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An Introduction to Infinite Ergodic Theory

Jon Aaronson

American Mathematical Society

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ABSTRACT. The book is about measure preserving transformations of infinite measure spaces. It could be of interest to mathematicians working in ergodic theory, probability and/or dynamical systems and should be accessible to graduate students.

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Preface

Infinite ergodic theory is the study of measure preserving transformations of infinite measure spaces (early references being [**Hop1**] and [**St**]). It is part of "non-singular ergodic theory", the more general study of non-singular transformations (since a measure preserving transformation is also a non-singular transformation).

Non-singular ergodic theory arose as an attempt to generalise the classical ergodic theory of probability preserving transformations. Its major success was the ratio ergodic theorem. Another side to the theory also developed concentrating on facts which are valid "in the absence of invariant probabilities".

This book is more concerned with properties specific to infinite measure preserving transformations.

It should be readable by anyone initiated to metric space topology and measure theoretic probability.

Some readers may like to begin by following an example and perhaps one of the simplest in the book is Boole's transformation $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = x - \frac{1}{x}$.

This is a conservative, exact measure preserving transformation of $\mathbb{R}$ equipped with Lebesgue measure; and for each absolutely continuous probability P on $\mathbb{R}$ and non-negative, integrable function $f : \mathbb{R} \to \mathbb{R}$ with unit integral,

$$P\Big(\Big[\sum_{k=0}^{n-1} f \circ T^k \leq \frac{\sqrt{2n}}{\pi} t\Big]\Big) \to \frac{2}{\pi} \int_0^t e^{-\frac{s^2}{\pi}} ds$$

as $n \to \infty$.

The book begins with an introduction to basic non-singular ergodic theory (chapters 1 and 2), including recurrence behaviour, existence of invariant measures, ergodic theorems and spectral theory. One of the results in §2.4 is the collapse of absolutely normalised pointwise ergodic convergence for ergodic measure preserving transformations of infinite measure spaces.

This leaves a wide range of possible "ergodic behaviour" which is catalogued in chapter 3 mainly according to the yardsticks of intrinsic normalising constants, laws of large numbers and return sequences (the return sequence of Boole's transformation is $\frac{\sqrt{2n}}{\pi}$).

The rest of the book (excepting chapter 5) consists of illustrations of these phenomena by examples.

Markov maps which arise both in probability theory and in smooth dynamics are treated in chapter 4. They illustrate distributional convergence phenomena (mentioned above) as do the inner functions of chapter 6. Geodesic flows on hyperbolic surfaces were one of the first examples considered ([**Hop1**]), and these are treated in chapter 7. Some of the extremely pathological examples in the subject can be found in the chapter on cocycles and skew products (chapter 8). In chapter 5, there is a modest beginning to the classification theory.

There is a small (but insufficient) amount of probability preserving ergodic theory in the book, and I recommend the uninitiated reader to take advantage of the excellent books available on this subject, including [**Cor-Sin-Fom**], [**De-Gr-Sig**], [**Fu**], [**Mañ**], [**Parr2**], [**Pet**], [**Rudo**], [**Wa**].

The reader will no doubt find that many (but hopefully not the reader's favourite) topics are conspicuous by their absence. By way of excuse I can only say that some of these are better covered elsewhere, while others are deemed too advanced for an introduction and yet others are too "fresh" for a book (there being no time to write about them).

Lastly I come to the thanks. I would like to thank the people who worked with me on the topics described in the book (see bibliography). Without them, none of this would have been possible. Also I would like to thank my colleagues Gilat and Lemańczyk; and my student Omri Sarig who found mistakes in early versions (any remaining errors being my sole responsibility having been introduced subsequently while correcting mistakes).

Jon. Aaronson
Tel Aviv, October 1996

Contents

CHAPTER 1

Non-singular transformations

§1.0 Standard measure spaces

Apart from the well known, classical theory of abstract measure spaces to be found (for example) in [**Halm1**], we'll also need certain results from the theory of *standard* measure spaces.

This section is a review of that theory. Some (but not all) proofs are supplied here. Complete treatments of the subject can be found in [**Coh**], [**Kec**], [**Kur**] and [**Part**].

DEFINITION: POLISH SPACE, BOREL SETS.

A *Polish space* is a complete, separable metric space. Let X be a Polish space X. The collection of *Borel sets* $\mathcal{B}(X)$ is the σ-algebra of subsets of X generated by the collection of open sets.

DEFINITION: STANDARD MEASURABLE SPACE.

A *standard measurable space* (or standard Borel space) $(X, \mathcal{B})$ is a Polish space X equipped with its collection of Borel sets $\mathcal{B} = \mathcal{B}(X)$.

DEFINITION: MEASURABLE FUNCTION.

Let X, X' be Polish spaces. A function $f : X \to X'$ is called *(Borel) measurable* if $f^{-1}\mathcal{B}(X') \subset \mathcal{B}(X)$.

Given a standard measurable space $(X, \mathcal{B})$, we consider the collection of probability measures defined on $(X, \mathcal{B})$

$$\mathcal{P} = \mathcal{P}(X, \mathcal{B}) := \{p : \mathcal{B} \to [0,1] : \ p \text{ a probability measure}\}.$$

Let $\mathcal{B}(\mathcal{P})$ be the smallest σ-algebra of subsets of $\mathcal{P}$ such that for each $A \in \mathcal{B}$, the function $\mu \mapsto \mu(A)$ is measurable ($\mathcal{P} \to [0,1]$). It follows that $(\mathcal{P}, \mathcal{B}(\mathcal{P}))$ is also a standard measurable space.

