PARTIAL DIFFERENTIAL EQUATIONS

An Introduction

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PREFACE

It has been the purpose and hope of the author in writing this book to help fill a serious need for introductory texts on the graduate level in the field of partial differential equations. The vastness of the field and—even more significantly—the absence of a comprehensive basic theory have been responsible, we believe, for the comparative scarcity of introductory books dealing with this subject. However, the importance of this field is so tremendous that the difficulties and pitfalls awaiting anyone who seeks to write such a book should be looked upon as a provocative and stimulating challenge. We hope that we have achieved some measure of success in meeting this challenge.

Any book dealing with a subject possessing substance and vitality is bound to reflect the particular interests and prejudices of the author. Even if the field is well organized and has been worked out with a considerable degree of completeness, the author's inclinations will be reflected in the manner in which the subject matter is presented. When, in addition, the subject is as extensive and incompletely developed as that here under consideration, they will also be reflected in the choice of material. We are well aware that many important topics are presented briefly or not at all. However, we are consoled by the thought that in the writing of a book of moderate size the omission of much significant material was inevitable, and by the hope that our presentation will be such as not only to interest the student in the topics presented here, but also to stimulate him to pursue some of them, as well as topics not touched upon here, in other books and in the research journals.

Throughout this book the stress has been on existence theory rather than on the effective determination of solutions of specific classes of problems. It is hoped that the presentation will complement usefully any text which emphasizes the more "practical" or "applied" aspects of the subject.

A word is in order concerning the intimate relationship between physics and the theory of differential equations, both ordinary and partial. Physics has certainly been the richest source of problems in this field, and physical reasoning has often been an invaluable guide to the correct formulation of purely mathematical problems and to the successful development of techniques for solving such problems. In this connection we would strongly urge every prospective student of differential equations (indeed, every prospective student of mathematics) to read, and deliberate on, the splendid preface to the Courant-Hilbert masterpiece, "Methods of Mathematical Physics." Although little is said in the following pages concerning the physical origins of many of the mathematical problems which are discussed, the student will find that his understanding of these problems will be heightened by an awareness of their physical counterparts.

The author has good reason to follow the tradition of acknowledging gratefully his wife's unselfish aid in the seemingly endless task of typing successive drafts of the manuscript. More important than the typing, however, was her constant encouragement to carry the writing task through to its completion. It is hoped that her encouragement was directed to a worthwhile objective.

Bernard Epstein

TERMINOLOGY AND BASIC THEOREMS

For convenience we list here a few terms, notations, and theorems that will be used frequently.

A domain is an open connected set (in the plane or in a higher-dimensional euclidean space); an equivalent definition is that a domain is an open set which cannot be expressed as the union of two disjoint nonvacuous open sets.

The "Kronecker delta" δ_{ij} assumes the values 1 and 0, according as the indices i, j are equal or unequal.

$\bar{S}$ denotes the closure of the set S.

A disc is the set of all points (in the plane) satisfying an inequality of the form $(x - x_0)^2 + (y - y_0)^2 < r^2$; a circle is the boundary of a disc. The unit disc is defined by the inequality $x^2 + y^2 < 1$, and the unit circle is its boundary.

When dealing with a curve we denote arc length, measured from some fixed point of the curve, by the letter s.

The symbols $\in$, $\subset$, $\cup$, $\cap$, are used with their customary set-theoretic significance: $a \in A$ means that a belongs to (is an element of) the set A, and $A \subset B$ means that every element which belongs to A also belongs to B (A is a subset of B). Note that $A \subset B$ does not imply that A is a *proper* subset of B. $A \cup B$ and $A \cap B$ denote the union and intersection, respectively, of the sets A, B.

The distance between two sets is the minimum distance between a pair of points, one from each set. If both sets are closed and at least one is bounded, then the minimum is actually attained.

The Heine-Borel theorem: Given a compact (i.e., bounded and closed) set S in a euclidean space and an open covering of S (i.e., a collection of open sets whose union contains S), then it is possible to extract from this covering a finite number of open sets which suffice to cover S.

A real-valued function which is defined and continuous on a compact set is uniformly continuous, is bounded above and below, and attains its maximum and minimum values.

