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Introduction to Operator Theory and Invariant Subspaces

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INTRODUCTION TO OPERATOR THEORY AND INVARIANT SUBSPACES

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Introduction

As a whole, Operator Theory does not exist yet. We may have satisfactory results for some classes of operators, or handle, in a convenient way, some limited situations, but most of the fundamental problems are still unsolved, and it is even not clear that they are correctly formulated.

Among classes which have received special attention during a recent past, the best known are the compact operators and the normal ones, for which some powerful tools are available, such as spectral decompositions. But even for these classes, some simple questions are still open.

The most fundamental question in Operator Theory is the Invariant Subspace Problem, which is still unsolved in Hilbert spaces. Most attempts, since the beginning of the century, have been made in the positive direction, that is trying to prove that every operator has a non-trivial Invariant Subspace. An obvious remark is that, in order to succeed, one has first to show that, for every operator, there is a point x, the orbit of which (that is : the set $\{T^n x \; ; \; n \in \mathbb{N}\}$) is not dense in the whole space. Indeed, if every orbit is dense, there is no Invariant Subspace. Unfortunately, this second problem is also still open, and, strangely enough, has not received much attention. Moreover, there are stronger versions of it which are also unsolved.

The Invariant Subspace Problem does not appear as the last open question in a well-polished theory. It even seems to me, conversely, that the paths which might lead to its solution have not been traced. Fundamental —and extremely natural— questions, such as the behavior of orbits of a given operator, are considered here for the first time.

Even the most indulgent reader will not be able to find in the whole Operator Theory (let's call it this way, for simplicity), a result which approaches, by far, the depth of Dvoretzky's Theorem, about sections of convex bodies, in the Geometry of Banach spaces, or of Carleson's Theorem, about convergence of Fourier series, in Harmonic Analysis. Obviously, the whole field is globally late, compared to many others in Analysis (not to mention, of course, other branches of mathematics).

To this fact, I can see three possible and complementary explanations. First, a basic understanding of Operator Theory relies on a rather deep knowledge of other areas, such as Fourier Analysis or Analytic Functions. Just to explain what are the Invariant Subspaces of the usual shift requires Beurling's theory of the decomposition of H^2 functions into inner and outer parts. This theory is not so old : 40 years only. Other tools, connected with boundary values of analytic functions, are fairly recent, and not well understood. The connection with spectral synthesis seems also to come up. In short, we may say that the analytic tools which are needed are not old, and have not made their way yet.

The second reason is that the theory has developed only, so far, into a small number of directions, which might not be the right ones. Emphasis has been laid upon a few constructions which have been religiously considered by their adepts, despite their limited range of applicability.

The third one is a strong lack of examples. Besides trivial ones, such as shifts of multiplication operators, very few examples of significant operators are known. Indeed, their construction is quite hard, but, precisely for this reason, it should become a kind of moral necessity for the specialits in the field, just to prove that the abstract results they present are not vacuous. A consequence of this lack of examples is that one never knows what is the range of application of the results which exist.

A striking case is that of Lomonossov's Theorem, proving the existence of hyperinvariant subspaces for operators commuting with a compact one (see Chapter IV). This theorem was proved in 1974. Only in 1981 was given an example of an operator to which it does not apply. Meanwhile, dozens of papers had been published, dealing with "extensions" of Lomonossov's Theorem, and these extensions did not carry a single example showing they were not vacuous.

The present status of Operator Theory may perhaps be compared to what Geometry of Banach Spaces was twenty years ago, before the creation of numerous examples of "pathological" Banach spaces, and the unified theories which followed.

When, some years ago, I wrote a book about Geometry of Banach Spaces, the task did not seem too hard : the theory in itself had reached a nice degree of achievement, but no elementary text-book was available to present it.

The present situation is exactly opposite. First, as I already stated, the theory does not yet exist, and, second, the books to present it are already quite numerous. So, why should I add one more item to a list which is already too long ? To such a good question, I can provide only a bad answer : I won't present things in the same manner.

