the theory of groups and quantum mechanics

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THE THEORY OF GROUPS AND QUANTUM MECHANICS

BY

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WITH 3[illegible] DIAGRAMS

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FROM THE AUTHOR'S PREFACE TO THE FIRST GERMAN EDITION

THE importance of the standpoint afforded by the theory of groups for the discovery of the general laws of quantum theory has of late become more and more apparent. Since I have for some years been deeply concerned with the theory of the representation of continuous groups, it has seemed to me appropriate and important to give an account of the knowledge won by mathematicians working in this field in a form suitable to the requirements of quantum physics. An additional impetus is to be found in the fact that, from the purely mathematical standpoint, it is no longer justifiable to draw such sharp distinctions between finite and continuous groups in discussing the theory of their representations as has been done in the existing texts on the subject. My desire to show how the concepts arising in the theory of groups find their application in physics by discussing certain of the more important examples has necessitated the inclusion of a short account of the foundations of quantum physics, for at the time the manuscript was written there existed no treatment of the subject to which I could refer the reader. In brief this book, if it fulfills its purpose, should enable the reader to learn the essentials of the theory of groups and of quantum mechanics as well as the relationships existing between these two subjects; the mathematical portions have been written with the physicist in mind, and vice versa. I have particularly emphasized the "reciprocity" between the representations of the symmetric permutation group and those of the complete linear group; this reciprocity has as yet been unduly neglected in the physical literature, in spite of the fact that it follows most naturally from the conceptual structure of quantum mechanics.

There exists, in my opinion, a plainly discernible parallelism between the more recent developments of mathematics and physics. Occidental mathematics has in past centuries broken away from the Greek view and followed a course which seems to have originated in India and which has been transmitted, with additions, to us by the Arabs; in it the concept of number appears as logically prior to the concepts of geometry. The result of this has been that we have applied this systematically developed number concept to all branches, irrespective of whether it is most appropriate for these particular applications. But the present trend in mathematics is clearly in the direction of a return to the Greek standpoint; we now look upon each branch of mathematics as determining its own characteristic domain of quantities. The algebraist of the present day considers the continuum of real or complex numbers as merely one "field" among many; the recent axiomatic foundation of projective geometry may be considered as the geometric counterpart of this view. This newer mathematics, including the modern theory of groups and "abstract algebra," is clearly motivated by a spirit different from that of "classical mathematics," which found its highest expression in the theory of functions of a complex variable. The continuum of real numbers has retained its ancient prerogative in physics for the expression of physical measurements, but it can justly be maintained that the essence of the new Heisenberg-Schrödinger-Dirac quantum mechanics is to be found in the fact that there is associated with each physical system a set of quantities, constituting a non-commutative algebra in the technical mathematical sense, the elements of which are the physical quantities themselves.

ZURICH, *August, 1928*

AUTHOR'S PREFACE TO THE SECOND GERMAN EDITION

DURING the academic year 1928-29 I held a professorship in mathematical physics in Princeton University. The lectures which I gave there and in other American institutions afforded me a much desired opportunity to present anew, and from an improved pedagogical standpoint, the connection between groups and quanta. The experience thus obtained has found its expression in this new edition, in which the subject has been treated from a more thoroughly *elementary* standpoint. Transcendental methods, which are in group theory based on the calculus of *group characteristics*, have the advantage of offering a rapid view of the subject as a whole, but true understanding of the relationships is to be obtained only by following an explicit elementary development. I may mention in this connection the derivation of the *Clebsch-Gordan* series, which is of fundamental importance for the whole of spectroscopy and for the applications of quantum theory to chemistry, the section on the *Jordan-Hölder* theorem and its analogues, and above all the careful investigation of the connection between the algebra of symmetric transformations and the symmetric permutation group. The reciprocity laws expressing this connection, which were proved by transcendental methods in the first edition, as well as the group-theoretic problem arising from the existence of spin have also been treated from the elementary standpoint. Indeed, the whole of Chapter V—which was, in the opinion of many readers, much too condensed and more difficult to understand than the rest of the book—has been entirely re-written. The algebraic standpoint has been emphasized, in harmony with the recent development of " abstract algebra," which has proved so useful in simplifying and unifying general concepts. It seemed impossible to avoid presenting the principal part of the theory of representations twice; first in Chapter III, where the representations are taken as given and their properties examined, and again in Chapter V, where the method of constructing the representations of a given group and of deducing their properties is developed. But I believe the reader will find this two-fold treatment an advantage rather than a hindrance.