A function defined in a domain D is said to be of class C^n if all partial derivatives of order up to and including the nth exist and are continuous throughout D.

If the functions $f(x,y)$, $g(x,y)$ are of class C^n in a neighborhood of (x_0,y_0), and if the Jacobian $f_x g_y - f_y g_x$ does not vanish at that point, then the equations $\xi = f(x,y)$, $\eta = g(x,y)$ can be solved in a sufficiently small neighborhood of (x_0,y_0) for x and y in terms of ξ and η, say $x = \phi(\xi,\eta)$, $y = \psi(\xi,\eta)$, and the functions $\phi(\xi,\eta)$, $\psi(\xi,\eta)$ are also of class C^n.

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CONTENTS

1. SOME PRELIMINARY TOPICS

Before entering on the subject of partial differential equations, it seems appropriate to devote a chapter to some concepts and theorems with which the reader is perhaps not yet acquainted. The reader may choose to study the topics covered in this chapter as the need for them arises later, rather than beginning by going through this chapter (or those parts of it which he has not previously encountered) systematically.

1. Equicontinuous Families of Functions

A basic difficulty that besets many mathematical investigations is the fact that there exists no simple extension to families of functions of the Bolzano-Weierstrass theorem, which asserts (in one of its several possible formulations) that from every bounded sequence of real numbers it is possible to extract a convergent subsequence. If, instead of a sequence of numbers, we consider a sequence of functions $\{f_n(x)\}$ defined on a fixed interval, say $0 \leq x \leq 1$, then, from the hypotheses that each function $f_n(x)$ is continuous on this interval and that these functions are uniformly bounded [i.e., there exists a positive number M independent of x and n such that $|f_n(x)| < M$], it does *not* follow that it is possible to extract a subsequence convergent throughout the interval. (Cf. Exercise 1.) However, there is a certain more stringent condition of continuity, which is frequently found to be satisfied by sequences (or families) of functions encountered in problems of analysis, and which suffices, when taken together with the condition of uniform boundedness, to assure the existence of a *uniformly* convergent subsequence; this more stringent condition will now be explained.

A basic theorem of analysis asserts that a function $f(x)$ continuous on a compact (bounded and closed) set S is uniformly continuous on S;

that is, given any positive number ϵ, there exists a positive number δ such that whenever the conditions $x_1 \in S$, $x_2 \in S$, $|x_1 - x_2| < \delta$ are satisfied, the inequality $|f(x_1) - f(x_2)| < \epsilon$ holds. Given any finite number of continuous functions defined on S and any positive number ϵ, it is possible to find a single number δ which will suffice for each of these functions, for one can determine a suitable δ for each of these functions and then choose the smallest of these numbers. However, this method fails for an infinite family of functions, for there may not exist a positive lower bound to the corresponding set of numbers δ. (This occurs in the family of functions considered in Exercise 1.) We may describe this situation as a lack of "over-all" uniformity, despite the uniform continuity of each individual function of the family. We are thus led to formulate the following more restrictive concept of uniform continuity, which refers to a *family* of functions, not to a single function.

DEFINITION 1. A family of functions defined on a set S of real numbers[1] is said to be "equicontinuous" if for every positive number ϵ there exists a positive number δ such that, for every function $f(x)$ of the family and every pair of numbers x_1, x_2 contained in S and satisfying the inequality $|x_1 - x_2| < \delta$, the inequality $|f(x_1) - f(x_2)| < \epsilon$ holds. (It should be emphasized, for clarity, that each function of an equicontinuous family is uniformly continuous. Also it may be noted that no restriction of boundedness or closure is imposed on S.)

A simple example of an equicontinuous family is furnished by any set of functions defined and continuous on a fixed interval (open or closed) and having, at all points of this interval, a first derivative whose absolute value never exceeds some fixed number C; for then, by the theorem of mean value, we have, for any function $f(x)$ of the family and any two numbers x_1, x_2 of the interval, the inequality

$$|f(x_1) - f(x_2)| = |f'(\xi)(x_1 - x_2)| \leq C|x_1 - x_2| \qquad (x_1 < \xi < x_2) \quad (1)$$

so that, for any given ϵ, the choice $\delta = \epsilon/C$ will suffice. (Note that in Exercise 1 the derivatives of the functions under consideration are not uniformly bounded.)

