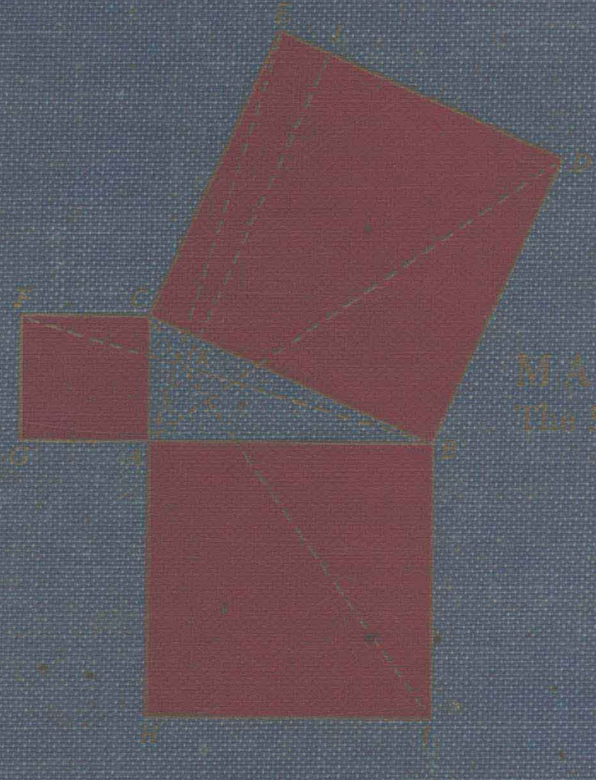

MATHEMATICS:
The Study of Axiom Systems

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MATHEMATICS:

The Study of Axiom Systems

A Blaisdell Book in the Pure and Applied Sciences

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Preface

An understanding of the nature of mathematics involves at least a knowledge of the following:

(i) Logic and sets as the foundations of mathematics.
(ii) The concept of the axiom system.
(iii) The distinction between intuitive and deductive argument, and
(iv) the distinction between mathematics and its applications.

The purpose of this book is to provide insight into these subjects for the mature reader who wishes to acquire some understanding of mathematics but who does not want to become a specialist in the field in order to accomplish this goal.

It is well known that one can develop great skill in the techniques of manipulation of mathematical symbols and yet fail to understand the nature of mathematics. Indeed, much of mathematics education in the past, at least below the graduate level, has tended to instill techniques rather than understanding, and the so-called "modern" approaches in the mathematical education of recent years have attempted to correct this deficiency.

The mature student who does not intend to specialize in mathematics must be provided with materials necessary to acquire an understanding of the subject. This problem is compounded by the fact that he can not begin at the beginning. As the product of his educational environment he has some skill (though frequently not much) in the techniques of elementary mathematics. In preparing such materials it is wrong to assume usable skills, since the student is frequently several years removed from his last school mathematics experience. It is also wrong to begin by re-teaching these skills, since they are not the primary goal.

The foregoing problem was faced at Western when it was decided that no person should be allowed to arrive at the baccalaureate level completely ignorant of the fundamental concepts of mathematical thought. The problem was to design a course that could present these concepts without assuming readily available, elementary mathematical skills.

Also, it was necessary to deal with the question of "pure" *versus* "applied" mathematics. This question was resolved by close adherence to the basic aim of teaching an understanding of mathematics. The question of mathematical application is thus left for other courses and other books. In making such a decision no value judgement is implied. The general knowledge of any cultured person should include an understanding of the uses and social implications of mathematics as well as an understanding of mathematics itself, but no single book or course can cover all subjects equally well.

The material included in Chapters 1, 2, and 3 has been used at Western in a general education course whose aims are as stated above. The course meets three times a week for one quarter, has no mathematics prerequisite, and is intended for freshmen who would not otherwise study mathematics in college.

Chapter 1 presents the rules of logic and the concept of an axiom system. The rules of logic are presented *as* an axiom system, which, itself, raises interesting questions of logic and philosophy. The purpose of such a procedure here is purely heuristic, however, and is based upon the opinion that logic, so organized, is the best approach for a brief and introductory treatment intended as a background for the study of mathematics.

