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Semi-Classical Analysis for the Schrödinger Operator and Applications

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§0 INTRODUCTION.

This course falls into two different parts. The first part (Sections 1-5) is the written version of ten lectures I gave in Nankai University in October 1985. It can be seen as an introduction to my work with J. Sjöstrand ($[\text{HE-SJ}]_{1-6}$). My purpose was to give in a simpler situation a relatively self-contained presentation of the tunneling effect. In fact, we have tried to refer only to the two basic books of Reed-Simon [RE-SI] and Abraham-Marsden. [AB-MA] (see also Abraham-Robin for the theory of the stable manifolds). The material presented here comes essentially from $[\text{HE-SJ}]_1$ but we have also used improvements that we found later in $[\text{HE-SJ}]_{2-6}$ and the proof presented here is, at least in the form, partly different (particularly for the B.K.W construction, where we present a simpler method, less general, but perhaps easier to understand for non-specialists in microlocal analysis).

Almost two years later, in June 1987, I was asked to complete these notes to permit a publication as a volume of the Springer Lecture Notes (Nankai Subseries). During these two years, we had applied these techniques, in collaboration with J. Sjöstrand or through students, to many other problems where the tunneling effect plays an important role: resonances, Schrödinger with periodic potential, Schrödinger with magnetic fields, etc... but it is probably too early to write a definite book on the subject. At the same time, a very good book on the Schrödinger operator by Cycon-Froese-Kirsch-Simon [C.F.K.S] has appeared. We have therefore chosen to present in § 6 and § 7 subjects which are complementary to this book and which are natural applications of the theory developed in the first 5 sections.
This book is organized as follows.

In § 1, we present a brief survey of semi-classical mechanics and recall basic facts on the Schrödinger operator. This material is more developed in the recent book by D. Robert [Ro] which we recommend to the reader interested in pseudodifferential techniques.
§ 2 is concerned with the B.K.W construction at the bottom. In § 3, we study the decay of the eigenfunctions in the spirit of Agmon [AG]. These results were developed in the semi-classical context by B. Simon $[\text{SI}]_{2-4}$ and B. Helffer-J. Sjöstrand $[\text{HE-SJ}]_{1-9}$.
§ 4 is concerned with the interaction between different wells. This is a mathematical version of the well-known L.C.A.O method used by chemists.
In § 5, we present briefly the application to Witten´s proof of the Morse inequalities [WIT]. There is an intersection with one chapter of the book [C.F.K.S] but we have tried to go a little further using the techniques of sections 2,3,4, however we still remain far from the best results (related to the method of instantons) obtained in $[\text{HE-SJ}]_4$. In § 6, we study the asymptotic behavior of the first band of

the Schrödinger operator with periodic potentials and present results obtained by B.Simon $[SI]_6$ and A. Outassourt [OU].

§ 7 is devoted to the study of some classical problems on the Schrödinger operator with magnetic fields: criteria for the compactness of the resolvent (after Helffer-Mohamed) [HE-MO]), multiplicity of the first eigenvalue (after Avron-Herbst-Simon [A-H-S], Lavine-O'Carroll [LA, O'CA], Helffer-Sjöstrand $[HE\text{-}SJ]_{10}$ and Helffer [HE]), effect of the flux of the magnetic field $[HE]_3$). The study of these problems is only beginning and we just give a flavor of some of the problems (see also the chapter in [C.F.K.S] devoted to these questions).

I have many people to thank at the end of this introduction. First of all the Universities of Wuhan and Nankai which organized this course in October 1985 with the help of the French "Ministère des Relations Extérieures" and particularly Professors Chi Min-yu, Wang Rou-hai and S.S. Chern.
I want also to thank J. Sjöstrand and D. Robert with whom I have collaborated in this field, M. Dauge who read a part of the text and C. Brunet and M. Coignac who typed the manuscript.
For the reader who does not appreciate my poor English, let me mention in closing that there exists a Chinese version of this course, written up by Professor Chi Min-yu and his students.

CONTENTS

§1 GENERALITIES ON SEMI-CLASSICAL ANALYSIS.