We now state the following striking theorem, which accounts for the important role played in analysis by the concept of equicontinuity.

Theorem 1. Ascoli Selection Theorem. Let F be an infinite, uniformly bounded, equicontinuous family of functions defined on a

[1] For simplicity, we formulate the definition only for functions of one real variable; the extension to more general classes of functions is quite straightforward. Similarly, the theorem of this section is stated only in the one-dimensional case, the extension to functions of any (finite) number of variables being clear.

finite closed interval S: $a \leq x \leq b$. Then from every sequence $\{f_n(x)\}$ chosen from F it is possible to select a uniformly convergent subsequence.

Proof. Select any countable dense subset S_1 of S, such as the set of all rational numbers in S, and enumerate them: $r_1, r_2, r_3, \ldots$. Let a sequence $\{f_n(x)\}$ be selected from F. Then the sequence $\{f_n(r_1)\}$ satisfies the hypothesis of the Bolzano-Weierstrass theorem, and so we may select a subsequence $\{f_{n1}(x)\}$ of the original sequence which converges at the point r_1. By applying the preceding argument to $\{f_{n1}(x)\}$ we obtain a sequence $\{f_{n2}(x)\}$ which converges at r_2 and also at r_1 (for any subsequence of a convergent sequence is also convergent, and has the same limit). Repeating this argument, we obtain further sequences $\{f_{n3}(x)\}$, $\{f_{n4}(x)\}$, . . . , each of which is a subsequence of the preceding one, and such that the kth sequence converges at $r_1, r_2, \ldots, r_k$. In order to obtain a single sequence which will converge at all the points of S_1, we employ the "diagonalization procedure," originally employed by Cantor to demonstrate the uncountability of the set of real numbers. Consider the sequence $\{f_{nn}(x)\}$ formed by taking the first function of the first subsequence, the second function of the second subsequence, etc. This last sequence is convergent at each point of S_1, for it is evidently a subsequence of the first sequence $\{f_{n1}(x)\}$ and, aside perhaps from the first $k - 1$ terms, a subsequence of the sequence $\{f_{nk}(x)\}$, $k = 2, 3, \ldots$.

It now remains to show that the sequence $\{f_{nn}(x)\}$ converges throughout S, and that the convergence is uniform. Given any $\epsilon > 0$, choose $\delta > 0$ such that, for every function f in F, and a fortiori for every function of the sequence $\{f_{nn}(x)\}$, the inequality $|x_1 - x_2| < \delta$ implies that $|f(x_1) - f(x_2)| < \epsilon$. Now we select a *finite* subset S_2 of S_1 such that each point of S differs by less than δ from at least one point of S_2. (This can be accomplished, for example, by dividing S into adjoining segments, each of length not exceeding δ, and selecting one point of S_1 in each of these segments.) We next determine a positive integer N so large that, for $n,m > N$, the inequality $|f_{nn}(y) - f_{mm}(y)| < \epsilon$ holds at each point y of S_2. Then for any point x of S we choose $y \in S_2$ such that $|x - y| < \delta$, and we obtain, for $n,m > N$, the chain of inequalities

$$|f_{nn}(x) - f_{mm}(x)| \leq |f_{nn}(x) - f_{nn}(y)| + |f_{nn}(y) - f_{mm}(y)| + |f_{mm}(y) - f_{mm}(x)| < 3\epsilon \quad (2)$$

Since the index N has been chosen independently of x, (2) implies the uniform convergence of the sequence $\{f_{nn}(x)\}$, and the proof is complete.

Two brief remarks may be helpful in clarifying the significance of the two hypotheses (uniform boundedness and equicontinuity). First, the proof of the existence of a subsequence of the original sequence which converges at all points of a preassigned countable dense set requires only the existence of a pointwise bound (*not* a uniform bound) on the family F, and does not involve equicontinuity. Secondly, the proof given above may be easily modified to establish the following corollary, which may be left as a simple exercise.

