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# HARMONIC ANALYSIS IN EUCLIDEAN SPACES

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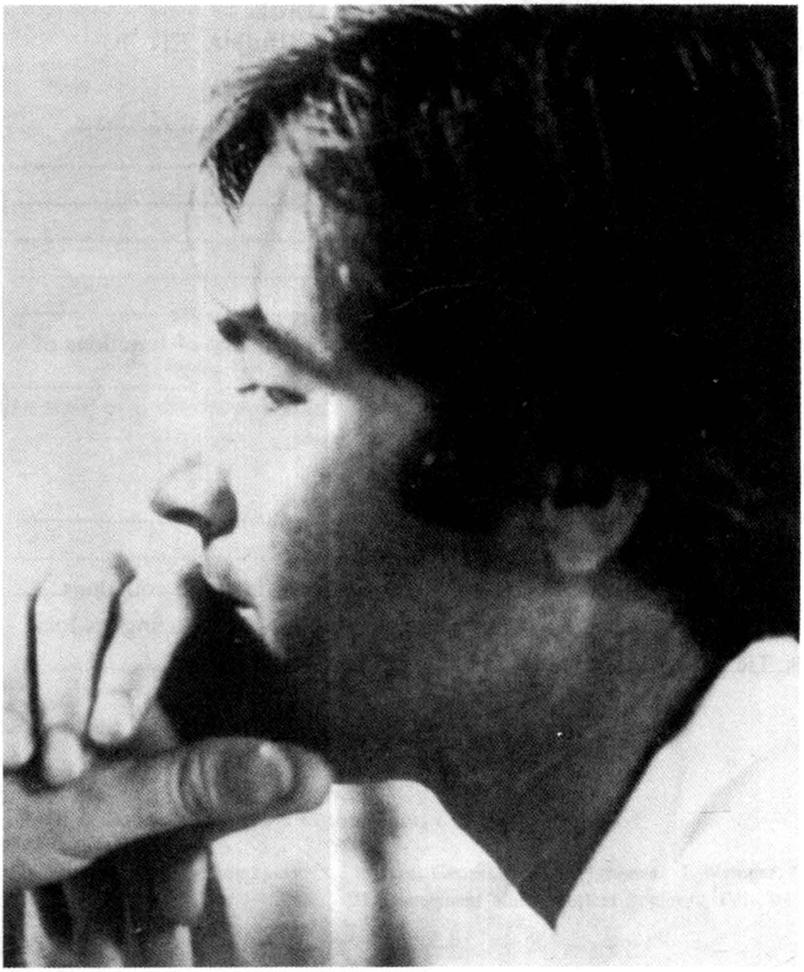
# HARMONIC ANALYSIS IN EUCLIDEAN SPACES

**Part 1**



These Proceedings are dedicated to

**NESTOR M. RIVIÈRE**

**1940–1978**

NESTOR M. RIVIÈRE
JUNE 10, 1940
JANUARY 3, 1978

It is striking to contemplate the influence of Nestor Rivière upon so many areas in analysis, and even more striking to think of his influence upon so many people. His graciousness was reflected in his mathematical work. He loved to work with people and to share his ideas. Many of us attending this conference have benefited in our own work from these ideas and from the breadth of his mathematical knowledge. His collaborations with others were always marked by a brilliance, a willingness to listen, and an optimism that created an unending flow of ideas.

Born and raised in Buenos Aires, Argentina, Nestor entered the University at the age of 16. He received his Licenciatura in mathematics in 1960, married Marisa Renda in 1961 and taught in Buenos Aires and Bariloche until December 1962. At that time, with the help of A.P. Calderón, he came to the University of Chicago to pursue his mathematical studies. Nestor received his Ph.D. degree in 1966 and in the Fall of that year became a member of the faculty at the University of Minnesota. In April 1974 Marisa and Nestor's daughter, Melisa, was born.

Nestor was naturally influenced by his education at Chicago. Real and harmonic analysis always remained his primary mathematical interest. At Minnesota the environment was perfect for the development of this interest and for the application of his knowledge to problems in other areas, especially to certain areas of partial differential equations. From 1966 Nestor's work in real and harmonic analysis went along hand-in-hand with his work in P.D.E. . Below we review some of Nestor's work in four major areas: Singular Integrals, Multiplier Theory, Interpolation Theory, and Partial Differential Equations.

