ADVANCED ALGEBRA

HAWKES

ADVANCED ALGEBRA

BY

HERBERT E. HAWKES, Ph.D.

Assistant Professor of Mathematics in Yale University

GINN AND COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON
ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

423.1

The Athenæum Press
GINN AND COMPANY · PRO-
PRIETORS · BOSTON · U.S.A.

PREFACE

This book is designed for use in secondary schools and in short college courses. It aims to present in concise but clear form the portions of algebra that are required for entrance to the most exacting colleges and technical schools.

The chapters on algebra to quadratics are intended for a review of the subject, and contain many points of view that should be presented to a student after he has taken a first course on those topics. Throughout the book the attention is concentrated on subjects that are most vital, pedagogically and practically, while topics that demand a knowledge of the calculus for their complete comprehension (as multiple roots, and Sturm's theorem) or are more closely related to other portions of mathematics (as theory of numbers, and series) have been omitted.

The chapter on graphical representation has been introduced early, in the belief that the illumination which it affords greatly enlivens the entire presentation of algebra. The discussion of the relation between pairs of linear equations and pairs of straight lines is particularly suggestive.

In each chapter the discussion is directed toward a definite result. The chapter on theory of equations aims to give a simple and clear treatment of the method of obtaining the real roots of an equation and the theorems that lead to that process. Similarly direct in its argument is the chapter on determinants, its object being the solution of non-homogeneous equations and the necessary evaluation of determinants.

I am under obligations to many friends and colleagues for suggestions, but especially to Professor P. F. Smith, who has read the book both in manuscript and proof and whose numerous suggestions have been invaluable.

New Haven, Connecticut
August, 1905

CONTENTS

ALGEBRA TO QUADRATICS

CHAPTER I

FUNDAMENTAL OPERATIONS

SECTION		PAGE
2.	Addition	1
3.	Subtraction	1
4.	Zero	2
5.	Multiplication	2
6.	Division	3
7.	Division by Zero	4
8.	Fundamental Operations	5
9.	Practical Demand for Negative and Fractional Numbers	5
10.	Laws of Operation	5
11.	Integral and Rational Expressions	6
12.	Operations on Polynomials	6
13.	Addition of Polynomials	7
14.	Subtraction of Polynomials	7
15.	Parentheses	8
16.	Multiplication	9
17.	Multiplication of Monomials by Polynomials	10
18.	Multiplication of Polynomials	10
19.	Types of Multiplication	11
20.	The Square of a Binomial	11
21.	The Square of a Polynomial	12
22.	The Cube of a Binomial	12
23.	Division	12
24.	Division of Monomials	13
25.	Division of a Polynomial by a Monomial	13
26.	Division of a Polynomial by a Polynomial	13
27.	Types of Division	15

CHAPTER II

FACTORING

SECTION		PAGE
28.	Statement of the Problem	16
29.	Monomial Factors	16
30.	Factoring by grouping Terms	17
31.	Factors of a Quadratic Trinomial	18
32.	Factoring the Difference of Squares	20
33.	Reduction to the Difference of Squares	20
34.	Replacing a Parenthesis by a Letter	21
35.	Factoring Binomials of the Form $a^n \pm b^n$	22
36.	Highest Common Factor	22
37.	H.C.F. of Two Polynomials	23
38.	Euclid's Method of finding the H.C.F.	23
39.	Method of finding the H.C.F. of Two Polynomials	24
40.	Least Common Multiple	26
41.	Second Rule for finding the Least Common Multiple	26

CHAPTER III

FRACTIONS

42.	General Principles	27
43.	Principle I	27
44.	Principle II	27
45.	Principle III	27
46.	Reduction	27
47.	Least Common Denominators of Several Fractions	28
48.	Addition of Fractions	29
49.	Subtraction of Fractions	29
50.	Multiplication of Fractions	29
51.	Division of Fractions	29

CHAPTER IV

EQUATIONS

52.	Introduction	32
53.	Identities and Equations of Condition	32
54.	Linear Equations in One Variable	33
55.	Solution of Problems	37
56.	Linear Equations in Two Variables	40
57.	Solution of a Pair of Equations	40