To come to the changes in the more physical portions, in Chapter IV the rôle of the group of virtual rotations of space is more clearly presented. But above all several sections have been added which deal with the energy-momentum theorem of quantum physics and with the quantization of the wave equation in accordance with the recent work of *Heisenberg and Pauli*. This extension already leads so far away from the fundamental purpose of the book that I felt forced to omit the formulation of the quantum laws in accordance with the general theory of relativity, as developed by *V. Fock* and myself, in spite of its desirability for the deduction of the energy-momentum tensor. The fundamental problem of the proton and the electron has been discussed in its relation to the symmetry properties of the quantum laws with respect to the interchange of right and left, past and future, and positive and negative electricity. At present no solution of the problem seems in sight; I fear that the clouds hanging over this part of the subject will roll together to form a new crisis in quantum physics. I have intentionally presented the more difficult portions of these problems of spin and second quantization in considerable detail, as they have been for the most part either entirely ignored or but hastily indicated in the large number of texts which have now appeared on quantum mechanics.

It has been rumoured that the "group pest" is gradually being cut out of quantum physics. This is certainly not true in so far as the rotation and Lorentz groups are concerned; as for the permutation group, it does indeed seem possible to avoid it with the aid of the *Pauli* exclusion principle. Nevertheless the theory must retain the representations of the permutation group as a natural tool in obtaining an understanding of the relationships due to the introduction of spin, so long as its specific dynamic effect is neglected. I have here followed the trend of the times, as far as justifiable, in presenting the group-theoretic portions in as elementary a form as possible. The calculations of perturbation theory are widely separated from these general considerations; I have therefore restricted myself to indicating the method of attack without either going into details or mentioning the many applications which have been based on the ingenious papers of *Hartree*, *Slater*, *Dirac* and others.

The constants c and h, the velocity of light and the quantum of action, have caused some trouble. The insight into the significance of these constants, obtained by the theory of relativity on the one hand and quantum theory on the other, is most forcibly expressed by the fact that they do not occur in the laws of Nature in a thoroughly systematic development of these theories. But physicists prefer to retain the usual c.g.s. units—principally because they are of the order of magnitude of the physical quantities with which we deal in everyday life. Only a wavering compromise is possible between these practical considerations and the ideal of the systematic theorist; I initially adopt, with some regret, the current physical usage, but in the course of Chapter IV the theorist gains the upper hand.

An attempt has been made to increase the clarity of the exposition by numbering the formulæ in accordance with the sections to which they belong, by emphasizing the more important concepts by the use of boldface type on introducing them, and by lists of operational symbols and of letters having a fixed significance.

H. WEYL.

GÖTTINGEN, *November, 1930*

TRANSLATOR'S PREFACE

THIS translation was first planned, and in part completed, during the academic year 1928-29, when the translator was acting as assistant to Professor Weyl in Princeton. Unforeseen delays prevented the completion of the manuscript at that time, and as Professor Weyl decided shortly afterward to undertake the revision outlined in the preface above it seemed desirable to follow the revised edition. In the preparation of this manuscript the German has been followed as closely as possible, in the conviction that any alterations would but detract from the elegant and logical treatment which characterizes Professor Weyl's works. While an attempt has been made to follow the more usual English terminology in general, this programme is limited by the fact that the fusion of branches of knowledge which have in the past been so widely separated as the theory of groups and quantum theory can be accomplished only by adapting the existing terminology of each to that of the other; a minor difficulty of a similar nature is to be found in the fact that the development of "fields" and "algebras" in Chapter V is accomplished in a manner which makes it appear desirable to deviate from the accepted English terminology.

It is a pleasure to express my indebtedness to Professor Weyl for general encouragement and assistance, to Professor R. E. Winger of Union College for the assistance he has rendered in correcting proof and in preparing the index, and to the publishers for their coöperation in adhering as closely as possible to the original typography.