Indeed, we will not try to develop the achievements of the theory, we will show the gaps : this spirit is already clear from what we said. Our goal is to be, at the same time, an *Introduction* to Operator Theory, and a *research* book. In the present case, these words should not look incompatible. Let's explain them more in detail.

Being an Introduction, the book is intended for students. It requires only a basic knowledge of classical Analysis : measure theory, analytic functions, Hilbert spaces, functional Analysis. The book is self-contained, except for a few technical tools, for which precise references are given.

Part I takes things at their beginning : finite-dimensional spaces, general spectral theory. But very soon (Chapter III), new material is presented, leading to new directions for research, and open questions are mentioned.

Part II concerns compactness and its applications : not only spectral theory for compact operators (with Invariant Subspaces, Lomonossov's Theorem), but also duality between the space of nuclear operators and the space of all operators on a Hilbert space, a result which is very seldom presented.

Part III deals with analytic functions, and contains only few new developments. It seemed necessary to us to include H^p spaces, Beurling's theory, etc, because they play a central role in Operator Theory. These results are of course presented in many well-known and well-written books, dealing with Fourier Analysis and Analytic functions. But, quite often, these books contain a general theory which is quite deep and goes far beyond what we need here. This is why we have chosen to present a simplified version, operator-oriented. But on this particular topic, what we present is obviously unsufficient : recent advances on the Invariant Subspace Problem seem to rely upon deep properties of analytic functions.

Part IV contains Algebra Techniques : Gelfand's Theory, and applications to Normal Operators. Here again, directions for research are indicated.

Part V presents dilations and extensions : Nagy–Foiaş dilation theory, and the author's work about C_1-contractions.

Part VI deals with the Invariant Subspace Problem : first, positive results derived from S. Brown's methods. Then, we turn to counter-examples. After P. Enflo's counter- example of an operator with no Invariant Subspaces, further examples were found, by C. Read and the author. C. Read's example was later simplified by himself and then by A.M. Davie ; this last version is now quite accessible, and is presented here for the first time.

As can be seen from this brief description, a lot of new material is presented, and the level of research is reached, on the Invariant Subspace Problem, both on the positive and the negative directions.

Each Chapter ends with some "Complements" which indicate further results, open problems, and give references. To help the reader, we have added exercises and an Index.

We also felt free to indicate what are the directions of research which look promising, and, conversely, what are the ones which have, more or less, reached their limits. These opinions are of course strictly personal, but they are motivated by quite a few years of experience !

There are, of course, several aspects of Operator Theory which we do not approach (such as approximation by compact operators) : we had in mind the Invariant Subspace Problem. But we would be glad if the topics presented could attract new people to an area which is quite fascinating and challenging.

The notation is rather customary, and is compatible with that of our previous books. A Hilbert space is denoted by H, a Banach space by E or X, an operator by T, and so on. However, in a few occasions, we found the existing terminology improper, confusing, and sometimes misleading. In such cases, we took the liberty to eliminate gothic letters, complicated symbols, in order to be left with the naked ideas. We do not think of the notation as a decoration, but as a mean of communicating the ideas, as simply as possible.

A few words about the presentation and the typing. In the bad old days, it was customary to thank our secretaries for their typing, with hypocritical tears in the eyes (see my previous books). Things have changed, and, for once, in the good sense.

This book was typeset by the author himself, using Plain TEX. Knuth's TEX is not easy to learn in its subtleties, but already an elementary knowledge of it allows (as one sees here) an output which is incomparably better than what anyone can produce on a typewriter. Most symbols, commands already exist in TeX, and require no further work. The only symbol I had to create myself was " $\not\to$ ". I am glad to offer to the mathematical community the definition of such an important feature (with no increase in the price of the book) :

```
\def\notto{\hbox{$~\rightarrow~\kern - 1.5em\hbox{/ }$ } }
```

So, if you put this line in your "macro file" (list of definitions), whenever you type " \notto ", you get $\not\to$.