To see this, choose a compact topology on X generating $\mathcal{B}$, then with respect to the corresponding vague topology (inherited from the weak $*$ topology on $C(X)^*$): $\mathcal{P}$ is compact metric space and $\mathcal{B}(\mathcal{P})$ is its collection of Borel sets.

DEFINITION: STANDARD MEASURE SPACE.

A *standard measure space* is a measure space $(X, \mathcal{B}, m)$ where $(X, \mathcal{B})$ is a standard measurable space.

We sometimes suppress the σ-algebra $\mathcal{B}$ in these notations, denoting the standard measurable space $(X, \mathcal{B}(X))$ by X, and the standard measure space $(X, \mathcal{B}(X), m)$ by (X, m).

A *standard probability space* is a probability space which is a standard measure space. A measure space is called *pure* if it is either non-atomic or purely atomic, and *symmetric* if it is either non-atomic or purely atomic with all atoms having the same measure.

The first result we review simplifies the treatment of measurable functions on standard spaces.

1.0.0 LUSIN'S THEOREM.
Suppose that $(X, \mathcal{B}, m)$ is a standard probability space, that X' is a Polish space, and that $f : X \to X'$ is measurable, then $\forall\ \epsilon > 0$, $\exists$ a compact set $K \subset X$ with $m(K) > 1 - \epsilon$, and such that f is continuous on K.

A Polish space is either finite, countable or has the cardinality of the continuum. The next result shows that there is essentially only one standard measurable space with the cardinality of the continuum.

1.0.1 KURATOWSKI'S ISOMORPHISM THEOREM.
Suppose that $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ are standard measurable spaces with the same cardinality, then $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ are isomorphic in the sense that there is a bijection $\pi : X \to X'$ such that $\pi\mathcal{B} = \mathcal{B}'$.

DEFINITION: ANALYTIC SET. Let $(X, \mathcal{B})$ be a standard measurable space. A subset $A \subset X$ is called *analytic* if $\exists$ another standard measurable space $(X', \mathcal{B}')$, a measurable function $f : X' \to X$ and a set $A' \in \mathcal{B}'$ such that $fA' = A$.

Clearly Borel sets are analytic.

It was shown by Souslin [**So**] that in any uncountable standard measurable space there are analytic sets which are not Borel. This fact (unknown to Lebesgue) contributes some subtlety to the subject and we shall therefore need the following three results.

1.0.2 UNIVERSAL MEASURABILITY THEOREM. *Let $(X, \mathcal{B}, m)$ be a standard measure space.*
If $A \subset X$ is analytic, then $\exists\ B, D \in \mathcal{B}$ such that $A \triangle B \subset D$ and $m(D) = 0$.

1.0.3 MEASURABLE IMAGE THEOREM. *Suppose that $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ are standard measurable spaces and that $f : X \to X'$ is measurable and 1-1, then $f(A) \in \mathcal{B}'\ \ \forall\ A \in \mathcal{B}$.*

1.0.4 ANALYTIC SECTION THEOREM. *Suppose that $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ are standard measurable spaces and that $f : X \to X'$ is measurable, then $\exists\ g : X' \to X$ which is analytically measurable (in the sense that $g^{-1}A$ is an analytic subset of X' whenever $A \in \mathcal{B}$) such that $f \circ g = Id_{X'}$.*

Let $(X, \mathcal{B})$ be standard.

Given $A \in \mathcal{B}$, let $\mathcal{B} \cap A := \{B \in \mathcal{B} :\ B \subset A\}$. There is a Polish topology on A so that Id: $A \to X$ is continuous. It follows from the measurable image theorem that $\mathcal{B} \cap A = \mathcal{B}(A)$, whence $(A, \mathcal{B} \cap A)$ is a standard measurable space.

Now let $(X, \mathcal{B}, m)$ be a standard measure space and let $A \in \mathcal{B}_+ := \{B \in \mathcal{B} : m(A) > 0\}$. The *induced* measure space is $(A, \mathcal{B} \cap A, m|_A)$ (where $\mathcal{B} \cap A := \{B \in \mathcal{B} :\ B \subset A\}$ and $m|_A(B) := m(B \cap A)$) and this is standard by the above.

Here, and throughout, we'll denote, for a collection $\mathcal{C} \subseteq \mathcal{B}$ of measurable sets, $\mathcal{C}_+ = \{C \in \mathcal{C} : m(C) > 0\}$, and, for $A \subset X$, $\mathcal{C} \cap A = \{C \in \mathcal{C} : C \subset A\}$.

DEFINITION: MEASURABLE MAP, AND INVERTIBLE MAP.

Let $(X, \mathcal{B}, m)$ and $(X', \mathcal{B}', m')$ be measure spaces, and let $A \in \mathcal{B},\ A' \in \mathcal{B}'$.

The map $f : A \to A'$ is *measurable* if $f^{-1}C \in \mathcal{B}\ \forall C \in \mathcal{B}'$.

The measurable map $f : A \to A'$ is called *invertible* on $B \in \mathcal{B} \cap A$ if f is $1-1$ on B, $fB \in \mathcal{B}'$, and $f^{-1} : fB \to B$ is measurable.