H. P. ROBERTSON

PRINCETON, *September*, 1931

INTRODUCTION

THE *quantum theory of atomic processes* was proposed by NIELS BOHR in the year 1913, and was based on the atomic model proposed earlier by RUTHERFORD. The deduction of the *Balmer* series for the line spectrum of hydrogen and of the *Rydberg* number from universal atomic constants constituted its first convincing confirmation. This theory gave us the key to the understanding of the regularities observed in optical and X-ray spectra, and led to a deeper insight into the structure of the periodic system of chemical elements. The issue of *Naturwissenschaften*, dedicated to BOHR and entitled "Die ersten zehn Jahre der Theorie von NIELS Bohr über den Bau der Atome" (Vol. **11,** p. 535 (1923)), gives a short account of the successes of the theory at its peak. But about this time it began to become more and more apparent that the BOHR theory was a compromise between the old "classical" physics and a new quantum physics which has been in the process of development since Planck's introduction of energy quanta in 1900. BOHR described the situation in an address on "Atomic Theory and Mechanics" (appearing in *Nature*, **116,** p. 845 (1925)) in the words: "From these results it seems to follow that, in the general problem of the quantum theory, one is faced not with a modification of the mechanical and electrodynamical theories describable in terms of the usual physical concepts, but with an essential failure of the pictures in space and time on which the description of natural phenomena has hitherto been based." The rupture which led to a new stage of the theory was made by HEISENBERG, who replaced Bohr's negative prophecy by a positive guiding principle.

The foundations of the new quantum physics, or at least its more important theoretical aspects, are to be treated in this book. For supplementary references on the physical side, which are urgently required, I name above all the fourth edition of SOMMERFELD's well-known "Atombau und Spektrallinien" (Braunschweig, 1924), or the English translation "Atomic Structure and Spectral Lines" (London, 1923) of the third edition, together with the recent (1929) "Wellenmechanischer Ergänzungsband" or its English translation "Wave Mechanics" (1930). An equivalent original English book is that of RUARK AND UREY, "Atoms, Molecules and Quanta" (New York, 1930), which appears in the "International Series in Physics," edited by RICHTMEYER. I should also recommend GERLACH's short but valuable survey "Experimentelle Grundlagen der Quantentheorie" (Braunschweig, 1921). The spectroscopic data, presented in accordance with the new quantum theory, together with complete references to the literature, are given in the following three volumes of the series "Struktur der Materie," edited by BORN AND FRANCK:—

F. HUND, "Linienspektren und periodisches System der Elemente" (1927);

E. BACK AND A. LANDÉ, "Zeemaneffekt und Multiplettstruktur der Spektrallinien" (1925);

W. GROTRIAN, "Graphische Darstellung der Spektren von Atomen und Ionen mit ein, zwei und drei Valenzelektronen" (1928).

The spectroscopic aspects of the subject are also discussed in PAULING AND GOUDSMIT's recent "The Structure of Line Spectra" (1930), which also appears in the "International Series in Physics."

The development of quantum theory has only been made possible by the enormous *refinement of experimental technique*, which has given us an almost direct insight into atomic processes. If in the following little is said concerning the experimental facts, it should not be attributed to the mathematical haughtiness of the author; to report on these things lies outside his field. Allow me to express now, once and for all, my deep respect for the work of the experimenter and for his fight to wring significant facts from an inflexible Nature, who says so distinctly "No" and so indistinctly "Yes" to our theories.