The purpose of Chapter 2 is to present the language of the most elementary set theory for use in later chapters and to establish the connection between sets and non-negative integers. Chapters 1 and 2 provide the most basic tools needed for the study of the specific axiom systems of the ensuing chapters.

Chapters 3 and 5 provide an introductory treatment of the rational number system, making frequent use of the concepts developed in Chapters 1 and 2.

Chapter 4 deals with abstract groups and fields, generalizing from the group and field axioms of the rational number system presented in Chapter 3, and can be omitted without loss of continuity.

Chapters 6 and 7 are mutually independent and either may be taken as an extension of the material of the earlier chapters. For example, Chapters 1, 2, 3, and 6 constitute as self-contained a route as seemed possible at this level and in this space from the foundations of logic and sets to the Pythagorean Theorem, with a glimpse of geometries that lie beyond Euclid's. Similarly, Chapters 1, 2, 3, 5, and 7 lead as quickly as possible to the Fundamental Theorem of Integral Calculus.

No attempt or pretense is made at completeness. Difficult theorems and those requiring extensive background are stated without proof. This is especially true in Chapter 7, which would be of book length and quite unsuitable to the present purpose if all theorems there were proved in detail. On the other hand, enough proofs are included to provide experience in this most important aspect of the work of mathematics and to provide some illumination of the mode of thought common in mathematical development.

Acknowledgments are due to Professor George Springer for his patient and helpful editing of the mathematical content of the book, to the author's colleagues and students at Western who have tolerated the material in various forms for several years, and to Mrs. Grace Delano for her excellent preparation of the final manuscript. To these and others the author owes gratitude, but he accepts responsibility for the text in all its shortcomings.

GEORGE E. WITTER

Bellingham, Washington
1964

List of Axioms, Definitions, and Theorems

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4. GROUPS AND FIELDS

5. ALGEBRA

6. GEOMETRY

7. ANALYSIS

Chapter 1

Logic

1.1 Decisions

Every day of our lives we are constantly faced with the task of making decisions. We make decisions about clothes, food, books, cars, candidates and careers. Whether our decisions are trivial or of far-reaching importance we do *not* make them blindly. At first we may think that decisions such as choosing food we eat are made with little or no thought; but if this were true, then we should be satisfied to have our food chosen by the spin of a roulette wheel. The fact that this procedure would not satisfy most of us is an indication that our choice of food is not haphazard but is dictated by preference, custom, knowledge of nutrition, cost, and so forth. We choose food so frequently that the choice becomes habit but reasons do exist, however obscure, for each choice that is made.

How is a decision made? Either consciously or subconsciously our decisions are affected by the sum total of previous experience upon which our minds can call, that is, by the sum total of our knowledge. Thus the structure of our lives is like a building, with each story supported by all those below it.

Logic is the *science* of decision-making and it involves the study of methods whereby decisions are made in a systematic manner. It is this systematic aspect of logic which distinguishes it from the ordinary thought processes of everyday life. This chapter will deal with the rudiments of logic necessary for the study of mathematics.

1.2 Propositions

Information is recorded and communicated in the form of *statements*. The sentence,

The Statue of Liberty is green.

is an example of a statement. For our purposes, *statement* will be an undefined term or concept and therefore the reader must rely upon his intuitive idea of its meaning. In this we have little choice. We could set out to develop a system in which every concept introduced is precisely defined in terms of previously defined concepts *within the system*, but we should ultimately meet with failure since there must be a beginning. There will always be a *first* concept that cannot be defined in terms of anything within the existing system. In this case the first such concept is that of *statement*.

Ideally, any logical system would have only one undefined concept. It is frequently very difficult to build such an ideal system and therefore we shall not hesitate to introduce additional undefined concepts if such devices facilitate the development. The important point to note is that in such a semi-formal system all undefined concepts must be clearly recognized as such.

The words "statement" and "proposition" are frequently used interchangeably in the literature of logic and mathematics and will be so used throughout this book.