SECTION PAGE
58. Independent Equations . . . 41
59. Solution of a Pair of Simultaneous Linear Equations . . . 42
60. Incompatible Equations . . . 42
61. Résumé . . . 43
62. Solution of Problems involving Two Unknowns . . . 45
63. Solution of Linear Equations in Several Variables . . . 47

CHAPTER V

RATIO AND PROPORTION

64. Ratio . . . 49
65. Proportion . . . 49
66. Theorems concerning Proportion . . . 49
67. Theorem . . . 50
68. Mean Proportion . . . 50

CHAPTER VI

IRRATIONAL NUMBERS AND RADICALS

69. Existence of Irrational Numbers . . . 52
70. The Practical Necessity for Irrational Numbers . . . 53
71. Extraction of Square Root of Polynomials . . . 53
72. Extraction of Square Root of Numbers . . . 54
73. Approximation of Irrational Numbers . . . 55
74. Sequences . . . 56
75. Operations on Irrational Numbers . . . 56
76. Notation . . . 57
77. Other Irrational Numbers . . . 57
78. Reduction of a Radical to its Simplest Form . . . 58
79. Addition and Subtraction of Radicals . . . 59
80. Multiplication and Division of Radicals . . . 60
81. Rationalization . . . 61
82. Solution of Equations involving Radicals . . . 63

CHAPTER VII

THEORY OF INDICES

83. Negative Exponents . . . 66
84. Fractional Exponents . . . 66
85. Further Assumptions . . . 67
86. Theorem . . . 67
87. Operations with Radical Polynomials . . . 69

QUADRATICS AND BEYOND

CHAPTER VIII

QUADRATIC EQUATIONS

SECTION		PAGE
88.	Definition	70
89.	Solution of Quadratic Equations	70
90.	Pure Quadratics	72
91.	Solution of Quadratic Equations by Factoring	75
92.	Solution of an Equation by Factoring	75
93.	Quadratic Form	77
94.	Problems solvable by Quadratic Equations	79
95.	Theorems regarding Quadratic Equations	82
96.	Theorem	83
97.	Theorem	84
98.	Nature of the Roots of a Quadratic Equation	84

CHAPTER IX

GRAPHICAL REPRESENTATION

99.	Representation of Points on a Line	87
100.	Cartesian Coördinates	88
101.	The Graph of an Equation	90
102.	Restriction to Coördinates	91
103.	Plotting Equations	91
104.	Plotting Equations after Solution	93
105.	Graph of the Linear Equation	94
106.	Method of plotting a Line from its Equation	96
107.	Solution of Linear Equations, and the Intersection of their Graphs	97
108.	Graphs of Dependent Equations	99
109.	Incompatible Equations	99
110.	Graph of the Quadratic Equation	100
111.	Form of the Graph of a Quadratic Equation	101
112.	The Special Quadratic $ax^2 + bx = 0$	103
113.	The Special Quadratic $ax^2 + c = 0$	104
114.	Degeneration of the Quadratic Equation	104
115.	Sum and Difference of Roots	106
116.	Variation in Sign of a Quadratic	107

CHAPTER X

SIMULTANEOUS QUADRATIC EQUATIONS IN TWO VARIABLES

SECTION PAGE
117. Solution of Simultaneous Quadratics 111
118. Solution by Substitution 111
119. Number of Solutions 113
120. Solution when neither Equation is Linear 114
121. Equivalence of Pairs of Equations 120
122. Incompatible Equations 121
123. Graphical Representation of Simultaneous Quadratic Equations 122
124. Graphical Meaning of Homogeneous Equations 123

CHAPTER XI

MATHEMATICAL INDUCTION

125. General Statement 125

CHAPTER XII

BINOMIAL THEOREM

126. Statement of the Binomial Theorem 128
127. Proof of the Binomial Theorem 129
128. General Term 129

CHAPTER XIII

ARITHMETICAL PROGRESSION

129. Definitions 133
130. The nth Term 133
131. The Sum of the Series 134
132. Arithmetical Means 134

CHAPTER XIV

GEOMETRICAL PROGRESSION

133. Definitions 137
134. The nth Term 137
135. The Sum of the Series 138
136. Geometrical Means 138
137. Infinite Series 140