Nestor's love for mathematics and his desire to share ideas made him an exceptional teacher. During his years at Minnesota he supervised the theses of a number of students, among them were Eleonor Harboure de Aquilera, Nestor Aquilera, Norberto Fava, Robert Hanks, Wally Madych, and Felipe Zo.

## Singular Integrals

At the time Nestor was a student in Chicago the Calderón-Zygmund theory of singular integral operators of elliptic type had already arrived to a well understood stage. The techniques of the 1952 paper, "On Certain Singular Integrals", were being used by B.F. Jones to study the $L^p$-continuity of singular integrals arising from parabolic equations. Rivière realized that the entire theory could be placed under one general setting dependent on a fixed notion of dilation, namely

$$\lambda^\alpha x = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n) \ , \ \lambda > 0$$

where $\alpha_1, \ldots, \alpha_n$ are given positive numbers.

Associated with the above (nonisotropic) dilation is the metric, $r(x)$, defined for $x \neq 0$ as the unique positive number satisfying

$$\sum_{\alpha=1}^{n} \frac{x_j^2}{r^{2\alpha_j}} = 1 \ .$$

$r(x)$ has the homogeneity property, $r(\lambda^\alpha x) = \lambda r(x)$ and there is a polar decomposition of $R^n$ relative to $r$, i.e. each $x \neq 0$ can be written as

$$x = r^\alpha \sigma \ , \ |\sigma| = 1$$

and

$$dx = r^{(\Sigma\alpha_i)-1} J(\sigma)\, dr d\sigma$$

with $0 < J(\sigma) \in C^\infty(\Sigma)$, $\Sigma = \{\sigma : |\sigma| = 1\}$.

In this setting one can mimic the techniques of the 1952 paper of Calderón-Zygmund and prove the $L^p$ continuity, $1 < p < \infty$, of convolution singular integrals of the form

$$(*) \qquad \lim_{\epsilon \to 0} \int_{r(x-y) > \epsilon} k(x-y) f(y) dy$$

where

i) $k(x) \in C^1(R^n \setminus \{0\})$,

ii) $k(\lambda^\alpha x) = \lambda^{-\Sigma\alpha_i} k(x)$, $\lambda > 0$, $x \neq 0$,

iii) $\int_\Sigma k(\sigma) J(\sigma) d\sigma = 0$ where $d\sigma$ is area measure on $\Sigma$.

The proofs of the above results appeared in article [1] and the extensions of the results to certain nonconvolution type operators were given in [3].

Nestor went on to considerably generalize the setting in which one could consider convolution singular integral operators. In [13] he attaches the notion of a singular kernel with a one parameter family, $\{U_r : r > 0\}$, of open bounded neighborhoods of the origin satisfying the conditions:

i) $U_r \subset U_s$, $r < s$, $\bigcap_{r>0} \overline{U}_r = \{0\}$,

ii) The algebraic difference $U_r - U_r \subset U_{\phi(r)}$, with $\phi:(0,\infty)\to(0,\infty)$ nondecreasing, continuous, $\phi(r)\uparrow\infty$ as $r\uparrow\infty$.

iii) The Lebesque measure of $U_{\phi(r)}$, denoted by $m(U_{\phi(r)})$, is $\leq Am(U_r)$, $A$ independent of $r$.

Associated to such a family Rivière defined the notion of a singular kernel as a function, $k(x)\in L^1_{loc}(R^n\setminus\{0\})$ with the properties,

i) $\int_{U_s\setminus U_r} k(x)dx$ is bounded independently of $s$ and $r$ and

$\lim_{r\to 0+} \int_{U_s\setminus U_r} k(x)dx$ exists for each fixed $s>0$,

ii) $\int_{U_{\phi(r)}\setminus U_r} |k(x)|\,dx \leq A$, independent of $r>0$

iii) There exists $A>0$ such that $\int_{R^n\setminus U_{\phi(r)}} |k(x-y)-k(x)|\,dx \leq A$ for all $y\in U_r$ and for all $r$.

He then naturally defined the doubly truncated singular integral operator

$$K_{r,s}(f)(x) = \int_{U_s\setminus U_r} k(y)f(x-y)dy$$

and proved the following theorem, which was new even for the elliptic case, i.e. $r(x)=|x|$ and $\phi(r)=2r$.