A test to determine whether or not a particular collection of words is a proposition is to ask, "Does it make sense to say that the collection of words is true or false?" If the answer is "yes," then the collection of words is a proposition. Note that we did *not* say that a proposition must be true. We said that to be able to talk about the truth or falsity of a collection of words indicates that the collection comprises a proposition or statement.

A. Cincinnati is in Iowa.
B. George Washington's horse at night.

The collection of words *A* is a proposition and we can discuss the truth or falsity of it. The collection of words *B* is not a proposition and it is meaningless to discuss its truth or falsity.

1.3 Truth

A question may be raised here concerning what is meant by the *truth* of a statement. Some statements lend themselves readily to examination.

Cincinnati is in Iowa.

It will probably be generally agreed that the above statement is false.

$x = 3$.
Twenty-five miles per hour is the correct speed to drive.
It is wrong to buy stock in a bear market.

The truth of these statements obviously depends upon circumstances. (Actually, the truth of "Cincinnati is in Iowa" depends upon circumstances.) Then there are statements such as:

Parallel lines do not meet, however far extended.

If the reader thinks that this statement is obviously true, let him read Chapter 6.

What, then, is *truth*? We could fill this book with a discussion of the question and still not arrive at its answer. For our purposes, *we do not care*. We shall work with propositions and *assume* them to be true or false as we choose. The *consequences* of such *assumptions* are what interest us.

1.4 Combining Propositions

Two propositions may be connected by the word *or* and a new proposition obtained. Consider the propositions:

A. Today is Tuesday.
B. I am verbose.

A new collection of words *A or B* can be formed. The sentence,

Today is Tuesday or I am verbose.

is a perfectly good proposition. Note that we can discuss its truth or falsity. (Under what conditions is it true? False? For example, is the proposition true if today happens to be Tuesday but I am not really verbose?)

It is apparent that two propositions could be combined to form a new one in many ways. Two propositions could be connected by words such as *and*, *but*, *if*, and many others. In each case we could investigate the truth or falsity of the resulting statement.

The question may now be raised: What if the combination of two propositions produces one which *does not make sense*? For example, does it make sense to say, "Today is Tuesday or I am verbose?" We cannot answer this question without making some assumptions about the experience that has taken place prior to the formulation of the statement. All we can say is that it is meaningful to call the statement true or false.

Obviously it is easy to concoct propositions which do not make sense in our everyday life. However, we would probably waste our time. We say "probably" because there remains the possibility that our "meaningless" statement might make very desirable sense under a certain set of circumstances.

To avoid the question of making sense, let us use *symbols* for propositions. For example, let A and B denote any two propositions and consider the new proposition *A or B*. To assert that this new proposition is true is to claim (exactly what it says) that either A is true or B is true. Also, let us *agree* that we shall call the proposition *A or B* true if A is true and if B is also true. (Common English usage of *or* is in the "exclusive" sense; that is, "*A or B* is true" means either A is true or B is true but not both. Here we agree to use the "inclusive" *or* in order to comply with the traditions of logic and mathematics. It should be noted that either meaning of *or* could be used; all that is necessary is to specify which one we have in mind.) These statements can be summarized in a *truth table*, Table 1.1, where line two, for example,

Table 1.1

A	B	A or B
T	T	T
F	T	T
T	F	T
F	F	F

means that if A is false and B is true, then the proposition *A or B* is true.

A. I shall attend a concert.
B. I shall go shopping.
A or *B*. I shall attend a concert or I shall go shopping.

A or B appears to be true if A is true or if B is true. It seems possible that it could be true if A and B were both true. Here we are simply justifying an agreement which we have already made. Table 1.1 may be thought of as a *definition* of the proposition *A or B*. It should be recognized that we could have adopted other rules and that the ones we did adopt are simply precise descriptions of what we mean *here* when we make a statement of the form *A or B*.

Now let us consider *A and B*. When is this proposition true? It asserts the simultaneous truth of A and of B. The corresponding definition is given in Table 1.2.

Table 1.2

A	B	A and B
T	T	T
F	T	F
T	F	F
F	F	F