COROLLARY. Let F be an infinite family of functions defined on an open interval S: $a < x < b$, equicontinuous on every closed subinterval, and bounded at some point ξ, $a < \xi < b$ (cf. Exercise 4); then from every sequence $\{f_n(x)\}$ chosen from F it is possible to select a subsequence which converges uniformly on every compact subset of S.

The important Montel selection theorem of the theory of analytic functions is closely related to (the two-dimensional version of) this corollary; the essential point of the proof of this theorem consists in showing that a family of analytic functions uniformly bounded in a domain is equicontinuous in every compact subset of the domain.

EXERCISES

1. Consider the sequence of functions $\{\sin n\pi x\}$ on the interval $0 \leq x \leq 1$, $n = 1, 2, \ldots$; these functions are uniformly bounded on this interval, for $|\sin n\pi x| \leq 1$. Prove that there does not exist a subsequence which converges uniformly at each point of the interval.

2. Consider the sequence of functions $\{x^n\}$ on the interval $0 \leq x \leq 1$, $n = 1, 2, \ldots$; in contrast to the preceding exercise, this sequence (and hence every subsequence) converges throughout the interval, but the limit function is discontinuous at the end point $x = 1$. Prove directly from the definition that this sequence of functions is not equicontinuous. (This fact also follows, of course, from the selection theorem.) Show that in any *smaller* interval, $0 \leq x \leq a < 1$, the above functions are equicontinuous, in agreement with the fact that the limit function is continuous in this smaller interval.

3. Prove that a sequence of continuous functions which converges uniformly on a compact set forms an equicontinuous family.

4. Show that a family of functions equicontinuous on any bounded set S and bounded at one point of S is uniformly bounded on S.

2. The Weierstrass Approximation Theorem

In many branches of analysis there are theorems whose proofs have to be given in two parts: First the theorem is proved subject to certain additional hypotheses, and then it is shown, by the use of suitable approximation techniques, that the additional hypotheses may be dropped. To cite only one example, we may mention the Riemann-Lebesgue lemma, of fundamental importance in the theory of Fourier series and integrals, which asserts that, for any function $f(x)$ which is (absolutely) integrable over an interval I, finite or infinite, the quantity $\int_I f(x)e^{i\lambda x}\,dx$ approaches zero as the real parameter λ becomes infinite. It is a simple matter to prove this theorem under the additional assumptions that the interval I is finite and the function $f(x)$ is continuously differentiable, for in this case an integration by parts yields the desired result immediately. One can then extend the proof to continuous functions by using the fact that a continuous function can be suitably approximated by continuously differentiable functions; then, similarly, one uses the fact that any integrable function can be suitably approximated by continuous functions; finally, the restriction to a finite interval is easily dropped.

One of the most important and striking approximation theorems is the following, which will be used subsequently a number of times.

Theorem 2. Weierstrass Approximation Theorem. Let $f(x_1,x_2, \ldots ,x_n)$ be defined and continuous on any compact set R. Given any positive ϵ, there exists a polynomial $P(x_1,x_2, \ldots ,x_n)$ such that the inequality

$$|f(x_1,x_2, \ldots ,x_n) - P(x_1,x_2, \ldots ,x_n)| < \epsilon \tag{3}$$

holds at all points of R.

Proof. For simplicity, we consider the case $n = 2$; the modifications required for any other value of n will be clear from the proof to be presented. (Cf. Exercise 6.) First we make the additional assumption that the set R is a rectangle and that f vanishes at all boundary points of R. We extend the function f over the entire plane by assigning it the value zero at all points outside R. Clearly the extended function is uniformly continuous, not only in R, but also over the entire plane. Consider the one-parameter family of functions:

$$f_t(x_1,x_2) = \iint_{-\infty}^{\infty} f(\xi_1,\xi_2)P(\xi_1 - x_1,\ \xi_2 - x_2,\ t)\,d\xi_1\,d\xi_2 \qquad (t > 0) \tag{4}$$

where

$$P(\xi_1 - x_1,\ \xi_2 - x_2,\ t) \text{ (or, for brevity, } P_t) = (\pi t)^{-1} \exp\{-t^{-1}[(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2]\}$$