Theorem. The operator $K_*f(x) = \sup_{r,s}|K_{r,s}(f)(x)|$ is bounded from $L^p\to L^p$, $1<p<\infty$, and from $L^1\to$ weak $L^1$.

In particular for $f\in L^p$, $1\leq p<\infty$ $\lim_{\substack{s\to\infty\\ r\to 0+}} K_{r,s}(f)(x)$ exist pointwise for almost every $x\in R^n$

In 1973 Nestor, together with Steve Wainger and Alex Nagel, returned to a problem in singular integrals which was first formulated in the study of the singular integral operators involving "mixed homogeneous" kernels defined earlier. The problem was to find a "method of rotation" for these operators corresponding to that developed by Calderón and Zygmund. In the latter case the $L^p$-continuity of a singular integral of elliptic type arising from an odd kernel was reduced to the continuity of the one dimensional Hilbert transform. The problem was to find the appropriate one-dimensional operator for the mixed homogeneous operators coming from an odd kernel. A candidate was formulated as early as 1966, namely for $x\in R^n$

$$T_\gamma f(x) = \lim_{\epsilon \to 0} \int_{|t|>\epsilon} f(x_1 - \text{sgn}\, t\,|t|^{\alpha_1}, \ldots, x_n - \text{sgn}\, t\,|t|^{\alpha_n}) \frac{dt}{t} ,$$

$\alpha_i > 0$ , $i = 1,\ldots,n$ .

This operator was called by Nagel, Rivière, and Wainger, the Hilbert transform of $f$ along the curve, $\gamma(t) = (\text{sgn}\, t\,|t|^{\alpha_1},\ldots,\text{sgn}\, t\,|t|^{\alpha_n})$ . In [24] they prove the continuity of $T_\gamma$ on $L^p(R^n)$ , $1 < p < \infty$ , and as a consequence, the continuity on $L^p(R^n)$ of the mixed homogeneous operators in the form (*) where the smoothness of $k(x)$ is replaced by the condition

$$\int_\Sigma |k(\sigma)| \log^+ |k(\sigma)| d\sigma < \infty .$$

Multiplier Theory

Rivière's interest in the theory of Fourier multipliers began as a graduate student in Chicago. In [1] there appears the extension of the Hörmander multiplier theorem to the case of multipliers, $m(x)$ , behaving like smooth functions with mixed homogeneity zero. More precisely if $r(x)$ denotes the metric, discussed in the previous part, corresponding to the given dilation

$$\lambda^\alpha x = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n) , \alpha_i > 0 ,$$

then $m(x)$ is a Fourier multiplier on all $L^p$ , $1 < p < \infty$ provided $m$ is bounded and

$$R^{2(\alpha\cdot\beta) - |\alpha|} \int_{R/2 \le r(x) \le 2R} |D^\beta m(x)|^2 dx \le C \text{ , independent of } R ,$$

for all $\beta$ , $|\beta| < N$ with $N > \frac{|\alpha|}{2}$ $(|\alpha| = \sum_i \alpha_i)$ .

When Nestor joined the faculty at Minnesota in 1966 he immediately began working with Walter Littman and Charles McCarthy on refinements of the Marcinkiewicz multiplier theorem in $R^n$ ([6]). At this same time he started studying a problem posed to him earlier by A.P. Calderón who asked if a bounded rational function on $R^d$ , $d > 1$ , was a Fourier multiplier on $L^p$ for some interval of p's around 2 . Already Littman, McCarthy, and Rivière had given in [7] an example of a bounded rational function on $R^2$ which was not a multiplier on $L^p(R^2)$ , $1 < p < \frac{4}{3}$ . The example was $\frac{1}{x^2 - y + i}$ . In [13] Nestor extended the Marcinkiewicz multiplier result of [6] to operator valued multipliers and proved that any bounded rational function on $R^d$ is a multiplier on the space of functions

$$L^p(R^{d-1}, L^2(R)) = \{f(t,x), t \in R, x \in R^{d-1} \text{ such that } \left(\int_{R^{d-1}} \left(\int |f(t,x)|^2 dt\right)^{p/2} dx\right)^{1/p} < \infty\} \quad (1 < p < \infty) .$$

Finally in [13] Nestor proved a version of the Hörmander multiplier theorem that not only considerably generalized the setting of the theorem but added an original twist which even for the elliptic case gave a very interesting result. It is in this setting that we would like to state the result.