DEFINITION: NON-SINGULAR MAP, AND MEASURE PRESERVING MAP. The measurable map $f : A \to A'$ is called (two-sided) *non-singular* if for $C \in \mathcal{B}' \cap A'$, $m(f^{-1}C) = 0$ iff $m'(C) = 0$; and *measure preserving* if $m(f^{-1}C) = m'(C)$ for $C \in \mathcal{B}' \cap A'$.

If $f : A \to A'$ is measurable, invertible, and non-singular on $B \in \mathcal{B} \cap A$, then by the Radon-Nikodym theorem, $\exists\ \phi \in L^1(A)_+$ such that $m(fC) = \int_C \phi dm$ for $C \in \mathcal{B}' \cap A'$. The function ϕ is called the *Radon-Nikodym derivative* of f on A and is denoted by $f' = \frac{dm \circ f}{dm}$. Evidently $f : A \to A'$ is measurable, invertible, and non-singular is measure preserving iff $f' \equiv 1$.

The chain rule for Radon-Nikodym derivatives applies. If $f : A \to B$ and $g : B \to C$ are measurable, invertible, and non-singular on A and B (respectively), then $g \circ f : A \to C$ is measurable, invertible, and non-singular on A and $(g \circ f)' = g' \circ f f'$.

In case $f : B \to fB$ is measurable, invertible, and non-singular, then $f^{-1} : fB \to B$ is non-singular and $f^{-1\prime} = \frac{1}{f' \circ f^{-1}}$.

EXAMPLE. Let $(X, \mathcal{B}, m)$ be the unit interval equipped with Borel sets and Lebesgue measure, and suppose that $T : X \to X$ is a strictly increasing homeomorphism; then $T : X \to X$ is invertible.

We have that T is nonsingular iff both T and T^{-1} are absolutely continuous functions, and $T' := \frac{dm \circ T}{dm} = |DT|$ where $DT(x) := \lim_{h \to 0} \frac{T(x+h) - T(x)}{h}$.

DEFINITION: FACTOR MEASURE SPACE, AND FACTOR MAP.

The measure space $(X', \mathcal{B}', m')$ is a *factor space* of $(X, \mathcal{B}, m)$ if there are subsets $Y \in \mathcal{B},\ Y' \in \mathcal{B}'$ such that $m(X \setminus Y) = m'(X' \setminus Y') = 0$, and a measurable, measure preserving map $\pi : Y \to Y'$. This map is called a *factor map* and we sometimes denote it $\pi : X \to X'$.

In the same situation, we sometimes call the measure space $(X, \mathcal{B}, m)$ an *extension* of $(X', \mathcal{B}', m')$.

DEFINITION: CARTESIAN PRODUCT SPACE. If $(\Omega, \mathcal{F}, p)$ is a probability space, then the *Cartesian product space* $(X \times \Omega, \mathcal{B} \otimes \mathcal{F}, m \times p)$ is always an extension of $(X, \mathcal{B}, m)$. Here $\mathcal{B} \otimes \mathcal{F} := \sigma(\mathcal{B} \times \mathcal{F})$ and $(m \times p)(A \times B) := m(A)p(B)$.

DEFINITION: ISOMORPHISM OF MEASURE SPACES.

An *isomorphism* between the measure spaces $(X, \mathcal{B}, m)$ and $(X', \mathcal{B}', m')$ is an invertible factor map $\pi : X \to X'$, and the measure spaces $(X, \mathcal{B}, m)$ and $(X', \mathcal{B}', m')$ are *isomorphic* if there is an isomorphism between them.

REMARK: NON-ATOMIC STANDARD SPACES.

Any non-atomic standard probability space is isomorphic with the unit interval $[0,1] \subset \mathbb{R}$ equipped with its Borel sets $\mathcal{B}$ and Lebesgue measure λ. This is proven using theorem 1.0.1 to obtain isomorphism with the unit interval $[0,1] \subset \mathbb{R}$ equipped with its Borel sets and some non-atomic probability p; and then using $\lambda(J) = p(J)$ for intervals $J \subset I$ where $\pi(x) := p([0,x])$ to obtain the final isomorphism with $([0,1], \mathcal{B}, \lambda)$.

This can be used to show that any non-atomic, σ-finite standard measure space whose total mass is infinite is isomorphic to $\mathbb{R}$ equipped with its Borel sets and Lebesgue measure.

DEFINITION: COMPLETION OF A MEASURE SPACE.

Given a measure space $(X, \mathcal{B}, m)$, the *completion* of $\mathcal{B}$ (with respect to m) is the collection (a σ-algebra)

$$\overline{\mathcal{B}}_m := \{A \subset X : \exists\ B, D \in \mathcal{B} \text{ such that } A \triangle B \subset D, \quad m(D) = 0\}.$$

The measure space $(X, \overline{\mathcal{B}}_m, m)$ is also known as the *completion* of $(X, \mathcal{B}, m)$. A measure space $(X, \mathcal{B}, m)$ is *complete* if $\overline{\mathcal{B}}_m = \mathcal{B}$.

DEFINITION: SEPARABLE MEASURE SPACE.

