a^2 + \sqrt{2}ab + b^2);$$
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + a^{n-k}b^k + \ldots + b^{n-1}),\ n \in N;$$
$$a^{2n+1} + b^{2n+1} = (a+b)(a^{2n} - a^{2n-1}b + \ldots - ab^{2n-1} + b^{2n}),\ n \in N.$$

Represent the following mixed infinite decimal periodic fractions as common fractions (**1-4**):

1. 7 5 (3).

2. 2.1 (32).

3. $\dfrac{0.23\,(7)+\dfrac{43}{450}}{0.5\,(61)-\dfrac{113}{495}}$.

4. Find x if $\dfrac{0.1\,(6)+0.(3)}{0.(3)+1.1\,(6)}\,x=10$.

Calculate (**5-31**).

5. $\left(\dfrac{1}{3}\right)^{-10}\cdot 27^{-3}+0.2^{-4}\cdot 25^{-2}+(64^{-1/9})^{-3}$.

6. $\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)^2$.

7. $3\left(\dfrac{2}{\sqrt{10}+5}+\dfrac{5}{\sqrt{10}-2}-\dfrac{7}{\sqrt{10}}\right)$.

8. $\dfrac{(5\sqrt{3}+\sqrt{50})\,(5-\sqrt{24})}{\sqrt{75}-5\sqrt{2}}$.

9. $1+\sec 20^\circ-\sqrt{3}\cot 40^\circ$.

10. $\dfrac{2^{-2}\cdot 5^3\cdot 10^{-4}}{2^{-3}\cdot 5^2\cdot 10^{-5}}$.

11. $(6-4\,(5/16)^0)^{-2}+(2/3)^{-1}-3/4$.

12. $\dfrac{2^{-2}+2^0}{(1/2)^{-2}-5\cdot(-2)^{-2}+(2/3)^{-2}}$.

13. $\dfrac{(0.6)^0-(0.1)^{-1}}{(3/2^3)^{-1}\cdot(3/2)^3+(-1/3)^{-1}}$.

14. $\dfrac{3\dfrac{1}{3}\sqrt{9/80}-\dfrac{5}{4}\sqrt{4/5}+5\sqrt{1/5}+\sqrt{20}-10\sqrt{0.2}}{3\dfrac{1}{2}\sqrt{32}-\sqrt{4\dfrac{1}{2}}+2\sqrt{1/8}+6\sqrt{2/9}-140\sqrt{0.02}}\sqrt{2/5}$.

15. $\dfrac{2.4\sqrt{8\dfrac{1}{3}}-9\sqrt{1/3}+\sqrt{2\dfrac{1}{2}}+\dfrac{1}{2}\sqrt{1/3}-\dfrac{1}{3}\sqrt{27}}{1\dfrac{1}{3}\sqrt{4\,(1/2)}-\sqrt{0.5}+1.5\sqrt{2}+20\sqrt{1/50}-\sqrt{32}}\sqrt{2/3}$.

16. $\left(\sqrt{\left(\sqrt{2}-\dfrac{3}{2}\right)^2}-\sqrt[3]{(1-\sqrt{2})^3}\right)^2+2^{-3/2}\cos\dfrac{3\pi}{4}$.

17. $\left(\sqrt{\left(\sqrt{5}-\dfrac{5}{2}\right)^2}-\sqrt[3]{\left(\dfrac{3}{2}-\sqrt{5}\right)^3}\right)^{1/2}-\sqrt{2}\sin\dfrac{7\pi}{4}$.

18. $\left(\sqrt{\sqrt{3}+\sqrt{2}}-(\sqrt{3}-\sqrt{2})^{1/2}\right)\left((\sqrt{3}+\sqrt{2})^{1/2}\right.$
$\left.+\sqrt{\sqrt{3}-\sqrt{2}}\right)^{-1}-\cos\dfrac{5\pi}{4}$.

19. $\left(\sqrt{2+\sqrt{3}}+\left(2-2\cos\frac{11\pi}{6}\right)\right)^{1/2}\Big)$

$$\times\left(\sqrt{2+2\cos(\pi/6)}-\sqrt{2-\sqrt{3}}\right)^{-1}.$$

20. $\sqrt{25^{1/\log_8 5}+49^{1/\log_6 7}}$.

21. $\cos\left(\frac{\pi}{10}\left(\log_3\frac{1}{9}+\log_{1/9}3\right)\right)$.

22. $\log_{1/2}(\log_3\cos(\pi/6)-\log_3\sin(\pi/6))$.

23. $\left(\left(\frac{3}{2}\right)^{-0.5}\cdot 3^{\frac{1}{2}\log_3 6}+1\right)^{1/2}\sin\frac{7\pi}{3}$.

24. $\left(7^{1/3}\cdot 3^{-\log_{27}7}\cdot\tan\frac{5\pi}{3}+3^{1/2}+2\right)^{1/2}\cdot\cos\frac{7\pi}{4}$.

25. $\left((128^{3/7}\cdot 27^{1/3}\cdot 10^{-\log 48})^{-1/2}+\cot^{-1}\frac{2\pi}{3}\right)^2+2\cdot 6^{1/2}$.