Theorem. Let $\beta_j$, $j = 1,\dots,d$, be positive integers such that $\sum_{j=1}^{d} \beta_j^{-1} < 2$. Assume $m \in L^\infty$, and

$$\sup_{\substack{n = 0, \pm 1, \pm 2, \\ j = 1,\dots,d}} 2^{(2\beta_j - n)d} \int_{2^{n-1} < |x| < 2^n} |D_{x_j}^{\beta_j} m(x)|^2 dx < \infty .$$

Then $m$ is a Fourier multiplier on $L^p(R^d)$, $1 < p < \infty$.

The novelty of the above reslut is the "trade-off" of smoothness of the individual variables. One may assume a weak smoothness in one or several of the variables by requiring sufficient smoothness in the remaining ones.

Interpolation

Nestor began his studies at the University of Chicago in the area of interpolation. His unpublished thesis extended the Riesz-Thorin or Complex method of interpolation from Banach spaces to topological vector spaces, $B$, with a metric topology defined through an s-norm, $0 < s \le 1$, i.e. a function $\| \ \|_s : B \to [0, \infty)$ such that

i) $\|x\|_s = 0 \iff x = 0$

ii) $\|x + y\|_s \le \|x\|_s + \|y\|_s$

iii) $\|\lambda x\|_s = |\lambda|^s \|x\|_s$.

The metric is of course defined as $d(x,y) = \|x - y\|_s$. These spaces are called s-Banach spaces and prime examples are the Lebesque and Hardy spaces, $L^s(X,d\mu)$ and $H^s(R^n)$, $0 < s < 1$. In the thesis Nestor identifies, via the complex method, the intermediate spaces of various s-Banach spaces of functions and in particular shows that

$$[L^{p_1}(X,d\mu), L^{p_2}(X,d\mu)]_\alpha = L^p(X,d\mu)$$

where $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{(1-\alpha)}{p_2}$, $0 < \alpha < 1$, $0 < p_1$, $p_2 \le \infty$.

In [14] Nestor extends the techniques of the Marcinkiewicz interpolation theorem and as a consequence proves that any sublinear operator mapping boundedly

$$L^\infty(R^n) \to BMO(R^n)$$

and

$$L^1(R^n) \to L(1,\infty)$$

must also map boundedly $L^p \to L^p$ for $1 < p < \infty$. Here BMO denotes the space of functions with bounded mean oscillation, as defined by F. John and L. Nirenberg, and $L(1,\infty)$ is the Lorentz space of functions commonly called "weak $L^1$". This work of Nestor's, published in 1971, was his one mathematical paper written in Spanish.

Together with Yoram Sagher in [17], Nestor calculated the intermediate spaces, $(H^1, C_\omega)_{\theta,p}$, for the Lions-Peetre or real method of interpolation. Here $H^1$ denotes the classical Hardy space of functions defined on $R^n$ and $C_\omega$ denotes the class of continuous functions on $R^n$ vanishing at $\infty$. They proved

$$(H^1, C_\omega)_{\theta,p} = L^p \quad \text{for} \quad \frac{1}{p} = (1-\theta), \quad 0 < \theta < 1 .$$

As a consequence, if $M$ = space of finite measures then

$$(BMO, M)_{\theta,q} = L(p',q), \quad \frac{1}{p} = 1-\theta, \quad \frac{1}{p} + \frac{1}{p'} = 1 .$$

($L(p',q)$ denotes the usual Lorentz space.)

The above results were extended in [19] where the equalities

$$(H^{p_0}, L^\infty)_{\theta,p} = H^p \quad \text{and} \quad (H^{p_0}, H^{p_1})_{\theta,p} = H^p$$

were proved respectively for $\frac{1}{p} = \frac{1-\theta}{p_0}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < p_0, p_1 < \infty$, $0 < \theta < 1$.