A measure space $(X, \mathcal{B}, m)$ is *separable* if $\exists$ a countable collection $\mathcal{A} \subset \mathcal{B}$ such that $\sigma(\mathcal{A}) = \mathcal{B} \mod m$ (that is: $\overline{\sigma(\mathcal{A})}_m = \overline{\mathcal{B}}_m$) which separates points in the sense that

$$1_A(x) = 1_A(y)\ \forall\ A \in \mathcal{A} \implies x = y.$$

DEFINITION: LEBESGUE SPACE.

A *Lebesgue space* is a complete measure space which is isomorphic to the completion of a standard measure space.

A Lebesgue space $(X, \mathcal{B}, m)$ is evidently separable and complete, and there is a subset $X_0 \in \mathcal{B}$, $m(X \setminus X_0) = 0$ endowed with a Polish topology such that $\mathcal{B} \cap X_0 = \overline{\mathcal{B}(X_0)}_m$.

Let $(X, \mathcal{C}, m)$ be separable and complete with $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$ as the countable generating collection. Consider the compact metric space $\Omega := \{0,1\}^{\mathbb{N}}$, and define $\pi : X \to \Omega$ by $\pi(x)_n := 1_{A_n}(x)$. Evidently, $\pi : X \to \pi(X))$ is 1-1, and measurable. In fact if $\mu := m \circ \pi^{-1} : \mathcal{B}(\Omega) \to [0, \infty)$, then π is an isomorphism of the measure spaces $(X, \mathcal{B}, m)$ and $(\pi(X), \overline{\mathcal{B}(\Omega) \cap \pi(X)}_\mu, \mu)$.

As shown in ([**Ro1**], [**Rudo**]), the space $(X, \mathcal{C}, m)$ is a Lebesgue space if and only if $\pi(X) \in \overline{\mathcal{B}(\Omega)}_\mu$. To see this, if $\pi(X) \in \overline{\mathcal{B}(\Omega)}_\mu$, then $\exists\ \Omega_0 \in \mathcal{B}(\Omega)$ such that $\Omega_0 \subset \pi(X)$ and $\mu(\Omega_0) = 1$ with the consequence that $(X, \mathcal{C}, m)$ is isomorphic with $(\Omega_0, \overline{\mathcal{B}(\Omega_0 m)}_\mu, \mu)$ - a Lebesgue space. Conversely, if $(X, \mathcal{C}, m)$ is a Lebesgue space then $\exists\ X_0 \in \mathcal{B}$, $m(X_0) = 1$ such that X_0 is Polish, $\mathcal{C} \cap X_0 = \overline{\mathcal{B}(X_0)}_m$ and $\pi : X_0 \to \Omega$ is Borel measurable. By the universal measurability theorem (or by Lusin's theorem) $\pi(X_0) \in \overline{\mathcal{B}(\Omega)}_\mu$ and since $\mu(\pi(X_0)) = 1$, $\pi(X_0) \subset \pi(X)$ we have $\pi(X) \in \overline{\mathcal{B}(\Omega)}_\mu$.

This shows that $\exists$ complete, separable measure spaces which are not Lebesgue spaces, for example $(X, \mathcal{C}, m)$ where $X \subset [0,1]$ has full outer measure and zero inner measure, $\mathcal{C} := \overline{B([0,1])}_m \cap X$ and m is outer measure.

DEFINITION: MEASURE ALGEBRA. Let $(X, \mathcal{B}, m)$ be a measure space, and define the relation $\sim$ on $\mathcal{B}$ by $A \sim B$ if $m(A \Delta B) = 0$, then (see [**Halm1**]) $\sim$ is an equivalence relation,

$A_n, A'_n \in \mathcal{B},\ A_n \sim A'_n \ \ (n \geq 1) \implies$

$$A_1^c \sim A_1'^c, \quad \bigcup_{n=1}^{\infty} A_n \sim \bigcup_{n=1}^{\infty} A'_n, \ \& \ \bigcap_{n=1}^{\infty} A_n \sim \bigcap_{n=1}^{\infty} A'_n.$$

The collection of equivalence classes

$$\mathcal{S}(X,\mathcal{B},m) := \{\{A' \in \mathcal{B}:\ A' \sim A\}:\quad A \in \mathcal{B}\}$$

is called the *measure algebra* of $(X,\mathcal{B},m)$.

DEFINITION: MEASURE ALGEBRA CONJUGACY. A *measure algebra conjugacy* between the measure spaces $(X,\mathcal{B},m)$ and $(X',\mathcal{B}',m')$ is a bijection $\pi :$ $\mathcal{S}(X,\mathcal{B},m) \to \mathcal{S}(X',\mathcal{B}',m')$ such that $\pi(A \setminus B) = \pi(A) \setminus \pi(B)$, $\pi\Big(\bigcup_{n=1}^{\infty} A_n\Big) =$ $\bigcup_{n=1}^{\infty} \pi(A_n)$ and $m' \circ \pi = m$.

Isomorphic measure spaces are measure algebra conjugate, a measure algebra conjugacy being induced by an isomorphism.

Conversely, any measure algebra conjugacy between Lebesgue spaces induces an isomorphism ([**Ro1**], [**Rudo**]).

All separable, non-atomic probability spaces are measure algebra conjugate ([**Halm1**]), and every separable measure space is measure algebra conjugate to a standard one ([**Fu**]).