Mathematical editing had decreased in typographical quality, many publishers presenting books which were not typeset, but only typewritten, and then reproduced photographically. Let's hope that TeX will open a new era.

The editor I used was *PC-Write*, which is, in my opinion, the best of all I tried. It allows to redefine the keyboard (so as to produce, for example, greek letters with one keystroke), to store long sequences of words as "macros", and many other things.

The book follows two courses I gave at the University of Paris VII, in 1984/85 and 86/87. The final version was mostly written during the fall 87, when I was Visiting Professor at the Ohio State University, Columbus. I found there nice working facilities, including a good library. Computer equipment, however, was quite unsatisfactory. Fortunately, I was invited, for short periods, by other Universities, where I could produce "hard versions" of my work. I acknowledge the warm hospitality of Kent State University, University of Columbia–Missouri, University of Illinois at Urbana–Champaign, which put at my disposition all the modern equipment I needed, such as computers with hard disks, laser printers, and so on.

It is a very pleasant duty to mention all people who helped me, in various ways, to prepare the manuscript : Sylvie Guerre, Valérie Martel, Françoise Piquard, Earl Berkson, Gilles Godefroy, Elias Saab. Many others contributed to specific chapters and are mentioned at proper place.

My interest in Operator Theory originally comes from conversations with Per Enflo, who had done in this area a pioneering work which was not fully understood, and from which I could benefit. I am glad to acknowledge an influence which is anyway so obvious that I could not hide it, even if I wanted to.

As G. B. Shaw said, let me now withdraw and lift the curtain. If you find that there is not much behind, please consider at least that I did not paint it with artificial colors.

Paris, december 1987

Bernard Beauzamy

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Oceano Tex

Elle contemple l'écran, tout vert dans les ténèbres,
Où les chiffres s'affichant jettent des lueurs d'algèbre.
Viendra-t-il, le chapitre inscrit dans les mémoires ?
O touches, que vous savez de lugubres histoires !
Combien de mots, de lignes, se sont évanouis,
Dans l'aveugle machine à jamais engloutis !

II

Mais le menu s'affiche, rassurant, familier,
Tiens, dit-elle en souriant, le voici tout entier.

III

Sur la table, entassé, l'article et ses progrès,
Certain, démontré presque, et probable à peu près,
Attend, bloc annoté, qu'on veuille bien l'imprimer
Symbole après symbole, comme Sisyphe son rocher.
Ratures, sens perdu, doute, feuillet manquant,
Partout la question triple : -Comment ? Où ? Quand ?

IV

Mais le mot, qu'on le sache, est un être vivant,
La main de l'auteur vibre et tremble en le frappant.
La pagination à l'infini s'échappe,
A chaque instant lacune, embûche, chausse-trappe.
Faut-il mettre un backslash ou bien une cédille,
Au bout du théorème et de ses codicilles ?
Regardons le manuel. Ces choses-là sont rudes,
Il faut pour les comprendre avoir fait ses études !

V

La tâche se termine, le dernier mot s'inscrit.
Elle appelle TEX. Il vient, suppute, renifle, écrit,
Réclame deux dollars avec une parenthèse,
Accentue, calcule en toute hypothèse
Le lieu mobile, obscur, capricieux, changeant,
Où se plaît la virgule aux nageoires d'argent.
Oh cliquetis des phrases, tohu-bohu, rumeur,
De l'univers des mots le terrible écumeur !

VI

TEX termine son travail ; après tant de souffrance,
Tant de labeur obtient enfin sa récompense.
L'imprimante achevant lentement ses devoirs
Eclaire tout à coup dans ses jambages noirs
Le théorème orné de tous ses corollaires,
Eblouissant Shakespeare et ravissant Euler !

Victor Hugo

p.c.c. Bernard Beauzamy