Nestor's interest in interpolation remained throughout his career. In 1976 his student, Robert Hanks, identified in his thesis the intermediate space

$$(H^p, BMO)_{\theta,p} \quad \text{as} \quad H^p \quad \text{for} \quad p = \frac{p_0}{1-\theta}, \quad 0 < p_0 < \infty .$$

As a consequence Nestor's result on sublinear operators described above was extended to the case

$$T: L^\infty \to BMO$$

$$T: H^1 \to L(1,\infty) .$$

## Partial Differential Equations

As a graduate student Nestor was very interested in the use of singular integral operators as a general tool to study existence, uniqueness, and regularity for a large class of equations modeled mostly from elliptic operators. The use of the symbolic calculus, developed by A.P. Calderón and A. Zygmund in the elliptic case was adapted to parabolic operators in [4]. Any such operator, say $L = \sum_{|\alpha|=2b} A_\alpha(x,t) D_x^\alpha - D_t$, was decomposed on smooth functions with support in $R_+^{n+1}$ as

$$L = S((-1)^{b+1} \Delta^b - D_t)$$

with $S$ a parabolic singular integral operator. Assuming boundedness and uniform continuity on the coefficients the operator $S$ was shown to be invertible on $L^p(S_T)$ with $S_T = R^n \times (0,T)$ and $1 < p < \infty$. From this followed easily the existence and uniqueness for the problem

$$Lu = f \quad \text{in} \quad S_T \ , \ u(x,0) = 0$$

with $f \in L^p(S_T)$ and $u$ in the class of functions having spatial derivatives of orders $\leq 2b$ and one time derivative in $L^p(S_T)$. Also in papers [28] and [29] one can again find the development of a symbolic calculus designed to give general algebraic conditions for the solvability of initial boundary value problems associated with the Navier-Stokes equations.

The final three years of Nestor's life were dedicated to problems in partial differential equations and some of his best work was done at this time. Together with Luis Caffarelli very precise regularity results in two dimensions were obtained for the free boundaries arising from the solution of the minimal energy problem above a given obstacle and from the solution of the minimal surface problem staying above an obstacle.

In the above situations we are given a bounded, connected domain $D \subset R^n$ and a function $\varphi$, the obstacle, defined on $\overline{D}$, satisfying:

a) $\varphi < 0$ on $\partial D$

b) $\Delta\varphi$ and $\nabla(\Delta\varphi)$ do not vanish simultaneously.

We let $v(x)$, $x \in D$, be the solution of a specific variational inequality satisfying $v \geq \varphi$ in $D$. For example in the case of minimizing energy

$$\int_D |\nabla v|^2 dx = \inf \{\int_D |\nabla u|^2 dx : u\big|_{\partial D} = 0 \ , \ u \geq \varphi \ \text{in} \ D\} .$$

In [26] it was shown that the set of coincidence,

$$\Lambda = \{x \in D : v(x) = \varphi)x)\} ,$$

has the following structure in 2 dimensions:

Theorem. If $\Delta\varphi \in C^{k,\alpha}$, $0 < \alpha < 1$, $k \geq 1$, then each component of the interior of $\Lambda$ is composed of a finite number of Jordan arcs each having a nondegenerate $C^{k+1,\alpha}$ parametrization. Moreover if $\Delta\varphi$ is real analytic the Jordan arcs are real analytic.

It was later shown by Caffarelli (even for the general n-dimensional case) that if $x_0 \in \Lambda$ is a point of positive density of $\Lambda$ then there exists a ball, $B(x_0)$, about $x_0$ such that $\partial\Lambda \cap B(x_0)$ is a $C^1$ curve. In [31] Rivière and Caffarelli studied the case when $x_0$ is a point of zero density and showed the existence of a neighborhood, $B(x_0)$, of $x_0$ in which $\Lambda \cap B(x_0)$ is contained between two tangent $C^1$ curves. In fact, with a proper choice of coordinates $\Lambda \cap B(x_0)$ is contained between the curves

$$y = \pm C_1 \, |x| \exp\{-C_2(\log |x|)^{1/2}\} .$$

The final work submitted for publication by Rivière and his co-authors was [34]. Here A.P. Calderón's recent results concerning the Cauchy integral over a $C^1$-curve were used to solve the Dirichlet and Neumann problems for Laplace's equation in a $C^1$-domain, $D$, contained in $R^n$. The data were assumed to belong to $L^p(\partial D)$, $1 < p < \infty$, and the solutions were written respectively in the form of the classical double and single layer potentials. In the Dirichlet case the nontangential maximal function associated with the solution was shown to belong to $L^p(\partial D)$ and, as a consequence, the solution converged nontangentially to the data at almost every point of the boundary. Similarly in the Neumann problem the nontangential maximal function associated to the gradient of the solution was shown to belong to $L^p(\partial D)$ and again the data was assumed in a pointwise nontangential sense at almost every point of the boundary.