A non-singular- (or measure preserving-) transformation is a non-singular- (or measure preserving-) map mod m. The next definitions make this precise.

DEFINITION: NON-SINGULAR TRANSFORMATION AND MEASURE PRESERVING TRANSFORMATION.

A *non-singular transformation* T of X is a measurable, non-singular map $T :$ $Y \to Y$ where $Y \in \mathcal{B}$ and $m(X \setminus Y) = 0$.

A *measure preserving transformation* T of X is a measurable, measure preserving map $T : Y \to Y$ where $Y \in \mathcal{B}$ and $m(X \setminus Y) = 0$.

1.0.5 PROPOSITION. *Suppose that $(X,\mathcal{B},m)$ is standard, $Y \in \mathcal{B}$, $m(X\setminus Y) = 0$ and $T : Y \to X$ is a non-singular map, then T is a non-singular transformation of X.*

PROOF. By the universal measurability theorem, $TY \in \overline{\mathcal{B}}_m$. Also,

$$0 = m(X \setminus Y) \geq m(X \setminus T^{-1}TY) = m(T^{-1}(X \setminus TY))$$

whence $m(X \setminus TY) = 0$ since $T : Y \to X$ is non-singular.

Choose $Z \in \mathcal{B}$, $Z \subset TY$, $m(X \setminus Z) = 0$ and set $U := \bigcap_{n=1}^{\infty} T^{-n}Z$, then $TU = Z \cap U$ and $TU \in \mathcal{B}$, $TU \subset U$, and $T : U \to TU$ is a nonsingular map.

By non-singularity of $T : Y \to X$, $m(X \setminus T^{-n}Z) = 0\ \ \forall\, n \geq 0$ whence
$m(X \setminus U) = m(X \setminus TU) = m(U \setminus TU) = 0$;
$T : U \to U$ is a non-singular map,
and T is a non-singular transformation of X. □

EXAMPLE 1.0.1, A NON-SINGULAR TRANSFORMATION. Let $(X,\mathcal{B},m)$ be the unit interval equipped with Borel sets and Lebesgue measure, and let $\{A_j : j \geq 1\}$ be a partition of X into open intervals (i.e. the $\{A_j : j \geq 1\}$ are disjoint open intervals, and $X = \bigcup_{j=1}^{\infty} A_j \bmod m$).

Given a collection $\{B_j : j \geq 1\}$ of open intervals, define for $j \geq 1$, $T : A_j \to B_j$ to be an absolutely continuous bijection whose inverse is also absolutely continuous (e.g. the increasing linear bijection).

Note that T is only defined on $U = \bigcup_{j=1}^{\infty} A_j$. Clearly $T : U \to TU = \bigcup_{j=1}^{\infty} B_j$ is measurable, and non-singular. Therefore by proposition 1.0.5, T is a non-singular transformation of X iff

$$TU = \bigcup_{j=1}^{\infty} B_j = X \qquad \text{mod } m.$$

If T is a non-singular transformation of a σ-finite measure space $(X, \mathcal{B}, m)$, and p is another measure on $(X, \mathcal{B})$ *equivalent* to m (denoted $p \sim m$) in the sense that

$$p(A) = 0 \ \Leftrightarrow \ m(A) = 0,$$

then T is a non-singular transformation of $(X, \mathcal{B}, p)$.

Thus, a non-singular transformation of a σ-finite measure space is actually a non-singular transformation of a probability space.

EXAMPLE 1.0.2, A PROBABILITY PRESERVING TRANSFORMATION. Let $X = [0,1]^{\mathbb{N}}$ and let $\mathcal{B}$ be the σ-algebra generated by *cylinder* sets of form $[A_1, \ldots, A_n] := \{\underline{x} \in X : x_j \in A_j, \ 1 \le j \le n\}$, where $A_1, \ldots, A_n \in \mathcal{B}(I)$ (the Borel subsets of $I = [0,1]$), and let the *shift* $T : X \to X$ be defined by $(Tx)_n = x_{n+1}$. Note that X is a compact metric space when equipped with the product topology, and $\mathcal{B}$ is its collection of Borel sets.

Define using Kolmogorov's existence theorem ([**Kol**], [**Part**]) a probability $m : \mathcal{B}(X) \to [0,1]$ by

$$m([A_1, ..., A_n]) := \prod_{k=1}^{n} |A_k| \ \ (A_1, \ldots, A_n \in \mathcal{B}(I))$$

where $|A|$ denotes the Lebesgue measure of $A \in \mathcal{B}(I)$). Evidently,

$$m(T^{-1}[A_1, ..., A_n]) = m([I, A_1, ..., A_n]) = m([A_1, ..., A_n]) \ \ (A_1, \ldots, A_n \in \mathcal{B}(I))$$

whence

$$m \circ T^{-1} = m.$$

The measure space X represents the set of (possibly random) *"configurations"* of some system, and T represents the change under *"passage of time"*. The non-singularity of T reflects the assumed property of the system that configuration sets that are impossible sometimes are always impossible. A probability preserving transformation would describe a system in a "steady state", where configuration sets occur with the same likelihood at all times.

One might conjecture that each non-singular transformation is obtained by starting with a measure preserving transformation, and then "passing" to some equivalent measure, however we'll see in §1.2 that this is not the case.

DEFINITION: INVERTIBLE, AND LOCALLY INVERTIBLE NON-SINGULAR TRANSFORMATIONS.