ADVANCED ALGEBRA

CHAPTER XV

PERMUTATIONS AND COMBINATIONS

SECTION		PAGE
138.	Introduction	143
139.	Permutations	144
140.	Combinations	146
141.	Circular Permutations	149
142.	Theorem	150

CHAPTER XVI

COMPLEX NUMBERS

143.	The Imaginary Unit	152
144.	Addition and Subtraction of Imaginary Numbers	153
145.	Multiplication and Division of Imaginaries	154
146.	Complex Numbers	155
147.	Graphical Representation of Complex Numbers	155
148.	Equality of Complex Numbers	155
149.	Addition and Subtraction	156
150.	Graphical Representation of Addition	156
151.	Multiplication of Complex Numbers	157
152.	Conjugate Complex Numbers	158
153.	Division of Complex Numbers	158
154.	Polar Representation	160
155.	Multiplication in Polar Form	160
156.	Powers of Numbers in Polar Form	161
157.	Division in Polar Form	162
158.	Roots of Complex Numbers	162

CHAPTER XVII

THEORY OF EQUATIONS

159.	Equation of the nth Degree	166
160.	Remainder Theorem	166
161.	Synthetic Division	167
162.	Proof of the Rule for Synthetic Division	169
163.	Plotting of Equations	170
164.	Extent of the Table of Values	171

SECTION PAGE
165. Roots of an Equation 172
166. Number of Roots 172
167. Graphical Interpretation 174
168. Imaginary Roots 174
169. Graphical Interpretation of Imaginary Roots 175
170. Relation between Roots and Coefficients 177
171. The General Term in the Binomial Expansion 178
172. Solution by Trial 178
173. Properties of Binomial Surds 179
174. Formation of Equations 180
175. To multiply the Roots by a Constant 183
176. Descartes' Rule of Signs 186
177. Negative Roots 189
178. Integral Roots 190
179. Rational Roots 190
180. Diminishing the Roots of an Equation 191
181. Graphical Interpretation of Decreasing Roots 193
182. Location Principle 194
183. Approximate Calculation of Roots by Horner's Method . . . 195
184. Roots nearly Equal 200

CHAPTER XVIII

DETERMINANTS

185. Solution of Two Linear Equations 203
186. Solution of Three Linear Equations 204
187. Inversion 208
188. Development of the Determinant 208
189. Number of Terms 210
190. Development by Minors 210
191. Multiplication by a Constant 213
192. Interchange of Rows and Columns 213
193. Interchange of Rows or Columns 214
194. Identical Rows or Columns 215
195. Proof for Development by Minors 215
196. Sum of Determinants 216
197. Vanishing of a Determinant 217
198. Evaluation by Factoring 218
199. Practical Directions for evaluating Determinants 219
200. Solution of Linear Equations 221
201. Solution of Homogeneous Linear Equations 223

CHAPTER XIX

PARTIAL FRACTIONS

SECTION PAGE
202. Introduction 225
203. Development when $\phi(x) = 0$ has no Multiple Roots 225
204. Development when $\phi(x) = 0$ has Imaginary Roots 229
205. Development when $\phi(x) = (x - \alpha)^n$ 232
206. General Case 233

CHAPTER XX

LOGARITHMS

207. Generalized Powers 235
208. Logarithms 236
209. Operations on Logarithms 237
210. Common System of Logarithms 239
211. Use of Tables 241
212. Interpolation 242
213. Antilogarithms 243
214. Cologarithms 245
215. Change of Base 247
216. Exponential Equations 251
217. Compound Interest 253

CHAPTER XXI

CONTINUED FRACTIONS

218. Definitions 256
219. Terminating Continued Fractions 256
220. Convergents 258
221. Recurring Continued Fractions 260
222. Expression of a Surd as a Recurring Continued Fraction . 263
223. Properties of Convergents 265
224. Limit of Error 267

CHAPTER XXII

INEQUALITIES

225. General Theorems 269
226. Conditional Linear Inequalities 271
227. Conditional Quadratic Inequalities 271

CHAPTER XXIII

VARIATION

SECTION PAGE
228. General Principles 273

CHAPTER XXIV

PROBABILITY

229. Illustration 276
230. General Statement 276

CHAPTER XXV

SCALES OF NOTATION

231. General Statement 279
232. Fundamental Operations 280
233. Change of Scale 281
234. Fractions 282
235. Duodecimals 284

ADVANCED ALGEBRA

ALGEBRA TO QUADRATICS

CHAPTER I

FUNDAMENTAL OPERATIONS

1. It is assumed that the elementary operations and the meaning of the usual symbols of algebra are familiar and do not demand detailed treatment. In the following brief exposition of the formal laws of algebra most of the proofs are omitted.