Our generation is witness to a development of physical knowledge such as has not been seen since the days of KEPLER, GALILEO AND NEWTON, and mathematics has scarcely ever experienced such a stormy epoch. Mathematical thought removes the spirit from its worldly haunts to solitude and renounces the unveiling of the secrets of Nature. But as recompense, mathematics is less bound to the course of worldly events than physics. While the quantum theory can be traced back only as far as 1900, the origin of the *theory of groups* is lost in a past scarcely accessible to history; the earliest works of art show that the symmetry groups of plane figures were even then already known, although the theory of these was only given definite form in the latter part of the eighteenth and in the nineteenth centuries. F. KLEIN considered the group concept as most characteristic of nineteenth century mathematics. Until the present, its most important application to natural science lay in the description of the symmetry of crystals, but it has recently been recognized that group theory is of fundamental importance for quantum physics; it here reveals the essential features which are not contingent on a special form of the dynamical laws nor on special assumptions concerning the forces involved. We may well expect that it is just this part of quantum physics which is most certain of a lasting place. Two groups, *the group of rotations in 3-dimensional space* and *the permutation group*, play here the principal rôle, for the laws governing the possible electronic configurations grouped about the stationary nucleus of an atom or an ion are spherically symmetric with respect to the nucleus, and since the various electrons of which the atom or ion is composed are identical, these possible configurations are invariant under a permutation of the individual electrons. The investigation of groups first becomes a connected and complete theory in *the theory of the representation of groups by linear transformations*, and it is exactly this mathematically most important part which is necessary for an adequate description of the quantum mechanical relations. *All quantum numbers, with the exception of the so-called principal quantum number, are indices characterizing representations of groups.*

This book, which is to set forth the *connection between groups and quanta*, consists of five chapters. The first of these is concerned with *unitary geometry*. It is somewhat distressing that the theory of linear algebras must again and again be developed from the beginning, for the fundamental concepts of this branch of mathematics crop up everywhere in mathematics and physics, and a knowledge of them should be as widely disseminated as the elements of differential calculus. In this chapter many details will be introduced with an eye to future use in the applications; it is to be hoped that in spite of this the simple thread of the argument has remained plainly visible. Chapter II is devoted to preparation on the physical side; only that has been given which seemed to me indispensable for an understanding of the meaning and methods of *quantum theory*. A multitude of physical phenomena, which have already been dealt with by quantum theory, have been omitted. Chapter III develops the elementary portions of *the theory of representations of groups* and Chapter IV *applies them to quantum physics*. Thus mathematics and physics alternate in the first four chapters, but in Chapter V the two are fused together, showing how completely the mathematical theory is adapted to the requirements of quantum physics. In this last chapter *the permutation group and its representations*, together with the groups of linear transformations in an affine or unitary space of an arbitary number of dimensions, will be subjected to a thorough going study.

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C. Explicit Algebraic Construction

CHAPTER I

UNITARY GEOMETRY

§ 1. The *n*-dimensional Vector Space

THE mathematical field of operation of quantum mechanics, as well as of the theory of the representations of groups, is the multi-dimensional affine or unitary space. The axiomatic method of developing the geometry of such a space is no doubt the most appropriate, but for the sake of clearness I shall at first proceed along purely algebraic lines. I begin with the explanation that a **vector** $\mathfrak{x}$ in the n-dimensional linear space $\mathfrak{R} = \mathfrak{R}_n$ is a set of n ordered numbers $(x_1, x_2 \cdots, x_n)$; vector analysis is the calculus of such ordered sets. The two fundamental operations of the vector calculus are *the multiplication of a vector* $\mathfrak{x}$ *by a number* a and the *addition of two vectors* $\mathfrak{x}$ *and* $\mathfrak{y}$. On introducing the notation

$$\mathfrak{x} = (x_1, x_2, \cdots, x_n), \quad \mathfrak{y} = (y_1, y_2, \cdots, y_n)$$

these operations are defined by the equations

$$a\mathfrak{x} = (ax_1, ax_2, \cdots, ax_n), \quad \mathfrak{x} + \mathfrak{y} = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n).$$

The fundamental rules governing these operations of multiplication by a number and addition are given in the following table of axioms, in which small German letters denote arbitrary vectors and small Latin letters arbitrary numbers:

(α) *Addition.*

1. $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$ (*commutative law*).
2. $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c})$ (*associative law*).
3. $\mathfrak{a}$ *and* $\mathfrak{c}$ *being any two vectors, there exists one and only one vector* $\mathfrak{x}$ *for which* $\mathfrak{a} + \mathfrak{x} = \mathfrak{c}$. *It is called the difference* $\mathfrak{c} - \mathfrak{a}$ *of* $\mathfrak{c}$ *and* $\mathfrak{a}$ (*possibility of subtraction*).