The purpose of the semi-classical analysis is to understand, from a mathematical point of view, the general correspondance principle of the Quantum mechanics saying that, when the Planck constant h tends to zero, we must recover, starting from the Quantum mechanics, the classical mechanics. The best references for this section are for example : Fedoryuk-Maslov [FE-MA], D. Robert [RO], J. Leray [LE] for the semi-classical aspects, Arnold [AR] and Abraham-Marsden [AB-MA] for the classical mechanics and Reed-Simon [RE-SI] for the study of the Schrödinger Operator.

§1.1 - The Classical mechanics (See section 3.3 in [AB-MA]).

In the most simple cases, the classical mechanics describes the motion of a point x(t) in a space $\mathbb{R}^n_x$ (x is the position) and more generally in a C^∞ manifold M. But adding the impulsion $\xi(t)$ of the point , we prefer to work in $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ and more intrinsically in $T^* M$ the cotangent bundle to M. $T^* M$ is a symplectic manifold, that means that we have on $T^* M$ a canonical non-degenerate, closed 2 -form ω. In the case of $T^* \mathbb{R}^n$, ω is defined by :

$$(1.1.1) \qquad \omega = \sum_j d\,\xi_j \wedge d\,x_j$$

In the case of $T^* M$, if (x_j) is a system of local coordinates and (ξ_j) is the dual system of coordinates, ω can be written in the same way.

More generally, if Ω is a symplectic manifold of dimension 2n, we can always find locally a system of coordinates (x,ξ) s.t ω is defined by (1.1.1). These coordinates are called the canonical coordinates.

In Hamiltonian Mechanics, the motion is described by a C^∞ function on $T^* M = \Omega$ called the hamiltonian :

$$(1.1.2) \qquad (x,\xi) \rightarrow p(x,\xi)$$

Associated to this hamiltonian, we define the hamiltonian vector H_p on Ω , which is given in canonical coordinates by :

$$(1.1.3) \qquad H_p = \left(\frac{\partial p}{\partial \xi}\ ,\ -\frac{\partial p}{\partial x}\right)$$

The motion of a point in Ω is described by the integral curves of H_p (called the bicharacteristics) which are the solutions of the system :

$$(1.1.4) \qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial p}{\partial \xi}(x,\xi) & x(0,y,\eta) = y \\[2ex] \dfrac{d\xi}{dt} = -\dfrac{\partial p}{\partial x}(x,\xi) & \xi(0,y,\eta) = \eta \end{cases}$$

It is not our purpose to give here precise existence theorems for the equations (1.1.4), but it is well-known that at least locally and for $|t|$ small, the solutions exist and we can define the Hamiltonian flow ϕ_t, by :

$$(1.1.5) \qquad \phi_t(y,\eta) = (x(t,y,\eta),\xi(t,y,\eta))$$

We are mainly interested in these lectures by the Hamiltonian :

$$p(x,\xi) = \xi^2 + V(x)$$

In the case where M is a Riemannian manifold, if $G = g^{ij}$ is the matrix of the metric in the coordinates x, we define $G^{-1} = g_{ij}$ and we must define ξ^2 as :

$$(1.1.6) \qquad \xi^2 \overset{\text{def}}{\equiv} \sum_{i,j} g_{ij}(x)\, \xi_i\, \xi_j$$

--

Then the motion is given (in the case of $\mathbb{R}^n$) by :

$$(1.1.7) \qquad \frac{dx}{dt} = 2\xi \ , \quad \frac{d\xi}{dt} = -\frac{\partial V}{\partial x}$$

so we recover the classical equation of the motion in M :

$$(1.1.8) \qquad \frac{d^2x}{dt^2} = -2\frac{\partial V}{\partial x} = -2 \text{ grad } V$$

(the number 2 appears because usually one takes $p(x,\xi) = \frac{\xi^2}{2} + V(x)$)