On November 23, 1977, during an informal gathering of harmonic analysts from the Midwest at the University of Chicago, Nestor spoke of some open problems which he considered exceptionally interesting. These problems are listed in this proceedings.

The last three years of Nestor's life were years of great personal growth. For each new crisis in his illness he found in himself new resources of courage. His sensitivity to other people increased, and his mathematical work continued unabated to the end. The grace he had shown under the most relentless pressure one has to face was his last, and greatest achievement.

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## Some Open Questions

1. Let $\{\mathcal{U}_t : t>0\}$ be a family of open bounded convex sets containing 0 such that $\mathcal{U}_s \subset \mathcal{U}_t$ for $s<t$, $\cap \overline{\mathcal{U}}_t = \{0\}$. If $\mu$ and $\nu$ are finite regular Borel measures is it true that

$$\lim_{t\to 0+} \frac{\mu(x+\mathcal{U}_t)}{\nu(x+\mathcal{U}_t)}$$ exists almost everywhere with respect to $\nu$ ?

2. Consider the fundamental solution, $m(x,y) = \dfrac{1}{x^2 - y + i}$ of the Schrödinger operator $-\dfrac{\partial^2}{\partial x^2} + i\dfrac{\partial}{\partial y} + i$. As a Fourier multiplier on $L^p(R^2)$ $m(x,y)$ is unbounded on $L^p$ for $p>4$. What can be said for the range $4/3 \le p \le 4$ ?

3. Suppose $P(x)$ and $Q(x)$ are polynomials on $R^n$ such that $\dfrac{P(x)}{Q(x)}$ is bounded. Does it follow that $\dfrac{P(x)}{Q(x)}$ is Fourier multiplier on $L^p(R^n)$ for some intervals of $p$'s around 2.

4. Assume $k_1(x)$ and $k_2(x)$, $x\in R^n$, are smooth functions on $R^n \setminus \{0\}$ such that $k_1(x)$ is an elliptic singular kernel and $k_2(x)$ is a parabolic kernel, i.e. $k_1(\lambda x) = \lambda^{-n} k_1(x)$, $\lambda>0$, $x\neq 0$ and its mean value over the unit sphere is zero; $k_2(\lambda x_1,\dots,\lambda x_{n-1},\lambda^2 x_n) = \lambda^{-n-1} k_2(x)$ and its appropriate mean value on the unit sphere is zero. Set $K_i f = k_i * f$. Does the composition $K_1K_2$ map $L^1 \to L^{1,\infty}$ ? (see [23]).

5. Suppose $T$ is a translation invariant operator mapping $L^p \to L^{p,\infty}$ for a given $p$, $1<p\le 2$. Does this imply $T : L^p \to L^{p,p'}$, $\frac{1}{p}+\frac{1}{p'}=1$ ? (The unknown cases are $1<p<2$.)

6. Let $Kf = k * f$ where $k(x)$, $x\in R^n$, is homogeneous of degree $-n$, mean value zero over the unit sphere, $\Sigma$, in $R^n$, and in $L \log^+ L(\Sigma)$. Does $K : L^\infty \to BMO$ ?

7. For a given bounded $C^1$-domain $D\subset R^n$, consider the boundary value problems

$$\Delta^2 u(x) = 0,\ x\in D, \text{ with } u|_{\partial D},\ \frac{\partial u}{\partial n}\Big|_{\partial D} \text{ given}$$

$$\Delta^2 u(x) = 0,\ x\in D, \text{ with } \frac{\partial u}{\partial n}\Big|_{\partial D},\ \frac{\partial^2 u}{\partial n^2}\Big|_{\partial D} \text{ given}$$

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Here $\dfrac{\partial^j u}{\partial n^j}\Big|_{\partial D}$ denotes the $j^{th}$ normal derivative of $u$ on $\partial D$. Prescribe classes of boundary data which give existence and uniqueness.

Since the meeting in Williamstown Carlos Kenig and Peter Tomas have answered problem 2 and, as a consequence, also problem 3. They have proved that $m(x,y)$ is only a multiplier on $L^2(R^2)$.