The non-singular transformation T of X is called *invertible* if T is invertible on some $Y \in \mathcal{B}$ with $m(X \setminus Y) = 0$, and *locally invertible* if there is are disjoint measurable sets $\{A_j : j \geq 1\}$ such that $m(X \setminus \bigcup_{j \geq 1} A_j) = 0$, and T is invertible on each A_j.

The non-singular transformation in example 1.0.1 is locally invertible (being invertible on each A_j). It is invertible iff $\{B_j : j \geq 1\}$ is a partition of X mod m.

Evidently, if T is a locally invertible, non-singular transformation of X, then T is *positively non-singular* in the sense that

$$A \in \mathcal{B},\ m(A) = 0 \implies m(TA) = 0.$$

The probability preserving transformation T in example 1.0.2 does not have this property. If $C \in \mathcal{B}(I)$ is a non-empty set of Lebesgue measure zero, and $D := [C]$, then $m(D) = 0$ and $TD = X$.

A non-singular transformation T of a standard probability space $(X, \mathcal{B}, m)$ is locally invertible iff $T^{-1}\{x\}$ is countable for m-a.e. $x \in X$.

Clearly if T is locally invertible, then $T^{-1}\{x\}$ is countable $\forall\ x \in X$. The converse follows from the

1.0.6 LOCAL INVERTIBILITY LEMMA.

Let $(X, \mathcal{B}, m)$ and $(Y, \mathcal{C}, \mu)$ be standard probability spaces, and suppose that $\pi : X \to Y$ is a measurable, measure preserving map with $\pi^{-1}\{x\}$ countable $\forall\ x \in X$, then $\exists$ a countable, measurable partition α of X such that $\pi : a \to Ta$ is non-singular and invertible $\forall\ a \in \alpha$.

We prove the local invertibility lemma using two results which will also be important in the sequel:
the exhaustion lemma 1.0.7; and the disintegration theorem 1.0.8.

DEFINITION: HEREDITARY COLLECTION, MEASURABLE UNION.

Let $(X, \mathcal{B}, m)$ be a measure space. A collection $\mathfrak{H} \subset \mathcal{B}$ is called *hereditary* if

$$C \in \mathfrak{H},\ B \subset C,\ B \in \mathcal{B} \implies B \in \mathfrak{H}.$$

A set $U \in \mathcal{B}$ is said to *cover* the hereditary collection $\mathfrak{H}$ if $A \subset U$ mod m $\forall\ A \in \mathfrak{H}$.

A hereditary collection $\mathfrak{H} \subset \mathcal{B}$ is said to *saturate* $A \in \mathcal{B}$ if $\forall\ B \in \mathcal{B},\ B \subset A,\ m(B) > 0,\ \exists\ C \in \mathfrak{H},\ m(C) > 0,\ C \subset B$.

The set $U \in \mathcal{B}$ is called a *measurable union* of the hereditary collection $\mathfrak{H} \subset \mathcal{B}$ if it both covers, and is saturated by $\mathfrak{H}$.

There is no more than one measurable union of a hereditary collection. To see this, let $U, U' \in \mathcal{B}$ be measurable unions of the hereditary collection $\mathfrak{H}$, and suppose that $m(U \setminus U') > 0$, then (since $\mathfrak{H}$ saturates U) $\exists\ C \in \mathfrak{H},\ m(C) > 0,\ C \subset U \setminus U'$ whence (since U' covers $\mathfrak{H}$) $C \subset U'$ mod m contradicting $m(C) > 0$. This shows that $U \subset U'$ mod m and by symmetry, $U = U'$ mod m.

The exhaustion lemma (below) shows existence of measurable unions.

1.0.7 EXHAUSTION LEMMA. *Let $(X, \mathcal{B}, m)$ be a probability space and let $\mathfrak{H} \subset \mathcal{B}$ be hereditary, then $\exists\ A_1, A_2, \cdots \in \mathfrak{H}$ disjoint such that $U(\mathfrak{H}) = \bigcup_{n=1}^{\infty} A_n$ is a measurable union of $\mathfrak{H}$.*

PROOF.
Let
$$\epsilon_1 := \sup\{m(A): \ A \in \mathfrak{H}\},$$
choose $A_1 \in \mathfrak{H}$ such that $m(A_1) \geq \frac{\epsilon_1}{2}$, and let
Let
$$\epsilon_2 := \sup\{m(A): \ A \in \mathfrak{H}, \ A \cap A_1 = \emptyset\},$$
choose $A_2 \in \mathfrak{H}$ such that $A_2 \cap A_1 = \emptyset$ and $m(A_2) \geq \frac{\epsilon_2}{2}$.

Continuing the process, we obtain a sequence of disjoint $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{H}$ and $\epsilon_n \downarrow$ such that
$$\epsilon_n := \sup\{m(A): \ A \in \mathfrak{H}, \ A \cap A_k = \emptyset \ \forall \ k < n\}, \ \ m(A_n) \geq \frac{\epsilon_n}{2}.$$
Clearly $\sum_{n=1}^{\infty} \epsilon_n \leq 2\sum_{n=1}^{\infty} m(A_n) \leq 2$, whence $\epsilon_n \to 0$.