On account of the uniform continuity of f we can choose $\delta > 0$, independent of x_1 and x_2, such that $|f(\xi_1,\xi_2) - f(x_1,x_2)| < \epsilon/3$ whenever $(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 < \delta^2$. Taking account of the fact that the right side of (4) has the value one when the function f is replaced by the constant function $f(x_1,x_2) \equiv 1$, we obtain the chain of inequalities

$$|f_t(x_1,x_2) - f(x_1,x_2)| = \left| \iint\limits_{-\infty}^{\infty} (f(\xi_1,\xi_2) - f(x_1,x_2))P_t \, d\xi_1 \, d\xi_2 \right|$$

$$\leq \iint\limits_{-\infty}^{\infty} |f(\xi_1,\xi_2) - f(x_1,x_2)| \, P_t \, d\xi_1 \, d\xi_2 \leq \frac{\epsilon}{3} \iint_\Delta P_t \, d\xi_1 \, d\xi_2$$

$$+ 2M \iint_{\Delta'} P_t \, d\xi_1 \, d\xi_2 < \frac{\epsilon}{3} \iint\limits_{-\infty}^{\infty} P_t \, d\xi_1 \, d\xi_2$$

$$+ \frac{2M}{\pi t} \int_0^{2\pi} \int_\delta^\infty e^{-r^2/t} r \, dr \, d\theta = \frac{\epsilon}{3} + 2Me^{-\delta^2/t} \quad (5)$$

[Here $M = \max |f|$, while Δ and Δ' denote the disc

$$(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 < \delta^2$$

and its complement, respectively.] We now choose t (independent of x_1 and x_2) such that $2M \exp(-\delta^2/t) < \epsilon/3$. With this choice of t, (5) yields the inequality

$$|f_t(x_1,x_2) - f(x_1,x_2)| < \frac{2\epsilon}{3} \quad (6)$$

(This inequality holds everywhere, not only in R.) We now write the Taylor expansion of the function e^{-u} in the form

$$e^{-u} = 1 - \frac{u}{1!} + \frac{u^2}{2!} - \cdots + \frac{(-u)^N}{N!} + R_N(u) \quad (7)$$

choosing N so large that $|R_N| < \epsilon\pi t/3MA$ whenever $|u| \leq \rho^2/t$, A and ρ denoting the area and diameter, respectively, of R. (That N can be so chosen follows either from the elements of the theory of analytic functions or from Taylor's theorem with remainder.) Letting

$$u = t^{-1}[(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2]$$

and
$$P(x_1,x_2) = (\pi t)^{-1} \iint_R f(\xi_1,\xi_2) \sum_{k=0}^{N} \frac{(-u)^k}{k!} \, d\xi_1 \, d\xi_2$$

we obtain

$$|f_t(x_1,x_2) - P(x_1,x_2)| \leq (\pi t)^{-1} \iint_R |f(\xi_1,\xi_2)| \cdot |R_N(u)| \, d\xi_1 \, d\xi_2$$

$$\leq \frac{M}{\pi t} \cdot \frac{\epsilon\pi t}{3MA} \cdot A = \frac{\epsilon}{3} \quad (8)$$

This is valid at all points of R. Combining (6) and (8), we obtain

$$|f(x_1,x_2) - P(x_1,x_2)| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad (9)$$

By inspection, it is clear that $P(x_1,x_2)$ is a polynomial, so that the proof of the theorem is complete, subject to the restrictions concerning R and f that were made at the beginning.

To eliminate these restrictions we require the following lemma.

LEMMA. LEBESGUE EXTENSION THEOREM. If a function $f(x_1,x_2, \ldots ,x_n)$ is defined and continuous on a compact set R, it is possible to extend f continuously to any larger set.

Momentarily accepting this lemma, we now choose a (closed) rectangle R' containing the compact set R in its interior, and extend f by defining it to be zero on the boundary of R'. Clearly f is still continuous under this extension, and its domain of definition is still compact. We invoke the lemma to extend f continuously to all of R'. Now the preceding argument is applicable, so that f can be approximated within ϵ throughout R', and a fortiori throughout R, by a polynomial.