§1.2 - The Quantum Mechanics

One of the problems is to find a natural Hilbert space. Here, because, we consider only the case when $\Omega = T^* M$, the natural choice is $L^2(M)$ (where the measure is the canonical measure associated to the Riemannian Structure ; because we consider only the case $M = \mathbb{R}^n$ or the case M = compact C^∞ Manifold, $L^2(M)$ is complete). We need also a dense subspace (usually $C^\infty(M)$ if M is compact and $C_0^\infty(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$ in the case of $\mathbb{R}^n$). Let us consider the case of $\mathbb{R}^n$. Under some conditions on p (See [HO], the Weyl-Calculus, for a general point of view), we associate to the Hamiltonian p an operator a priori defined on $\mathcal{S}(\mathbb{R}^n)$ by the so-called Weyl Quantification :

$$p \to Op^W(p) = p^W(x,h\, D_x)$$

$$(1.2.1) \qquad \begin{cases} Op^W(p).f = h^{-n} \iint e^{\frac{i}{h}\langle x-y,\xi\rangle}\, p\left(\frac{x+y}{2},\xi\right) f(y)\, dy\, đ\xi \\ \text{for } f \in \mathcal{S}(\mathbb{R}^n) \ , \ h \in \,]0,h_0] \\ \text{with the convention } đ\xi = (2\pi)^{-n}\, d\xi \end{cases}$$

via a theory of pseudodifferential operators on $\mathbb{R}^n$.

This choice is not the only possible but it is very convenient because you get for example that, if p is real, $Op^W(p)$ is formally self-adjoint, that means :

(1.2.2) $(p^W(x,hD)\,u/v) = (u|\,p^W(x,hD)v)$ $\forall u \in \mathcal{S}(\mathbb{R}^n)$ $\forall v \in \mathcal{S}(\mathbb{R}^n)$

where (/) denotes the L^2 scalar product.

For our purpose, we are interested in the extension of this formally self-adjoint operator defined on $\mathcal{S}(\mathbb{R}^n)$ as a self-adjoint non-bounded operator on $L^2(\mathbb{R}^n)$. When this extension exists and is unique, $p^W(x,hD)$ is called essentially self-adjoint. General criteria to verify that $p^W(x,hD)$ is essentially self-adjoint for h small enough are given in the more general context of the " admissible operators " (associated to hamiltonians depending on h

$$p(x,\xi,h) \sim \sum_{j=0}^{\infty} h^j\, p_j(x,\xi))$$

are given in $[RO]_3$ and $[HE\text{-}RO]_{1 \text{ to } 3}$.

In the case of a Riemannian Manifold, the h-pseudodifferential calculus can also be defined but you lose the notion of the Weyl-Calculus at least if you don't add a group structure (See [MELI]). But, in these lectures, we will be mainly interested in the study of the Schrödinger operator associated to the Hamiltonian $\xi^2 + V(x)$, where V is a C^∞ real function. In this particular case, there is a natural quantification given by the geometry. We associate to ξ^2 the Laplace-Beltrami operator on the Manifold M $h^2(-\Delta) = h^2(d^*+d)^2$, which is given in coordinates by :

(1.2.3) $$-h^2\,\Delta = -\,h^2 \sum_{i,j} \sqrt{g}\,\frac{\partial}{\partial x_i}\,\frac{1}{\sqrt{g}}\cdot g_{ij}\,\frac{\partial}{\partial x_j}$$

where $g = (\det G)$

and we get the Schrödinger operator :

(1.2.4) $$-\,h^2\,\Delta + V$$

which is formally self-adjoint on $C_0^\infty(M)$ where the scalar product is given by $(u,v) \to \int u\,.\,\bar{v}\,g^{-1/2}\,dx$ for u and v with support in the chart.

§1.3 - Some basic results on the Schrödinger operators

In the case where M is a C^∞ compact riemannian manifold, the Schrödinger operator $(-h^2\,\Delta + V)$ defined on $C^\infty(M)$ by (1.2.3) has a unique self-adjoint extension whose domain is the Sobolev space $H^2(M)$. Then we know that the injection of $H^2(M)$ in $L^2(M)$ is compact. In this case, it is well-known that we have a orthonormal basis of $L^2(M)$ constituted by eigenfunctions (in $C^\infty(M)$) $\varphi_j(h)$. $(j \in \mathbb{N})$ satisfying to :

$$(1.3.1)\quad \begin{cases} (-h^2\ \Delta + V)\ \varphi_j(h) = \lambda_j(h)\ \varphi_j(h) \\ \lambda_j(h) \leqslant \lambda_{j+1}(h) \end{cases}$$