2. Addition. The process of adding two positive integers a and b consists in finding a number x such that

$$a + b = x.$$

For any two given positive integers a single **sum** x exists which is itself a positive integer.

3. Subtraction. The process of subtracting the positive number b from the positive number a consists in finding a number x such that

$$b + x = a. \tag{1}$$

This number x is called the **difference** between a and b and is denoted as follows:

$$a - b = x,$$

a being called the **minuend** and b the **subtrahend.**

If $a > b$ and both are positive integers, then a single positive integer x exists which satisfies the condition expressed by equation (1)

If $a < b$, then x is not a positive integer. In order that the process of subtraction may be possible in this case also, we introduce negative numbers which we symbolize by $(-a)$, $(-b)$, etc. When in the difference $a-b$, a is less than b, we define $a-b=(-(b-a))$. The processes of addition and subtraction for the negative numbers are defined as follows:

$$(-a)+(-b)=(-(a+b)).$$
$$(-a)+b=(-(a-b)).$$
$$a+(-b)=a-b.$$
$$(-a)-(-b)=(-(a-b)).$$
$$(-a)-b=(-(a+b)).$$
$$a-(-b)=a+b.$$
$$(-(-a))=a.^*$$

4. Zero. If in equation (1), $a=b$, there is no positive or negative number which satisfies the equation. In order that in this case also the equation may have a number satisfying it, we introduce the number zero which is symbolized by 0 and defined by the equation

$$a+0=a,$$

or

$$a-a=0.$$

The processes of addition and subtraction for this new number zero are defined as follows, where α stands for either a positive or a negative number

$$0+\alpha=\alpha\pm 0=\alpha.$$
$$0-\alpha=-\alpha.$$
$$0\pm 0=0.$$

5. Multiplication. The process of multiplying a by b consists in finding a number x which satisfies the equation

$$a\cdot b=x.$$

* The symbol for a positive integer might be written $(+a)$, $(+b)$, etc., consistently with the notation for negative numbers. Since, however, no ambiguity results, we omit the + sign. Since the laws of combining the + and − signs given in this and the following paragraphs remove the necessity for the parentheses in the notation for the negative number, we shall omit them where no ambiguity results.

When a and b are positive integers x is a positive integer which may be found by adding a to itself b times. When the numbers to be multiplied are negative we have the following laws,

$$(-a)\cdot(-b)=a\cdot b,$$
$$(-a)\cdot b=a\cdot(-b)=-(a\cdot b),$$
$$0\cdot\alpha=\alpha\cdot 0=0, \tag{1}$$

where α is a positive or negative number or zero.

These symbolical statements include the statement of the following

PRINCIPLE. *A product of numbers is zero when and only when one or more of the factors are zero.*

This most important fact, which we shall use continually, assures us that when we have a product of several numbers as

$$a\cdot b\cdot c\cdot d=e,$$

first, if e equals zero, it is certain that one or more of the numbers a, b, c, or d are zero; *second*, if one or more of the numbers a, b, c, or d are zero, then e is also zero.

6. Division. The process of dividing α by β consists in finding a number x which satisfies the equation

$$x\cdot\beta=\alpha, \tag{1}$$

where α and β are positive or negative integers, or α is 0.

When α occurs in the sequence of numbers

$$\cdots-3\beta,\ -2\beta,\ -\beta,\ 0,\ \beta,\ 2\beta,\ 3\beta,\ \cdots,$$

x is a definite integer or 0, that is, it is a number such as we have previously considered. If α is not found in this series, but is between two numbers of the series, then in order that in this case the process may also be possible we introduce the fraction which we symbolize by $\alpha\div\beta$ or $\frac{\alpha}{\beta}$ and which is defined by the equation

$$\frac{\alpha}{\beta}\cdot\beta=\alpha.$$