It remains to prove the lemma. Again we may restrict attention to the case of two independent variables. First, we consider the very simple particular case of extending a continuous function defined on the boundary of a square to the interior. We merely assign to the center of the square the mean of the values at the four vertices, and then define the function along each line segment connecting the center to a boundary point by linear interpolation. It should be noted that this method of extending the given function assigns to each interior point a value between the minimum and maximum values which are assigned on the boundary. Now let a continuous function f be defined on any compact set R, and let R be contained in the interior of a square S whose boundary we denote by Γ. We then extend f continuously to the compact set $R \cup \Gamma$ by defining f to vanish everywhere on Γ. To prove the lemma it will suffice to show that f can be extended continuously to all of S, for if this can be done, we can accomplish the continuous extension of f to the entire plane, and hence to any specified set in the plane, by defining f to vanish everywhere outside Γ. Let G denote the set of points inside Γ which do not belong to R. Since G is open (and nonvacuous), it is possible to construct a network of equally spaced horizontal and vertical lines sufficiently fine that at least one square of this network lies, together with its boundary, entirely in G. Then the original network is refined by adding horizontal and vertical lines midway between those originally constructed. Those squares (if any) of the finer network which lie (together with their boundaries) in G but whose interiors are disjoint from the square or squares previously selected from the original network are now determined. By repeatedly refining the network and selecting squares, we evidently break G down into a countable union of closed squares whose interiors are disjoint. Let V be defined as the set of all points of G which appear as vertices of this collection of squares. We define f at each point Q of V as the minimum value of f at those points of $R \cup \Gamma$ which are closest to Q. (Since these points constitute a compact set, this definition is meaningful.) It is readily seen from the manner in which the squares were chosen that f has been defined at the vertex of each square, and perhaps also at a *finite* number of additional points on the boundary of each square. Let f be defined along the boundary of each square by linear interpolation between successive points of the set V, and then let f be defined inside each square by the method described earlier in this paragraph. Then f has evidently been defined throughout G and is continuous there, but it must still be shown that this function possesses the proper behavior near the boundary of G. Let any boundary point of G, say T, be selected, and let a positive number ϵ be given. Then $\delta(>0)$ can be so chosen that, for all points T' of $R \cup \Gamma$ whose distance from T does not exceed δ, the inequality $|f(T) - f(T')| < \epsilon$ holds. For any point P of V whose distance from T is less than $\frac{1}{2}\delta$, it is evident that those points of $R \cup \Gamma$ closest to P all lie within a distance less than δ from T, so that $|f(T) - f(P)| < \epsilon$. If, finally, P lies within a distance less than $\frac{1}{4}\delta$ from T, it is readily seen that P lies inside or on the boundary of a square such that all the points of V on the boundary of this square lie within a distance less than $\frac{1}{2}\delta$ from T. From the manner in which f was defined on the boundary and inside each square, it follows that $|f(T) - f(P)| < \epsilon$, and the proof is complete.

EXERCISES

5. Carry out in detail the proof of the Riemann-Lebesgue lemma sketched at the beginning of this section, for any function absolutely integrable over the real axis (in the Riemann sense).

6. Modify the proof given in the text to apply equally well to any value of n.

7. Prove the Lebesgue extension theorem in the one-dimensional case. (This case is decidedly simpler than in higher dimensions.)

8. Carry out in detail the proof of the Weierstrass theorem in one dimension which is outlined here: By the preceding exercise, we may assume that the function f is defined on a closed interval, rather than on an arbitrary compact set. By uniform continuity, f can be approximated uniformly by a polygonal function (i.e., a function which is continuous and sectionally linear, with only a finite number of "corners"). This polygonal function can be expressed as a finite sum of polygonal functions, each having only one "corner." Each such function, in turn, can be expressed as the sum of a linear function and a function of the form constant $\cdot\, |x - a|$. It therefore suffices to prove that the function $|x|$ can be uniformly approximated by polynomials on the interval $-1 \leq x \leq 1$. To do this, consider the identity $|x| = [1 - (1 - x^2)]^{1/2}$ and the Taylor series of the function $(1 - u)^{1/2}$ about the point $u = 0$. (This proof is due to Lebesgue.)

9. Prove the following extension of the Weierstrass theorem: If $f(x)$ is of class C^n on the interval $a \leq x \leq b$ [i.e., $f^{(n)}(x)$ exists and is continuous on the interval], then,