Moreover, for h fixed, $\lambda_j(h) \xrightarrow[j \to \infty]{} \infty$

The spectrum of the Schrödinger operator on a non compact manifold is more complicate. We will restrict ourselves in these notes in the case of $\mathbb{R}^n$ and we assume in this case that the following hypothesis is satisfied for some constant C_0 :

$$(1.3.2)\qquad V \in C^\infty(\mathbb{R}^n)\ ,\quad V \geqslant - C_0$$

Under this hypothesis, one can define a self-adjoint operator on $L^2(\mathbb{R}^n)$ by taking the Friedrichs extension starting of $C_0^\infty(\mathbb{R}^n)$. Moreover this is the unique self-adjoint extension of $- h^2\ \Delta + V$ starting of $C_0^\infty(\mathbb{R}^n)$ (See [RE-SI] Vol. II, Th. X.28 and p. 340 ex 24). Let us define :

$$(1.3.3)\qquad C = \varliminf_{|x| \to \infty} V$$

Then, the restriction of the spectrum to $]-\infty, C[$ is constituted of eigenvalues with finite multiplicity (See Reed-Simon[RE-SI], § XIII.4 Cor. 2 p. 113 and Th. XIII.16).

In the future, we make the convention that $C = +\infty$ in the compact case.

As an example of a semi-classical result we shall need after, let us present a spectral result.

For $\lambda < C$, let us define :

$$(1.3.4)\qquad N_h(\lambda) = \{\#\ j,\ \lambda_j(h) \leqslant \lambda\}$$

The problem is to study the asymptotic behavior of $N_h(\lambda)$ when h tends to 0. It is a classical result that :

$$(1.3.5)\qquad \lim_{h \to 0} h^n\ N_h(\lambda) = \int_{\xi^2 + V(x) < \lambda} dx\ d\!\!{}^{-}\xi$$

(See [RE-SI], [HE-RO], [CDV] for references).

In the last years, many mathematicians have tried to give the best estimate for :

$$(1.3.6)\qquad R_h(\lambda) = h^n\ N_h(\lambda) - \int_{\xi^2 + V(x) < \lambda} dx\ d\!\!{}^{-}\xi$$

Let us give the best known theorem

Theorem 1.3.1

Let V verifying (1.3.2).

Suppose that $\lambda < C$ and that λ is not a critical value for V then :

(1.3.7) $R_h(\lambda) = 0(h)$

This theorem is proved in the compact case by Colin de Verdière $[C.D.V]_1$. In the case of $\mathbb{R}^n$, $R_h(\lambda) = 0(h^{\mathcal{J}})$ was proved by Tulovski-Šubin [ŠU] (for $\mathcal{J} < 1/2$), Hörmander $[H0]_2$ ($\mathcal{J} < 2/3$), Combes-Schrader-Seiler [C.S.S], and Helffer-Robert $[HE\text{-}RO]_3$ but always under additionnal hypothesis of the potential of the following type :

$$(1.3.8) \quad \begin{cases} \exists\, C_\alpha \quad \text{s.t} \quad |\partial_x^\alpha V| \leqslant C_\alpha (V + C_0 + 1) \\ \exists\, D,M \quad \text{s.t} \quad |V(x)| \leqslant D(V(y) + C_0 + 1)\ (1 + |x-y|)^M \\ \qquad\qquad\qquad \forall x \in \mathbb{R}^n,\ \forall y \in \mathbb{R}^n . \end{cases}$$

Ivrii has announced theorem 1.3.1. without hypothesis (1.3.8). We shall explain in section 4.2, Remark 4.2.4. how to deduce the Theorem 1.3.1. in the general case from the Theorem (1.3.1) in the particular casewhere (1.3.8) is satisfied.

Remark 1.3.2

In the following we need only a weak version of (1.3.5) :

(1.3.9) $N_h(\lambda) = \mathcal{O}(h^{-N_0})$ for some N_0.

but also in the study of Dirichlet problems.

This type of results is an easy consequence of the min-max principle ([RE-SI] Vol. IV, Th. XIII.1 and XIII.2 and probl. 1 p. 364) which gives, if $\lambda \leqslant \underline{\lim}_{|x| \to \infty} V_0$, the inequality :

(1.3.10) $N_h^{V_0}(\lambda) \geqslant N_h^{V_1}(\lambda)$ if $V_0 \leqslant V_1$

Remark 1.3.3

One could think that it is rather stupid to improve the estimate of $R_h(\lambda)$.