26. $\left(3^{1/2}\cdot 8^{\log_2 3}\cdot\sin\frac{2\pi}{3}+\left(\frac{1}{3}\right)^{-4}\right)^{1/2}\cos^{-1}\frac{5\pi}{6}$.

27. $(x^{1/3}+y^{1/3})(x^{2/3}-x^{1/3}y^{1/3}+y^{2/3})$ for $x=4\frac{5}{7}$, $y=5\frac{2}{7}$.

28. $\frac{x-1}{x^{3/4}+x^{1/2}}\cdot\frac{x^{1/2}+x^{1/4}}{x^{1/2}+1}\cdot x^{1/4}+1$ for $x=16$.

29. $$\frac{a^3-a-2b-\frac{b^2}{a}}{\left(1-\sqrt{\frac{1}{a}+\frac{b}{a^2}}\right)(a+\sqrt{a+b})}:\left(\frac{a^3+a^2+ab+a^2b}{a^2-b^2}+\frac{b}{a-b}\right)$$
for $a=23$, $b=22$.

30. $x^3+3x-14$ for

$$x=\sqrt[3]{7+5\sqrt{2}}-\frac{1}{\sqrt[3]{7+5\sqrt{2}}}.$$

31. The difference $\sqrt{|40\sqrt{2}-57|}-\sqrt{40\sqrt{2}+57}$ is an integer. Find that integer.

Remove the irrationality in the denominator **(32-33).**

32. $\frac{1}{1+\sqrt{2}+\sqrt{3}}$.

33. $\frac{1}{\sqrt{2\sqrt{3}-\sqrt{2}\sqrt{\sqrt{2}+\sqrt[3]{3}}}}$.

34. Compare the following two numbers: $a=\frac{9}{\sqrt{11}-\sqrt{2}}$ and $b=\frac{6}{3\sqrt{3}}$.

35. Given: $1<a<b+c<a+1$, $b<c$. Prove that $a<b$.

36. What is larger, $\log_3 108$ or $\log_5 375$?

Arrange the following numbers in an increasing order (**37-39**).

37. 0; $\sqrt{0.8}$; 1.2; $11/30$; 0.91846.

38. 1; 0.37; $65/63$; $61/59$; $\tan 33^\circ$; $\tan(-314^\circ)$.

39. 0.02; 1; 0.85; $\sqrt{3}/2$; $\sqrt{0.762}$; $-\cos 571^\circ$.

40. Prove that 53^{53}-33^{33} is divisible by 10.

Factor (**41-43**).

41. n^4+4.

42. $1+n^4+n^8$.

43. $1+x^5$.

Simplify the following expressions (**44-166**).

44. $\frac{a^2-b^2}{a-b}-\frac{a^3-b^3}{a^2-b^2}$.

45. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$.

46. $\left(a+\frac{ab}{a-b}\right)\left(\frac{ab}{a+b}-a\right):\frac{a^2+b^2}{a^2-b^2}$.

47. $\left(\frac{4(a+b)^2}{ab}-16\right)\left(\frac{(a+b)^2-ab}{ab}\right):\frac{a^3-b^3}{ab}$.

48. $\left(\frac{a+3b}{(a-b)^2}+\frac{a-3b}{a^2-b^2}\right):\frac{a^2+3b^2}{(a-b)^2}$.

49. $\left(m+n-\frac{4mn}{m+n}\right):\left(\frac{m}{m+n}-\frac{n}{n-m}-\frac{2mn}{m^2-n^2}\right)$.

50. $\left(\frac{1}{(m+n)^2}\left(\frac{1}{m^2}+\frac{1}{n^2}\right)+\frac{2}{(m+n)^3}\left(\frac{1}{m}+\frac{1}{n}\right)\right)m^2n^2$.

51. $\left(\frac{a\sqrt{2}}{(1+a^2)^{-1}}-\frac{2\sqrt{2}}{a^{-1}}\right)\frac{a^{-3}}{1-a^{-2}}$.

52. $\frac{\frac{1}{a}+\frac{1}{b+c}}{\frac{1}{a}-\frac{1}{b+c}}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)(a+b+c)^{-2}$.

53. $\frac{\frac{1}{a}-\frac{1}{b+c}}{\frac{1}{a}+\frac{1}{b+c}}\left(1+\frac{b^2+c^2-a^2}{2bc}\right):\frac{a-b-c}{abc}$.

54. $\frac{x^3+y^3}{x+y}:(x^2-y^2)+\frac{2y}{x+y}-\frac{xy}{x^2-y^2}$.

55. $\left(\left(\left(\frac{a+1}{a-1}\right)^2+3\right):\left(\left(\frac{a-1}{a+1}\right)^2+3\right)\right):\frac{a^3+1}{a^3-1}-\frac{2a}{a-1}$.

56. $\left(\left(\frac{y}{y-x}\right)^{-2}-\frac{(x+y)^2-4xy}{x^2-xy}\right)\frac{x^4}{x^2y^2-y^4}$.

57. $\left(\frac{a}{a^2-4}-\frac{8}{a^2+2a}\right)\frac{a^2-2a}{4-a}+\frac{a+8}{a+2}$.