We claim that $U := \bigcup_{n=1}^{\infty} A_n = X$ is a measurable union of $\mathfrak{H}$. Evidently $\mathfrak{H}$ saturates U. To see that U covers $\mathfrak{H}$ assume otherwise, then $\exists \ A \in \mathfrak{H}, \ m(A) > 0$ such that $A \cap A_n = \emptyset \ \ \forall \ n \geq 1$ whence $m(A) \leq \epsilon_n \to 0$ contradicting $m(A) > 0$. □

We denote the measurable union of the hereditary collection $\mathfrak{H}$ by $U(\mathfrak{H})$.

1.0.8 DISINTEGRATION THEOREM. *Let $(X, \mathcal{B}, m)$, $(Y, \mathcal{C}, \mu)$ be standard probability spaces and suppose that $\pi : X \to Y$ is a measurable, and $m = \mu \circ \pi^{-1}$, then $\exists \ Y_0 \in \mathcal{C}$ such that $\mu(Y_0) = 1$ and $\exists$ a measurable function $y \mapsto m_y \ \ (Y_0 \to \mathcal{P}(X, \mathcal{B}))$ such that $m_y(\pi^{-1}\{y\}) = 1 \ \forall \ y \in Y_0$ and*
$$m(A \cap \pi^{-1}B) = \int_B m_y(A) d\mu(y) \ \forall \ A \in \mathcal{B}, \ B \in \mathcal{C}.$$

The measure m_y is called the *fibre measure* over $\pi^{-1}\{y\}$.

PROOF. For each $f \in L^1(m)_+$, define a measure ν_A on $\mathcal{C}$ by
$$\nu_f(C) := \int_{\pi^{-1}C} f dm.$$
The measure ν_f is μ-absolutely continuous and so, by the Radon-Nikodym theorem there is a measurable function $E(f|\pi) = \frac{d\nu_f}{d\mu} \in L^1(\mu)$, such that $\int_C E(f|\pi) d\mu = \int_{\pi^{-1}C} f dm$.

Set
$$u_y(A) = E(1_A|\pi)(y),$$
then
$$\int_C u_y(A) d\mu(y) = m(A \cap \pi^{-1}C) \quad \forall \ A \in \mathcal{B}, \ C \in \mathcal{C}.$$
Also, if $A_1, ... \in \mathcal{B}$ are disjoint, then
$$u_y(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} u_y(A_k),$$
for m-a.e. $y \in Y$.

Since X is standard and uncountable, we may assume by Kuratowski's isomorphism theorem that $X = \{0, 1\}^{\mathbb{N}}$. Let $\mathcal{A}$ denote the algebra of finite unions of cylinder sets in Ω, then $\mathcal{A}$ is countable and generates $\mathcal{B}$, and each set in $\mathcal{A}$ is both open and compact.

Since $\mathcal{A}$ is countable, there is a set $Y_0 = Y \bmod m$ such that

$$u_y(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} u_y(A_k) \quad \forall y \in Y_0$$

whenever $A_1, ..., A_n \in \mathcal{A}$ are disjoint.

For $y \in Y_0$ and $E \subset X$ define

$$m_y(E) = \inf\{\sum_{n=1}^{\infty} u_y(A_n) : \bigcup_{n=1}^{\infty} A_n \supseteq E, \ A_n \in \mathcal{A}\}.$$

By Caratheodory's theorem, $m_y : \mathcal{B} \to [0,1]$ is a measure. Since sets in $\mathcal{A}$ are both compact and open, we have that $u_y(A) = m_y(A) \ \forall \ A \in \mathcal{A}$.

Clearly $y \mapsto m_y(A)$ is measurable ($Y_0 \mapsto [0,1]$) $\forall \ A \in \mathcal{A}$. A monotone class argument shows measurability $\forall \ A \in \mathcal{B}$ and that $m_y(A) = u_y(A)$ for a.e. $y \in Y_0$.

To see that $m_y(\pi^{-1}\{y\}) = 1$ for a.e. $y \in Y$, note first that for $C \in \mathcal{C}$ fixed,

$$\int_C m_y(\pi^{-1}C) d\mu(y) = m(\pi^{-1}C) = \int_C m_y(X) d\mu(y) = \mu(C)$$

whence $m_y(\pi^{-1}C) = 1$ for a.e. $y \in C$.

Now fix a metric on Y and let $\beta_n \uparrow$ be an increasing sequence of countable, measurable partitions on Y such that $\sup_{b \in \beta_n} \operatorname{diam} b \to 0$ as $n \to \infty$. For $y \in Y, \ n \geq 1$ write $y \in b_n(y) \in \beta_n$. Choose $Y_1 \in \mathcal{C} \cap Y_0, \ \mu(Y_1) = 1$ such that

$$m_y(\pi^{-1}b) = 1 \ \forall \ y \in b \cap Y_1, \ b \in \beta_n, \ n \geq 1.$$

We now have for $y \in Y_1$ that

$$m_y(\pi^{-1}\{y\}) \ \leftarrow \ m_y(\pi^{-1}b_n(y)) = 1.$$

□

REMARK: FIBRE EXPECTATIONS AND CONDITIONAL EXPECTATIONS.

Let $(X, \mathcal{B}, m)$, $(Y, \mathcal{C}, \mu)$ be standard probability spaces and suppose that $\pi : X \to Y$ is measurable, and $m = \mu \circ \pi^{-1}$, with fibre measures $y \mapsto m_y \in \mathcal{P}(\pi^{-1}\{y\})$.