But O(h) is an important step because this estimate cannot be improved without adding extra-hypotheses on the flow ϕ_t on the energy level : $\xi^2 + V = \lambda$. Ivrii [IV] and V. Petkov--D. Robert [PE-RO] have given conditions to get 0(h).

Remark 1.3.4

Interesting questions remain, in the case of Magnetic fields, that means for operators of the type :

$$P(h) = - \sum_{j=1}^{n} (h\, \partial_{x_j} - i\, a_j)^2 + V$$

We refer to papers of Avron-Herbst-Simon [A.H.S], J.M. Combes - R. Schrader - R. Seiler [C.S.S], D. Robert $[RO]_2$, B. Simon $[SI]_1$ and more recently to three papers of H. Tamura [TA], J.P. Demailly [DE] and Y. Colin de Verdière $[C.D.V]_2$.

Another interesting point is that you can get, under convenient hypotheses, a compact resolvant without the hypotheses that $V \to \infty$.

For example, one can deduce from my results with Nourrigat [HE-NO] that, if

$$V = \sum_{i=1}^{N} q_i^2(x)$$

and if the vector space of the polynomials generated by the $\partial_x^\alpha q_i$ contains all the polynomials of d° 1, the resolvant is compact.

Example : $V = x_1^2\, x_2^2$ on $\mathbb{R}^2_{x_1,x_2}$

The estimate of $N_h(\lambda)$ (for $\lambda \to \infty$ (h fixed) or $h \to 0$ (λ fixed)) is a difficult task in general (See for particular cases $[RO]_2$, $[SI]_6$).

Another interesting example is :

$$-(h\, \partial_{x_1} - x_2^2)^2 - (h\, \partial_{x_2})^2 + x_1^2$$

which is also with compact resolvant.

Remark 1.3.5.

There are many papers studying the asymptotic behavior of $N_h(\lambda)$ when λ tends to ∞. I prefer to refer to my book [HE] where many references are given.

§2 B.K.W CONSTRUCTION FOR A POTENTIAL NEAR THE BOTTOM IN THE CASE OF NON-DEGENERATE MINIMA.

In all this section, we work with some C^∞ potential V which admits a local non-degenerate minimum at a point.
By changing of coordinates, we can suppose that :

$$(2.0) \qquad V(0) = 0, \; V'(0) = 0, \; V''(0) > 0$$

where $V''(0)$ is the Hessian of V at 0.
In this section we will not try to follow the most direct way to get the results but we prefer to see how the different technics work.

§.2.1 - The Harmonic oscillator

Before to study the general situation, it is convenient to recall the basic properties of the Harmonic oscillator.
Let us consider in $\mathbb{R}^n$:

$$(2.1.1) \qquad P_0(h) = - \sum_{j=1}^{n} h^2 \frac{d^2}{dx_j^2} + \sum_{j=1}^{n} \mu_j x_j^2 \qquad \text{with } \mu_j > 0$$

This is of course a Schrödinger operator whose potential is given by :

$$(2.1.2) \qquad V_0(x) = \sum_j \mu_j x_j^2$$

Let us recall very briefly how to compute the spectrum and the eigenfunctions of P_0 (h).

Step 1 The spectrum of $(- \frac{d^2}{dx^2} + x^2)$ is constituted by eigenvalues (2j+1) ($j \in \mathbb{Z}^+$).
The first eigenfunction is given by $(\sqrt{2\Pi})^{-1} e^{-x^2/2} = u_0(x)$.
The (j+1)th eigenfunction u_j corresponding to the eigenvalue (2j+1) is deduced from u_{j-1} by the relation :

$$(2.1.3) \qquad u_j(x) = \alpha_j \cdot (- \frac{d}{dx} + x) u_{j-1} \qquad \text{where } \alpha_j > 0$$

and is chosen to normalize u_j.
You get easily that :

$$(2.1.4) \qquad u_j(x) = P_j(x) \cdot e^{-x^2/2}$$

where $P_j(x)$ is a polynomial of order j

Step 2 By easy manipulations, you get that the spectrum of $P_0(1)$ is given by

(2.1.5) $$\lambda_\alpha = \sum_{j=1}^{n} \sqrt{\mu_j}\ (2\alpha_j + 1) \quad , \quad \alpha \in (\mathbb{Z}^+)^n$$