If $f : X \to \mathbb{R}$ is bounded and measurable, the function $y \mapsto \int_X f dm_y$ is called the *fibre expectation* of f on $\pi^{-1}\{y\}$. Evidently $\int_X f dm_y = E(f|\pi)(y)$ (as defined above) for μ-a.e. $y \in Y$. It follows that the conditional expectation of f with respect to $\pi^{-1}\mathcal{C}$ is given by

$$E(f|\pi^{-1}\mathcal{C}) = E(f|\pi) \circ \pi \quad m- \text{ a.e. on } X.$$

PROOF OF THE LOCAL INVERTIBILITY LEMMA.

Let $y \mapsto m_y \in \mathcal{P}(\pi^{-1}\{y\})$ $(y \in Y_0)$ be the fibre measures on $\pi^{-1}\{y\}$ as in the disintegration theorem.

Since $m_y(\pi^{-1}\{y\}) = 1 \ \ \forall \ y \in Y_0$, and $\pi^{-1}\{y\}$ is countable $\forall \ y \in Y_0$, the probabilities m_y $(y \in Y_0)$ are purely atomic, and we may assume (possibly discarding a null set) that $m_{\pi x}(\{x\}) > 0 \ \forall \ x \in X$.

Call a set $A \in \mathcal{B}$ a *π-section* if $\pi A \in \mathcal{B}$ and $\pi : A \to \pi A$ is measurable, non-singular and invertible, and call a section $A \in \mathcal{B}$ *onto* if $\pi A = Y \bmod \mu$. Denote the collection of π-sections by $\mathfrak{S}$. It is enough to show that there is a partition of X into π-sections.

We claim that the hereditary collection $\mathfrak{S}$ saturates X. To see this let $B \in \mathcal{B}$, $m(B) > 0$, then by the universal measurability theorem
$\exists\, C \in \mathcal{C},\ C \subset \pi B,\ \mu(C) > 0$ and by the analytic section theorem $\exists\ f : C \to B$ analytically measurable such that $\pi \circ f = \mathrm{Id}|_C$. It follows that $f(C) = B \cap \pi^{-1}C =: A$ and that $A \in \mathfrak{S}$. To see that $m(A) > 0$,

$$m(A) = \int_Y m_y(A)d\mu(y) = \int_C m_y(A)d\mu(y) = \int_C m_y(\{f(y)\})d\mu(y) > 0$$

because $\mu(C) > 0$ and $m_{\pi x}(\{x\}) > 0\ \forall\ x \in X$.

The result now follows from the exhaustion lemma. □

If T is a non-singular transformation of $(X, \mathcal{B}, m)$, and $A \in \mathcal{B}$, $m(A) > 0$, $T^{-1}A = A$, then T is a non-singular transformation of $(A, \mathcal{B} \cap A, m|_A)$. The concept of irreducibility for non-singular transformations is called *ergodicity.*

DEFINITION: ERGODIC.

A non-singular transformation T is called *ergodic* if $A \in \mathcal{B}$, $T^{-1}A = A \bmod m$ implies $m(A) = 0$ or $m(A^c) = 0$.

This condition actually implies a stronger condition.

1.0.9 PROPOSITION. *Suppose that T is an ergodic non-singular transformation. If $f : X \to \mathbb{R}$ is measurable, and $f \circ T = f$ a.e., then*

$$\exists c \in \mathbb{R} \text{ such that } f = c \text{ a.e.}$$

PROOF. For $c \in \mathbb{R}$ the set $[f \le c]$ is T-invariant, and hence $[f \le c] = \emptyset,\ X \bmod m$. If

$$c_0 = \inf\{c \in \mathbb{R} :\ [f \le c] = X \mod m\},$$

then

$$[f = c_0] = [f \le c_0] \setminus \bigcup_{n \ge 1} [f \le c_0 - \frac{1}{n}] = X \mod m.$$

□

DEFINITION: FACTOR TRANSFORMATION, ISOMORPHISM.

The non-singular transformation T' of $(X', \mathcal{B}', m')$ is a *factor* of the non-singular transformation T of $(X, \mathcal{B}, m)$ if there are sets $Y \in \mathcal{B}$, $Y' \in \mathcal{B}'$ such that $m(X \setminus Y) = m'(X' \setminus Y') = 0$, $TY \subset Y$, $T'Y' \subset Y'$; and there is a measurable, measure preserving map $\pi : Y \to Y'$ so that $\pi \circ T' = T \circ \pi$ on Y.

The map $\pi : Y \to Y'$ is called a *factor map* and is sometimes denoted $\pi : T \to T'$.

An *isomorphism* between the non-singular transformations T of $(X, \mathcal{B}, m)$ and T' of $(X', \mathcal{B}', m')$ is a factor map $\pi : T \to T'$ which is invertible in the sense that there are sets $Y \in \mathcal{B}$, $Y' \in \mathcal{B}'$ such that $m(X \setminus Y) = m'(X' \setminus Y') = 0$, $TY \subset Y$, $T'Y' \subset Y'$; and such that $\pi : Y \to Y'$ is invertible.

The non-singular transformations T of $(X, \mathcal{B}, m)$ and T' of $(X', \mathcal{B}', m')$ are *isomorphic* if is an isomorphism between them.