with corresponding eigenfunction

(2.1.6) $$u_\alpha(x) = P_{\alpha_1}(\mu_1^{1/4} x_1) \cdot \ldots \cdot P_{\alpha_n}(\mu_n^{1/4} x_n) \cdot e^{-\sum_{j=1}^{n} \sqrt{\mu_j}\ x^2{}_j/2} \cdot (\mu_1 \cdot \ldots \cdot \mu_n)^{-1/8}$$

Step 3 We observe now that if $\Psi(x)$ is a normalized eigenfunction for $P_0(1)$ associated to the eigenvalues λ, $\Psi(x/\sqrt{h}) \cdot h^{-n/4}$ is a normalized eigenfunction for $P_0(h)$ associated to the eigenvalue λh. We then get finally that the eigenvalues of $P_0(h)$ are given by :

(2.1.7) $$\lambda_\alpha(h) = h\,\lambda_\alpha(1) = (\sum_{j=1}^{n} \sqrt{\mu_j}\ (2\alpha_j+1))\ h \qquad (\alpha \in (\mathbb{Z}^+)^n)$$

with corresponding eigenfunction :

(2.1.8) $$u_\alpha(h)(x) = h^{-n/4}\, u_\alpha(x/\sqrt{h}) = h^{-n/4}\, (\mu_1 \ldots \mu_n)^{-1/8}\, P^\alpha(\frac{\mu^{1/4} . x}{\sqrt{h}}) . e^{-\sum_{j=1}^{n} \sqrt{\mu_j}\, x^2{}_j/2h}$$

--

We keep from (2.1.8) that $u_\alpha(h)(x)$ has the following form :

(2.1.9) $$u_\alpha(h)(x) = h^{-n/4}\, a_\alpha(x,h)\, e^{-\varphi_0(x)/h}$$

where

(2.1.10) $$\varphi_0(x) = \sum_{j=1}^{n} \frac{\sqrt{\mu_j} \cdot x_j^2}{2}$$

(2.1.11) $$a_\alpha(x,h) = c_\alpha\, h^{-|\alpha|/2}\, [x^\alpha + \sum_{\substack{|\beta| < |\alpha| \\ |\beta| + \gamma/2 = |\alpha|}} c^\beta_{\alpha\gamma}\, x^\beta\, h^\gamma\,] \qquad \text{with } c_\alpha \neq 0$$

Let us also remark that $\varphi_0(x)$ satisfies to the " eiconal " equation :

(2.1.12) $$|\nabla \varphi_0(x)|^2 = V_0(x)$$

§.2.2 - Approximate solutions starting from the Harmonic oscillator.

For a more sophisticated version in this spirit, we refer to B. Simon $[SI]_2$ (and to his references). We make the hypothesis (2.0), and suppose moreover that :

(2.2.1) $$V''(0) = 2 \begin{pmatrix} \mu_1 & & 0 \\ & \mu_2 & \\ 0 & & \ddots\ \mu_n \end{pmatrix} \quad ; \quad \mu_j > 0$$

and we suppose (See 1.2.3) that the metric is chosen s.t :

(2.2.2) $\quad g = 1 \,, \quad g_{ij}(x) = \delta_{ij} + O(|x|)$

To simplify, we shall only look at the first eigenvalue of the Schrödinger operator : $P(h) = -h^2 \Delta + V$ (see 1.2.3).
Then starting from :

$$u_0(h)(x) = c_0 . h^{-n/4} e^{-\varphi_0(x)/h} \qquad (c_0 \neq 0)$$

(2.2.3)

$$\text{s.t } \| u_0(h) \| = 1$$

we introduce, for $\varepsilon > 0$ fixed sufficiently small :

(2.2.4) $\quad \Psi_0(h)(x) = \chi_\varepsilon(\varphi_0(x)) \,.\, u_0(h)(x)$

where χ_ε is a C^∞ function s.t $\chi_\varepsilon(t) = 1$ for $t \in [-\varepsilon/2, +\varepsilon/2]$, and $\operatorname{supp} \chi_\varepsilon \subset [-\varepsilon, \varepsilon]$.
Let us look now at :

$$P(h)(\Psi_0(h)(x))$$

We have :

(2.2.5) $\quad P(h)(\Psi_0(h)(x)) = \lambda_0(h)\, \Psi_0(h)(x) + R_0(h)(x) + R_1(h)(x)$

with

(2.2.6) $\quad R_0(h)(x) = [P, \chi_\varepsilon]\, (u_0(h)(x)$

(2.2.7) $\quad R_1(h)(x) = \chi_\varepsilon (P - P_0(h))(u_0(h)(x))$

It is immediate to see that :

(2.2.8) $\quad \| R_0(h)(x) \|_{L^2} \leqslant C\, h^{-n/4} \,.\, e^{-\varepsilon/4h}$

Let us look now at $R_1(h)(x)$ which is the sum of two terms :

$$R^\circ{}_1(h)(x) = -h^2 \chi_\varepsilon \, (\sum_{i,j} \frac{d}{dx_i} \, (g_{ij}(x) - g_{ij}(0)) \frac{d}{dx_j}) \, u_0(h)(x)$$

and

$$R^1{}_1(h)(x) = \chi_\varepsilon(\varphi_0(x)) \, (V(x) - V_0(x)) \, u_0(h)(x)$$

Using (2.2.2) and the property $(V(x)-V_0(x)) = 0(|x|^3)$
we have to estimate in the L^2-norm

$$(a) = h^2\, 0(|x|)\, \left(\frac{d^2}{dx_i\, dx_j}\, u_0(h)(x)\right) \cdot \chi_\varepsilon(\varphi_0(x))$$

$$(b) = h^2\, \left(\frac{d}{dx_i}\, u_0(h)(x)\right) \cdot \chi_\varepsilon(\varphi_0(x))$$

and

$$(c) = 0(|x|^3) \cdot u_0(h)(x) \cdot \chi_\varepsilon(\varphi_0(x))$$

$$\|(a)\|^2_{L^2} \leqslant C\, h^{-n/2} \int_{\mathbb{R}^n} |x|^6 \cdot e^{-2\varphi_0(x)/h}\, dx + C\, h^{2-n/2} \int_{\mathbb{R}^n} |x|^2\, e^{-2\varphi_0(x)/h}\, dx$$

$$\|(b)\|^2_{L^2} \leqslant C\, h^{-n/2} \cdot h^2 \int_{\mathbb{R}^n} |x|^2\, e^{-2\varphi_0(x)/h}\, dx$$

$$\|(c)\|^2_{L^2} \leqslant C\, h^{-n/2} \int_{\mathbb{R}^n} |x|^6\, e^{-2\varphi_0(x)/h}\, dx$$

and we get easily that :

$$\|(a)\|^2 + \|(b)\|^2 + \|(c)\|^2 \leqslant \tilde{C}\, h^3$$

(2.2.9) $\qquad \|R_1(h)(x)\|_{L^2} \leqslant \tilde{C}\, h^{3/2}$

We remark also very easily that :

(2.2.10) $\qquad \|\Psi_0(h)(x)\|_{L^2} = 1 + \mathcal{O}(e^{-\varepsilon/4h})$

Finally, we get a C^∞ function $\Psi_0(h)(x)$ s.t :

(2.2.11) $\qquad (P(h) - \lambda_0\ (h))\, \Psi_0(x,h) = \mathcal{O}(h^{3/2})\, \|\Psi_0\| \qquad$ in $L^2(\mathbb{R}^n)$

Suppose now that

(2.2.12) $\qquad \underline{\lim}_{|x|\to+\infty}\ V > 0$

then using a well known property of the self-adjoint operators :

(2.2.13) $\qquad d(\lambda, \mathrm{Sp}\ P(h))\, \|u\|_{L^2} \leqslant \|(P(h)-\lambda)u\|_{L^2}$

for $\lambda \in \mathbb{C}$ and where Sp P(h) is the spectrum of P(h), we get from (2.2.11) that :

(2.2.14) $\qquad d(\lambda_0(h), \mathrm{Sp}\ P(h)) \leqslant \tilde{C} \cdot h^{3/2}$